The CLT for Regenerative Processes

Let

$$R(t) = \int_0^t f(X(s))ds$$

be the total reward associated with running X over $[0,t]$. To approximate $P(R(t) > x)$ for t large, we seek a CLT of the form

$$t^{1/2}\left(\frac{1}{t}\int_0^t f(X(s))ds - \alpha\right) \Rightarrow \sigma N(0,1)$$

as $t \to \infty$ (for suitable constants α and σ^2). This suggests the approximation

$$P(R(t) > x) \approx P(N(0,1) > (x - \alpha t)/\sigma\sqrt{t})$$

for t large.

Remark This approximation tends to be good when x differs from αt by only a few standard deviations $\sigma\sqrt{t}$. Otherwise, one should use a "large deviations" approximation instead.

Theorem 1 *Let $X = (X(t) : t \geq 0)$ be an S–valued regenerative process (with regeneration times $(T(n) : n \geq 0)$). Suppose $f : S \to R$ satisfies $Y_0(|f|) < \infty$ a.s. and $E[Y_1(|f|)^2 + \tau_1^2] < \infty$. Then*

$$t^{1/2}\left(\frac{1}{t}\int_0^t f(X(s))ds - \alpha\right) \Rightarrow \sigma N(0,1)$$

as $t \to \infty$, where $\alpha = EY_1(f)/E\tau_1$, $\sigma^2 = EZ_1^2/E\tau_1$, and $Z_1 = Y_1(f) - \alpha\tau_1$.

The proof hinges on two facts.

Proposition 1 *Let $(Z_i : i \geq 1)$ be a sequence of iid rv's with $EZ_1 = 0$ and $\eta^2 = varZ_1 < \infty$. Let $(\gamma(t) : t \geq 0)$ is an integer–valued stochastic process for which*

$$\frac{1}{t}\gamma(t) \Rightarrow \kappa$$

as $t \to \infty$, where κ is deterministic. Then,

$$t^{-1/2}\left(\sum_{i=1}^{\gamma(t)} Z_i - \sum_{i=1}^{\lfloor \kappa t \rfloor} Z_i\right) \Rightarrow 0$$

as $t \to \infty$.

Proof. We need to show that for each $\epsilon > 0$,

$$P\left(\left|\sum_{i=1}^{\gamma(t)} Z_i - \sum_{i=1}^{\lfloor \kappa t \rfloor} Z_i\right| > \epsilon t^{1/2}\right) \to 0$$

as $t \to \infty$. But

$$
\begin{aligned}
&P\left(\left|\sum_{i=1}^{\gamma(t)} Z_i - \sum_{i=1}^{\lfloor \kappa t\rfloor} Z_i\right| > \epsilon t^{1/2}\right) \\
&\le P\left(\left|\sum_{i=1}^{\gamma(t)} Z_i - \sum_{i=1}^{\lfloor \kappa t\rfloor} Z_i\right| > \epsilon t^{1/2};\ (\kappa-\delta)t \le \gamma(t) \le \lfloor \kappa t\rfloor\right) \\
&\quad + P\left(\left|\sum_{i=1}^{\gamma(t)} Z_i - \sum_{i=1}^{\lfloor \kappa t\rfloor} Z_i\right| > \epsilon t^{1/2};\ \lfloor \kappa t\rfloor \le \gamma(t) \le \lfloor (\kappa+\delta)t\rfloor\right) \\
&\quad + P(|\gamma(t) - \kappa t| > \delta t) \\
&\le P\left(\max_{0\le j\le \lfloor \delta t\rfloor}\left|\sum_{i=\lfloor \kappa t\rfloor - j}^{\lfloor \kappa t\rfloor} Z_i\right| > \epsilon t^{1/2}\right) + P\left(\max_{0\le j\le \lfloor \delta t\rfloor}\left|\sum_{i=\lfloor \kappa t\rfloor}^{\lfloor \kappa t\rfloor + j} Z_i\right| > \epsilon t^{1/2}\right) \\
&\quad + P(|\gamma(t) - \kappa t| > \delta t) \\
&\le 2P\left(\max_{0\le j\le \lfloor \delta t\rfloor}\left|\sum_{i=1}^{j} Z_i\right| > \epsilon t^{1/2}\right) + P(|\gamma(t) - \kappa t| > \delta t).
\end{aligned}
$$

We now apply the maximal inequality for sums of mean zero finite variance rv's :

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Z_i\right| > x\right) \le \frac{n\mathrm{var} Z_1}{x^2}.$$

(We will prove this later when we discuss martingales.) This yields the bound

$$P\left(\left|\sum_{i=1}^{\gamma(t)} Z_i - \sum_{i=1}^{\lfloor \kappa t\rfloor} Z_i\right| > \epsilon t^{1/2}\right) \le \frac{2\lfloor \delta t\rfloor \mathrm{var} Z_1}{\epsilon^2 t} + P(|\gamma(t) - \kappa t| > \delta t).$$

