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ON THE RANGE OF RECURRENT MARKOV CHAINS

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Abstract: Let $\{X_n\}_0^\infty$ be an irreducible recurrent Markov Chain on the nonnegative integers. A result of Chosid and Isaac (1978) gives a sufficient condition for $n^{-1}R_n \to 0$ w.p. 1. where R_n is the range of the chain. We give an alternative proof using Kingman's subadditive ergodic theorem (Kingman, 1973). Some examples are also given.

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The main purpose of this note is to supply of proof of the following theorem using Kingman's subadditive ergodic theorem.

Theorem. *Let $\{X_n\}_0^\infty$ be an irreducible recurrent Markov Chain on the nonnegative inters. Let $R_n \equiv$ cardinality of $\{X_0, X_1, \dots X_n\}$ be the range sequence. Let $T_a = \inf\{n\colon\ n \geqslant 1,\ X_n = a\}$ be the first hitting time of the point. a. Then*

$$E_0(R_{T_0}) < \infty \textit{ implies } n^{-1}R_n \to 0 \textit{ w.p. } 1. \qquad (1)$$

for any initial distribution λ.

This theorem is a variant of a theorem due to Chosid and Isaac [2]. Their hypothesis is $E_\pi R_{T_0} < \infty$ where π is the unique invariant measure and is only apparently stronger than ours. (See Remark 2 below.) Their method uses Birkhoff's ergodic theorem along the lines of Kesten, Spitzer and Whitman as quoted in Chosid and Isaac (1978). We use Kingman's subadditive ergodic theorem (Kingman, 1973). (We need it only to prove convergence w.p. 1. – convergence in probability is easy. See the proof and Remark 1 below.)

Proof. Let $\{T_{0,i}\}_0^\infty$ be times of successive returns to 0 by the chain $\{X_n\}$. That is, we set $T_{0,0} \equiv T_0$, $T_{0,i+1} = \inf\{n\colon\ n > T_{0,i},\ X_n = 0\}$ for $i \geqslant 0$. Let for $l > k$, Y_{kl} be the cardinality of $\{X_n\colon\ T_{0,k} \leqslant n \leqslant T_{0,l}\}$, i.e., the number of distinct sites visited by the chain between the kth and the lth return to 0. Then, $\{Y_{kl}\}$ satisfies the following postulates of Kingman's subadditive ergodic theorem, namely,

(i) For $k < l < m$, $Y_{km} \leqslant Y_{kl} + Y_{lm}$ (subaddivity),

(ii) $Y_{kl} \sim Y_{k+1,l+1}$ (stationarity) ($\sim$ means distributed as),

(iii) $0 \leqslant EY_{0l} \leqslant lE_0(R_{T_0}) < \infty$ (integrability).

Thus, by Kingman's theorem,

$$\lim_{n\to\infty} n^{-1}Y_{0n} \equiv \xi \qquad (2)$$

exists w.p. 1.

Now we will show that $\xi = 0$ w.p. 1. This will imply the given result since, for any initial distribution λ,

$$R_n \leqslant R_{T_0} + Y_{0(N_n+1)} \quad \text{for } n \geqslant T_0,$$

where N_n is the number of returns to 0 by time n and hence

$$n^{-1}R_n \leqslant n^{-1}R_{T_0} + n^{-1}(N_n+1)(N_n+1)^{-1}Y_{0,N_n+1}$$

which tends to zero by (2) and the facts that

$N_n \leqslant n$, and that due to recurrence, $T_0 < \infty$ and $N_n \to \infty$ w.p. 1.

To see that $\xi \equiv 0$ w.p. 1., we bring in the sequence of random variables $\{n_j\}_1^\infty$ where n_j is the number of new sites visited during the jth excursion, i.e., between $T_{0,j-1}$ and $T_{0,j}$, $j = 1, 2, \ldots$. Notice that these n_n's are not independent but they satisfy $n_j = Y_{0,j} - Y_{0,j-1}$ (ignoring the visits before $T_{0,0}$). Now $En_j = \Sigma_k \theta_k (1-\theta_k)^{j-1}$ where θ_k is the probability that the state k is visited at all during any given excursion. The hypothesis says $En_1 = \Sigma_k \theta_k < \infty$. Now by irreducibility and recurrence we have $\theta_k > 0$ for all k. By the dominated convergence theorem we have $\lim_{j\to\infty} En_j \Sigma_k \theta_k \cdot 0 = 0$. Thus

$$E\left(n^{-1} Y_{0n}\right) \leqslant n^{-1} \sum_1^n En_j \to 0 \tag{3}$$

implying $n^{-1} Y_{0n} \to 0$ in probability. This with (2) says that $\xi = 0$ w.p. 1. □

Remarks 1. It is clear from (3) and the above proof that if (1) is weakened to convergence in probability then we do not need to use the subadditive ergodic theorem at all.

2. A standard argument from Markov Chain theory will show that $E_a R_{T_a}$ is finite for all a or for no a. It can also be shown that $E_0 R_{T_0} < \infty$ implies $E_\pi R_{T_0} < \infty$ thus making our hypothesis equivalent to that of Chosid and Isaac (1978).

