High Frequency Trading

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Stanford University

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Long Term Goals: Design and implement a trading strategy based on high frequency stocks data.

Data:

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Figure: A glimpse into AAPL (2021-01-05)
**Question**: From the point of view of a market maker, what is the fair price given the state of the order book?

If we know that fair price $\hat{P}$, we are able to place an order $(P_{t+1}^b, P_{t+1}^a)$ such that:

$$P_t^b \leq P_{t+1}^b \leq \hat{P} \leq P_{t+1}^a \leq P_t^a$$ (1)

Thus, we provide liquidity all the while being covered against price variations.
State of the Art

- **Mid-Price**

\[ M_t = \frac{P^b_t + P^a_t}{2} \quad (2) \]

- **Weighted Mid-Price**

\[ W_t = I_t P^a_t + (1 - I_t) P^b_t \quad (3) \]

\[ I_t = \frac{Q^b_t}{Q^a_t + Q^b_t} \quad (4) \]

where \( P^a_t \) (resp. \( P^b_t \)) denotes the price at best ask (resp. best bid) and \( Q^a_t \) (resp. \( Q^b_t \)) denotes the volume at best ask (resp. at best bid)
State of the Art (2)

Figure: Bid, ask, mid and weighted mid prices AAPL stock (2021-01-05)
The Micro Price

Goal
Build a fair estimator of the price $P_t$ of the stock at time $t$.

Definitions

- **Times when the mid-price changes:**
  \[ \tau_1 = \inf \{ u > t, M_u - M_u^- \neq 0 \} \] (5)
  \[ \tau_{i+1} = \inf \{ u > \tau_i, M_u - M_u^- \neq 0 \} \] (6)

- **Micro-price:**
  \[ P_t^{\text{micro}} := \lim_{i \to \infty} \mathbb{E} [ M_{\tau_i} | \mathcal{F}_t ] \] (7)

**Interpretation:** If we are at time $t$, we consider that the fair price is the conditional expectation of future mid-prices based on the current state of the order book (analogy with Black-Scholes theory).
The Micro Price (2)

Assumptions

▶ The information in the order book is determined by the processes of the mid, the imbalance and the spread:

\[ \mathcal{F}_t = \sigma(M_s, I_s, S_s; s \leq t) \]  \tag{8} 

▶ Mid price increments are independent from mid-price level:

\[
\begin{align*}
E[M_{\tau_{i+1}} - M_{\tau_i} | M_t = M, I_t = I, S_t = S] &= \\
&= E[M_{\tau_{i+1}} - M_{\tau_i} | I_t = I, S_t = S]
\end{align*}
\]
The Micro Price (3)

Theorem:
Given these two assumptions, the prediction of the \( i^{th} \) mid-price can be written as:

\[
E [M_{\tau_i} | \mathcal{F}_t] = M_t + \sum_{k=1}^{i} g^k(I_t, S_t) \tag{9}
\]

where

\[
g^1(I, S) = E [M_{\tau_1} - M_t | I_t = I, S_t = S] \tag{10}
\]

and

\[
g^{i+1}(I, S) = E [g^i(I_{\tau_1}, S_{\tau_1}) | I_t = I, S_t = S] \tag{11}
\]
The Micro Price (4)

The finite-space model

We denote by $X_t := (I_t, S_t)$ the state of the order book. For a finite number of possible values of the imbalance $I_t$ and spread $S_t$, we can express the micro-price as:

$$P_{t}^{\text{micro}} = M_t + \sum_{k=1}^{\infty} B^k G^1$$  \hspace{1cm} (12)

where:

$$B := (I - Q)^{-1} T$$ \hspace{1cm} (13)

$$G^1 := (I - Q)^{-1} RK$$ \hspace{1cm} (14)

$$Q_{xy} := P(M_{t+1} - M_t = 0 \cap X_{t+1} = y | X_t = x)$$ \hspace{1cm} (15)

$$T_{xy} := P(M_{t+1} - M_t \neq 0 \cap X_{t+1} = y | X_t = x)$$ \hspace{1cm} (16)

$$R_{xk} := P(M_{t+1} - M_t = k | X_t = x)$$ \hspace{1cm} (17)
Results adjustment

Figure: adjustments
Results state distribution
Out of sample Predictions
Out of sample wMid vs MicroP
Second Strategy: With inventory

Assumptions

- The mid price is a geometric brownian motion with volatility $\sigma$, the agent has no opinion on the drift or any autocorrelation structure of the stock

$$dS_u = \sigma dW_u$$ (18)

- The agent’s value function is:

$$v(x, s, q, t) = E \left[ -\exp(-\gamma(x + qS_T)) \right]$$ (19)

It can be written as $-\exp(-\gamma x) \exp(-\gamma qs) \exp \left( \frac{\gamma^2 q^2 \sigma^2 (T-t)}{2} \right)$

where $x$ is the initial wealth in dollars.
Reservation bid and ask

Definition:

- The reservation bid $r^b$ is defined by:

\[ v(x - r^b(s, q, t), s, q + 1, t) = v(x, s, q, t) \]  \hspace{1cm} (20)

It is the price that would make the agent indifferent between his current portfolio and his current portfolio plus one stock.

