Appendix A: Convergence Concepts for Random Variables

A.1 Convergence Definitions

Recall that a real-valued rv X is a function $X : \Omega \to \mathbb{R}$. Given that a rv is a function, there are many different mechanisms for determining whether a sequence of rvs/functions $(X_n : n \geq 0)$ converges to a limit rv/function X_∞.

Almost Sure Convergence:

We say that $(X_n : n \geq 1)$ *converges almost surely* to X_∞ if $P(A) = 1$, where

$$A = \{\omega : X_n(\omega) \to X_\infty(\omega) \quad \text{as } n \to \infty\}$$

and write $X_n \to X_\infty$ a.s. as $n \to \infty$ when this convergence holds. This type of convergence is equivalently called: convergence with probability one (written $X_n \to X_\infty$ w.p. 1 as $n \to \infty$); convergence almost everywhere (written $X_n \to X_\infty$ a.e. as $n \to \infty$); convergence almost certainly (written $X_n \to X_\infty$ a.c. as $n \to \infty$).

Remark A.1.1 Note that the event A depends on the infinite-dimensional joint distribution of $(X_n : 1 \leq n \leq \infty)$. As a result, rigorous discussion of probability assignments to events like A needed to await the development of measure-theoretic probability in the early 20th century.

Convergence in p'th Mean:

We say that $(X_n : n \geq 1)$ *converges in p'th mean* (for $p > 0$) to X_∞ if $E|X_n|^p < \infty$ for $n \geq 1$ and $\|X_n - X_\infty\|_p \to 0$ as $n \to \infty$, where

$$\|Y\|_p = E^{1/p}|Y|^p$$

for $E|Y|^p < \infty$. When this convergence holds, we write $X_n \xrightarrow{L^p} X_\infty$ as $n \to \infty$.

Convergence in Probability

We say that $(X_n : n \geq 1)$ *converges in probability* to X_∞ if, for each $\epsilon > 0$,

$$P(|X_n - X_\infty| > \epsilon) \to 0$$

as $n \to \infty$, in which case we write $X_n \xrightarrow{p} X_\infty$ as $n \to \infty$.

Remark A.1.2 Convergence in probability and convergence in p'th mean is a statement about the joint distribution of the two rv's X_n and X_∞ for n large.

Remark A.1.3 Convergence in p'th mean implies convergence in probability, since Markov's inequality implies that

$$P(|X_n - X_\infty| > \epsilon) \leq \frac{E|X_n - X_\infty|^p}{\epsilon^p}.$$

Exercise A.1.1 Prove that almost sure convergence implies convergence in probability.

Weak Convergence

We say that $(X_n : n \geq 1)$ *converges weakly* to X_∞ if X_∞ is a finite-valued rv for which

$$Ef(X_n) \to Ef(X_\infty)$$

as $n \to \infty$ for each bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$, in which case we write

$$X_n \Rightarrow X_\infty$$

as $n \to \infty$. Weak convergence is equivalently called "convergence in distribution".

Remark A.1.4 Weak convergence is a statement about the distribution of X_n when n is large.

Remark A.1.5 Weak convergence can be equivalently formulated as: $X_n \Rightarrow X_\infty$ as $n \to \infty$ if and only if X_∞ is a finite-valued rv for which

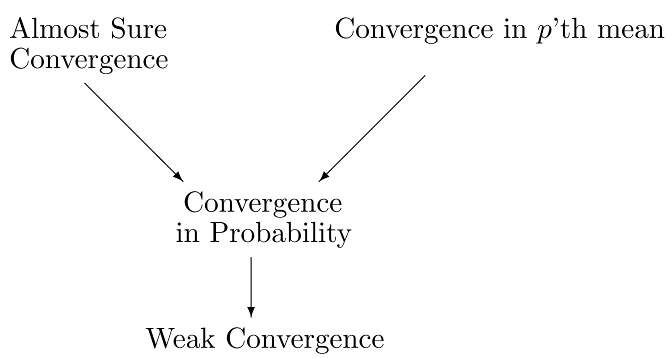
$$P(X_n \leq x) \to P(X_\infty \leq x)$$

as $n \to \infty$ whenever x is a continuity point of $P(X_\infty \leq \cdot)$.

Exercise A.1.2 Prove that $X_n \xrightarrow{p} X_\infty$ as $n \to \infty$ implies that $X_n \Rightarrow X_\infty$ as $n \to \infty$.