3. As noted in Chosid and Isaac (1978), the hypothesis $E_0(R_{T_0}) < \infty$ is only a sufficient condition. The relevant example is the simple symmetric random walk. See Remark 5 below.

4. Chosid and Isaac (1978) have an example where $E_0(R_{T_0}) = \infty$ and $\lim_n n^{-1} R_n > 0$ w.p. 1. thus showing that we could not dispense with our hypothesis. Finding a necessary and sufficient condition for $n^{-1} R_n$ to go to zero w.p. 1. is an interesting open problem.

5. We strengthen Chosid and Isaac's example mentioned above with an example where $E_0 R_{T_0} = \infty$ but $n^{-1} R_n$ does not go to zero in probability as well.

Let $\xi_2, \xi_2, \ldots$ be i.i.d. aperiodic positive integer valued random variables and $S_0 = 0$, $S_k = \xi_1 + \xi_2 + \ldots + \xi_k$, $k \geqslant 1$. Let $N_n = j$ and $X_n = n - S_j$ if $S_j \leqslant n < S_{j+1}$. Then $\{X_n\}_0^\infty$, called the age chain, is a recurrent Markov Chain on $\{0, 1, 2, \ldots\}$. Let $M_k = \max_{1 \leqslant i \leqslant k} \xi_i$. Then

$$R_n = \begin{cases} M_{N_n} & \text{if } X_n \leqslant M_{N_n}, \\ X_n & \text{if } X_n > M_{N_n}. \end{cases}$$

Now let $P(\xi_1 = r) = cr^{-3/2}$ for $r = 1, 2, \ldots$ where $c = (\Sigma_1^\infty r^{-3/2})^{-1}$. Then,

$$\begin{aligned} p\left(N_n \geqslant x\sqrt{n}\right) &= P\left(S_{[x\sqrt{n}]} \leqslant n\right) \\ &= P\left([x\sqrt{n}]^{-2} S_{[x\sqrt{n}]} \leqslant [x\sqrt{n}]^{-2} n\right) \to G_{2^{-1}}(x^{-2}) \end{aligned}$$

where $G_\alpha(x)$ is the one sided stable law of order α on $(0, \infty)$. Hence for any $\varepsilon > 0$, there exists a $\delta > 0$ and n_0 such that, for $n \geqslant n_0$, $P(N_n > \delta\sqrt{n}) \geqslant 1 - \varepsilon$. Also

$$\begin{aligned} &P\left(k^{-2} M_k \leqslant x\right) \\ &\quad = \left(P\left(\xi_1 \leqslant xk^2\right)\right)^k = \left\{1 - \left(1 - P\left(\xi_1 > xk^2\right)\right)\right\}^k \\ &\quad > \exp(-2c\sqrt{x}) \quad \text{for } x > 0. \end{aligned}$$

This implies that, for any $\varepsilon > 0$, there exist an $\eta > 0$ and n_1 such that for $k \geqslant n_1$, $P(k^{-2} M_k \geqslant \eta) \geqslant 1 - \varepsilon$. Finally, it is known (see Erickson, 1970) that $n^{-1} X_n$ converges to a distribution with [0,1] as its support set. Thus given $\varepsilon > 0$ there exist $\gamma > 0$ and n_2 such that for $n \geqslant n_2$, $P(X_n > \gamma_n) \geqslant 1 - \varepsilon$. Choosing $n > \max(n_0, n_1, n_2)$ we see that

$$P(X_n > \gamma_n) \geqslant 1 - \varepsilon \quad \text{and} \quad P\left(M_{N_n} \geqslant \delta^2 \eta n\right) \geqslant 1 - \varepsilon$$

implying that

$$P(R_n \geqslant \beta n) \geqslant 1 - 2\varepsilon \quad \text{where } \beta = \min(\delta^2\eta, \gamma).$$

This rules out the convergence of $n^{-1} R_n$ to zero in probability.

6. It is clear that the above argument goes through whenever $P(\xi_1 > x) \sim cx^{-\alpha}$ where $0 < \alpha < 1$.

7. The case $\alpha = 1$, however, is more interesting. In this case, $n^{-1}(\log n) R_n$ converges in distribution to an exponential and hence $n^{-1} R_n$ does go to zero in probability. To see this, we need the following facts:

(i) $n^{-1}(\log n) N_n$ converges in probability to a finite positive constant.

(ii) $n^{-1} M_n$ converges in distribution to an exponential.

(iii) $(\log n)^{-1}\log X_n$ converges in distribution to the uniform distribution on [0,1].

For (i) see Anderson and Athreya (1984), (ii) is straightforward and (iii) is proved in Erickson (1970).

From (i) and (ii) it follows that $n^{-1}(\log n)M_{N_n}$ converges in distribution to an exponential and hence $(\log n)^{-1}\log M_{N_n}$ goes to one in probability. Thus,

$$\begin{aligned} P(X_n > M_{N_n}) &= P(\log X_n > \log M_{N_n}) \\ &= P\left((\log n)^{-1}(\log X_n) > (\log n)^{-1}\log M_{N_n}\right) \end{aligned}$$

which goes to zero by (iii). Thus $P(R_n = M_{N_n}) \to 1$ and the given assertion follows.

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