Appendix B: Interchanging Limits and Expectation

In this appendix, we discuss conditions that guarantee that the interchange of limit and expectation is valid. To see that some conditions are necessary, suppose that $N = (N(t) : t \geq 0)$ is a Poisson process having unit rate. Note that if t is not a jump epoch of $N(\cdot)$, $N(t+h) = N(t)$ for h sufficiently small and positive. It follows that

$$\lim_{h \downarrow 0} \frac{N(t+h) - N(t)}{h} = 0 \quad \text{a.s.}$$

On the other hand,

$$\lim_{h \downarrow 0} E\left(\frac{N(t+h) - N(t)}{h}\right) = 1.$$

So, the limit interchange fails.

One of the contributions of measure-theoretic probability is a set of sufficient conditions guaranteeing the validity of the limit interchange.

The Bounded Convergence Theorem (BCT): Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$ and that there exists $c \in \mathbb{R}$ such that

$$|X_n(\omega)| \leq c < \infty$$

for $n \geq 0$ and $\omega \in \Omega$. Then, $EX_n \to EX_\infty$ as $n \to \infty$.

The Dominated Convergence Theorem (DCT): Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$ and that there exists a rv Y (called the "dominating rv") such that $EY < \infty$ and

$$|X_n(\omega)| \leq Y(\omega)$$

for $n \geq 0$ and $\omega \in \Omega$. Then, $EX_n \to EX_\infty$ as $n \to \infty$.

The Monotone Convergence Theorem (MCT): Suppose that $(X_n : n \geq 0)$ is a sequence of non-negative rv's for which $X_n(\omega) \leq X_{n+1}(\omega)$ for $n \geq 0$ and $\omega \in \Omega$. Then,

$$X_\infty = \lim_{n \to \infty} X_n$$

exists (and is possibly ∞-valued) and $EX_n \nearrow EX_\infty$ as $n \to \infty$.

Fatou's Lemma: If $(X_n : n \geq 0)$ is a sequence of non-negative rv's, then

$$E \liminf_{n \to \infty} X_n \leq \liminf_{n \to \infty} EX_n.$$

When the X_n's are non-negatve, a necessary and sufficient condition guaranteeing validity of the interchange can be identified.

Definition B.1 A sequence $(X_n : n \geq 0)$ is said to be *uniformly integrable* if, for each $\epsilon > 0$, there exists $c = c(\epsilon)$ such that

$$\limsup_{n\to\infty} E|X_n|I(|X_n| \geq c) < \epsilon.$$

Suppose that $(X_n : n \geq 0)$ is a sequence of non-negative rv's for which $X_n \to X_\infty$ a.s. as $n \to \infty$, where $EX_\infty < \infty$. Then, $EX_n \to EX_\infty$ as $n \to \infty$ if and only if $(X_n : n \geq 0)$ is uniformly integrable.

If $X_n \Rightarrow X_\infty$ as $n \to \infty$, we can often reduce the limit interchange problem to one involving almost sure convergence by invoking the following result.

Exercise B.1 Suppose that $X_n \Rightarrow X_\infty$ as $n \to \infty$. Prove that there exists a probability space supporting rv's $(X'_n : 1 \leq n \leq \infty)$ satisfying:

- $X_n \stackrel{\mathcal{D}}{=} X'_n$ for $1 \leq n \leq \infty$
- $X'_n \to X'_\infty$ a.s. as $n \to \infty$

(Hint: Consider $X'_n = F^{-1}_{X_n}(U)$, where U is uniform on $[0, 1]$, where $F^{-1}_{X_n}(x) = \sup\{x : F_{X_n}(z) \leq x\}$.)

Finally, note that the MCT implies that if the X_n's are non-negative rv's then

$$E\sum_{n=1}^{\infty} X_n = \sum_{n=1}^{\infty} EX_n.$$

This is one version of *Fubini's theorem.* Other variants:

- If $(X_n : n \geq 1)$ is a sequence of rv's for which

  $$E\sum_{n=1}^{\infty} |X_n| = \sum_{n=1}^{\infty} E|X_n| < \infty,$$

  then

  $$E\sum_{n=1}^{\infty} X_n = \sum_{n=1}^{\infty} EX_n.$$

- If $(X(t) : t \geq 0)$ is a non-negative process, then

  $$E\int_0^\infty X(t)dt = \int_0^\infty EX(t)dt.$$

- If $(X(t) : t \geq 0)$ is a process for which

  $$E\int_0^\infty |X(t)|dt = \int_0^\infty E|X(t)|dt < \infty,$$

  then

  $$E\int_0^\infty X(t)dt = \int_0^\infty EX(t)dt$$