Remark A.1.6 Convergence in probability does not imply almost sure convergence. Note that if $Y = (Y_n : n \geq 0)$ is a nearest neighbor symmetric random walk on $\mathbb{Z}$, then the recurrence of Y implies that $X_n \triangleq I(Y_n = 0) = 1$ infinitely often a.s. so $X_n \nrightarrow 0$ a.s. as $n \to \infty$. But $P(X_n = 0) = P(Y_n \neq 0) \to 1$ as $n \to \infty$, so $X_n \xrightarrow{p} 0$ as $n \to \infty$.

The following diagram makes clear the relationship between these convergence concepts:



A.2 Basic Facts on Almost Sure Convergence

Fact 1: Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$ and $Y_n \to Y_\infty$ a.s. as $n \to \infty$, where X_∞ and Y_∞ are finite-valued. If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then $h(X_n, Y_n) \to h(X_\infty, Y_\infty)$ a.s. as $n \to \infty$.

Fact 2: Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$ and $(T_n : n \geq 1)$ is a sequence of $\mathbb{Z}_+$-valued rv's for which $T_n \to \infty$ a.s. as $n \to \infty$. Then,

$$X_{T_n} \to X_\infty \quad \text{a.s.}$$

as $n \to \infty$.

A.3 Basic Facts on Convergence in Probability

Fact 1: Suppose that $X_n \xrightarrow{p} X_\infty$, where X_∞ is finite-valued. If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h(X_n) \xrightarrow{p} h(X_\infty)$ as $n \to \infty$.

Remark A.3.1 If $X_n \xrightarrow{p} X_\infty$ and $Y_n \xrightarrow{p} Y_\infty$ as $n \to \infty$ and $((X_n, Y_n) : 1 \leq n \leq \infty)$ are all defined on a common sample space (i.e. are jointly distributed), then $(X_n, Y_n) \xrightarrow{p} (X_\infty, Y_\infty)$ as $n \to \infty$.

Fact 2: Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$ and $T_n \xrightarrow{p} \infty$ as $n \to \infty$. Then, $X_{T_n} \xrightarrow{p} X_\infty$ as $n \to \infty$.

Remark A.3.2 If $X_n \xrightarrow{p} X_\infty$ and $T_n \xrightarrow{p} \infty$ as $n \to \infty$, it is not always the case that $X_{T_n} \xrightarrow{p} X_\infty$ as $n \to \infty$.

A.4 Basic Facts on Weak Convergence

Fact 1: Suppose that $X_n \Rightarrow X_\infty$ as $n \to \infty$. If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h(X_n) \Rightarrow h(X_\infty)$ as $n \to \infty$. In fact, if $P(X_\infty \in D_h) = 0$ (where $D_h = \{x \in \mathbb{R} : h(\cdot) \text{ is discontinuous at } x\}$), then $h(X_n) \Rightarrow h(X_\infty)$ as $n \to \infty$.

Remark A.4.1 The above result is called the *continuous mapping principle*.

Fact 2: Suppose that $X_n \Rightarrow X_\infty$ as $n \to \infty$ and $Y_n \xrightarrow{p} c$ as $n \to \infty$, where c is deterministic. If $h : \mathbb{R}^2 \to \mathbb{R}$ is such that $P((X_\infty, c) \in D_h) = 0$, then $h(X_n, Y_n) \Rightarrow h(X_\infty, c)$ as $n \to \infty$.

Remark A.4.2 The above fact implies that if $X_n \Rightarrow X_\infty$ and $Y_n \xrightarrow{p} c$, then $X_n + Y_n \Rightarrow X_\infty + c$ and $X_n Y_n \Rightarrow c X_\infty$ as $n \to \infty$.

Remark A.4.3 If $X_n \Rightarrow X_\infty$ and $Y_n \xrightarrow{p} Y_\infty$ as $n \to \infty$, it is not always the case that $h(X_n, Y_n) \Rightarrow h(X_\infty, Y_\infty)$ (even if $h : \mathbb{R}^2 \to \mathbb{R}$ is continuous everywhere).

Remark A.4.4 If $X_n \Rightarrow X_\infty$ as $n \to \infty$ and $T_n \xrightarrow{p} \infty$ as $n \to \infty$, it is not always the case that $X_{T_n} \Rightarrow X_\infty$ as $n \to \infty$.