Choose $\delta = r\epsilon^2/2\mathrm{var}Z_1$ and let $t \to \infty$. We get

$$\limsup_{t\to\infty} P\left(\left|\sum_{i=1}^{\gamma(t)} Z_i - \sum_{i=1}^{\lfloor \kappa t\rfloor} Z_i\right| > \epsilon t^{1/2}\right) \le r.$$

Since r was arbitrary, we can send r to zero, finishing the argument. □

Proposition 2 *Let $(W_n : n \ge 1)$ be a sequence of iid rv's with $E|W_1|^p < \infty$. Then*

$$n^{-1/p} W_n \to 0 \quad a.s.$$

as $n \to \infty$.

Proof. This is equivalent to proving that $n^{-1}|W_n|^p \to 0$ a.s. But

$$n^{-1}|W_n|^p = n^{-1}\sum_{i=1}^{n} |W_j|^p - (n-1)^{-1}\sum_{j=1}^{n-1} |W_j|^p \left(\frac{n-1}{n}\right) \to 0 \quad \text{a.s.} \quad \square$$

Proof of the CLT. Put $f_c(x) = f(x) - \alpha$ and note that

$$t^{-1/2}\left(\int_0^t f(X(s))ds - \alpha t\right) = t^{-1/2}\sum_{i=0}^{N(t)} Z_i + t^{-1/2}\int_{T(N(t))}^t f_c(X(s))ds.$$

Since $N(t)/t \to \lambda \triangleq 1/E\tau_1$ a.s. as $t \to \infty$, it follows from Proposition 1 that

$$t^{-1/2}\sum_{i=0}^{N(t)} Z_i \Rightarrow \sigma N(0,1).$$

On the other hand,

$$\begin{aligned} t^{-1/2}\left|\int_{T(N(t))}^t f_c(X(s))ds\right| &\leq t^{-1/2}\left(Y_{N(t)+1}(|f|) + |\alpha|\tau_{N(t)+1}\right) \\ &= \sqrt{\frac{N(t)+1}{t}}\left(\frac{Y_{N(t)+1}(|f|)}{\sqrt{N(t)+1}} + \frac{|\alpha|\tau_{N(t)+1}}{\sqrt{N(t)+1}}\right). \end{aligned}$$

Now, apply Proposition 2 first with $W_i = Y_i(|f|)$ and $p = 2$ and then with $W_i = \tau_i$ (and $p = 2$) to establish that

$$t^{-1/2}\left|\int_{T(N(t))}^t f_c(X(s))ds\right| \Rightarrow 0$$

as $t \to \infty$. □

This yields the following CLT for Markov chains.

Theorem 2 *Let $X = (X_n : n \geq 0)$ be an irreducible DTMC for which*

$$E_z\left(\sum_{j=0}^{\tau(z)-1} |f(X_j)| + 1\right)^2 < \infty.$$

Then,

$$n^{1/2}\left(\frac{1}{n}\sum_{j=0}^{n-1} f(X_j) - \pi f\right) \Rightarrow \sigma N(0,1)$$

where

$$\sigma^2 = E_z\left(\sum_{j=0}^{\tau(z)-1} f_c(X_j)\right)^2 / E_z\tau(z).$$

<u>Remark.</u> Note that $E_z\left(\sum_{j=0}^{\tau(z)-1} f_c(X_j)\right)^2$ can be computed as the solution to a linear system (via first transition analysis). This permits closed form or numerical computation for simple

models. For more difficult models, one may be able to bound this quantity using Lyapunov methods. Note that

$$
\begin{aligned}
u^*(x) &\triangleq E_x\left(\sum_{j=0}^{\tau(z)-1} f_c(X_j)\right)^2 \\
&= f_c(x)^2 + 2f_c(x)\sum_{y\neq x} P(x,y)E_y \sum_{j=0}^{\tau(z)-1} f_c(X_j) \\
&\quad + \sum_{y\neq z} P(x,y)E_y\left(\sum_{j=0}^{\tau(z)-1} f_c(X_j)\right)^2
\end{aligned}
$$

so that $u^* = k + Bu^*$, where

$$
k(x) = f_c(x)^2 + 2f_c(x)\sum_{y\neq z} P(x,y)E_y \sum_{j=0}^{\tau(z)-1} f_c(X_j).
$$

Given a suitable Lyapunov function g, we can find a bound on k, call it $\widetilde{k}$. We then try to find a second Lyapupov function $h \geq 0$ for which $Bh \leq h - \widetilde{k}$. Given such a Lyapunov function h, we then obtain our desired bound on u^*.