- The reservation ask $r^a$ is defined by:

\[ v(x + r^a(s, q, t), s, q - 1, t) = v(x, s, q, t) \]  \hspace{1cm} (21)

It is the price that would make the agent indifferent between his current portfolio and his current portfolio minus one stock.
Goal
Our goal in general will be to estimate $r^a$ and $r^b$.

Analytic Solution
In our very simple framework we have:

$$r^b(s, q, t) = s - (1 + 2q)\frac{\gamma\sigma^2(T - t)}{2}$$  \hspace{1cm} (22)

$$r^a(s, q, t) = s + (1 - 2q)\frac{\gamma\sigma^2(T - t)}{2}$$  \hspace{1cm} (23)

In this framework, we define the **reservation price**
$$r(s, q, t) = s - q\gamma\sigma^2(T - t)$$ as the mid of these two prices. It is an adjustment of the midprice which accounts for the inventory held by the agent.
With Limit Orders

- We allow the agent to trade the stock through limit orders that he sets around the mid-price.
- $p^b$ denotes the bid quote ($\delta^b = s - p^b$)
- $p^a$ denotes the ask quote ($\delta^a = s - p^a$)

Market impact

- Let us assume the agent places an order to buy $Q$ stocks.
- Let $p^Q$ be the price of the highest limit order executed in the trade.
- We define $\Delta p = p^Q - s$ to be the temporary market impact of the trade.
Trading intensity

Poisson model

- Orders to sell stock will hit the agent’s buy limit order at Poisson rate $\lambda^b(\delta^b)$, a decreasing function of $\delta^b$.
- Orders to buy stock will hit the agent’s sell limit order at Poisson rate $\lambda^a(\delta^a)$, a decreasing function of $\delta^a$.

Stochastic wealth process

The wealth and inventory are now stochastic

$$dX_t = p^a dN_t^a - p^b dN_t^b$$  \hspace{1cm} (24)

where $N_t^b$ is the amount of stocks bought by the agent, $N_t^a$ is the amount of stocks sold by the agent.

The inventory is thus defined as:

$$q_t = N_t^b - N_t^a$$  \hspace{1cm} (25)
New Optimization Problem

The objective of the agent who can set limit orders is:

$$\max_{\delta^a, \delta^b} E_t \left[ -\exp(-\gamma(X_T + q_T S_T)) \right]$$  \hspace{1cm} (26)

Trading intensity

We will assume from now on that we have symmetric, exponential arrival rates:

$$\lambda^a(\delta) = \lambda^b(\delta) = Ae^{-k\delta}$$  \hspace{1cm} (27)

which corresponds to $\Delta p \simeq ln(Q)$
Intuition on reservation price

- Basic strategy of creating symmetrical bid ask order around bid price will not work when price is trending.
- Inventory risk will build up

**Figure: downtrend stock**
Intuition on reservation price (2)

$r(s, q, t) = s - q\gamma\sigma^2(T - t)$

- $q$ is our inventory, if we have positive inventory, our reservation price is lower than the mid price, vice versa for negative inventory
- Gamma is our risk aversion to inventory
- $T-t$ is the time till the end of trading day, as trading day getting closer to the end, reservation price is more "aggressive" on rebalancing inventory
Intuition on reservation price (3)

Figure: reserved price
Implementation

- 1000 simulations of different stock paths
- recall that the rate of market order is assume to be Poisson with intensity $\lambda(\delta) = A \cdot \exp(-k\delta)$
- we will place limit order at optimal ask and optimal bid and it will get executed with probability $\lambda dt$
- pnl will be calculated. at each time step. profit for each limit sell order executed is $s + \delta^a$ and loss for each buy order executed is $s - \delta^b$
- **Control Strategy**: will place orders symmetrically around the mid price. it will get executed with a fixed probability $p$
## Results

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<th>Std(Profit)</th>
<th>Final q</th>
<th>Std(q)</th>
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**Figure:** pnl with small Gamma
### Results (2)

**Figure:** pnl with big Gamma

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Next Steps

- Order quantity not defined in the paper
- Assumption that volatility is constant might be too strong
- Order book liquidity parameter is constant