Section 9: Renewal Theory

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9.1 Renewal Equations

Let $X = (X(t) : t \geq 0)$ be a non-delayed regenerative process and suppose that $f : S \to \mathbb{R}_+$. Then, if

$$a^*(t) = \mathrm{E}f(X(t)),$$

$a^*(t)$ can be expressed as follows:

$$a^*(t) = \mathrm{E}f(X(t))\mathrm{I}(\tau_1 > t) + \int_{[0,t]} \mathrm{E}f(X(t))\mathrm{I}(\tau_1 \in ds).$$

But

$$\begin{aligned}
\int_{[0,t]} \mathrm{E}f(X(t))\mathrm{I}(\tau_1 \in \mathrm{d}s) &= \int_{[0,t]} \mathrm{E}[f(X(\tau_1 + t - s))|\tau_1 = s]\mathrm{P}(\tau_1 \in \mathrm{d}s) \\
&= \int_{[0,t]} \mathrm{E}f(X(t-s))\mathrm{P}(\tau_1 \in \mathrm{d}s) \\
&= \int_{[0,t]} a^*(t-s)\mathrm{P}(\tau_1 \in \mathrm{d}s).
\end{aligned}$$

In other words, $a^* = (a^*(t) : t \geq 0)$ satisfies the linear integral equation

$$a(t) = b(t) + (F * a)(t),$$

for $t \geq 0$, where

$$F(t) = \mathrm{P}(\tau_1 \leq t),$$

$$b(t) \triangleq \mathrm{E}f(X(t))\mathrm{I}(\tau_1 > t),$$

and

$$(G * h)(t) \triangleq \int_{[0,t]} h(t-s)G(\mathrm{d}s)$$

is the convolution operation (that is well-defined for any non-decreasing function G and non-negative h).

Definition 9.1.1 Given a function b and a non-negative G, a linear integral equation of the form

$$a = b + G * a$$

is called a *renewal equation* for a non-decreasing function G. Define the n-fold convolution of G via

$$G^{(n)}(t) = \mathrm{I}(t \geq 0)$$

for $n = 0$ and

$$G^{(n)}(t) = \int_{[0,t]} G(t-s)G^{(n-1)}(\mathrm{d}s)$$

for $n \geq 1$. Note that when G is the distribution function F of a positive rv τ_1,

$$F^{(n)}(t) = \mathrm{P}(T(n) \leq t),$$

where $T(n) = \tau_1 + \tau_2 + \cdots + \tau_n$, and the τ_i's are iid copies of τ_1.

Proposition 9.1.1 The function $(a^*(t) : t \geq 0) = (\mathrm{E}f(X(t)) : t \geq 0)$ is the minimal non-negative solution of

$$a = b + F * a$$

and is given by

$$a^* = \sum_{n=0}^{\infty} F^{(n)} * b.$$

Here are some additional examples of renewal equations:

Example 9.1.1 Let $a^*(t) = \mathrm{P}(T(N(t)+1) - t > x)$ be the "tail probability" of the "residual life" process $T(N(t)+1)-t$ corresponding to a non-delayed renewal counting process $N = (N(t) : t \geq 0)$. Note that a^* satisfies the renewal equation

$$a = b + F * a, \tag{9.1.1}$$

where

$$b(t) = \mathrm{P}(\tau_1 > t + x)$$

and

$$F(t) = \mathrm{P}(\tau_1 \leq t).$$

The function a^* is the minimal non-negative solution of (9.1.1) and is given by

$$a^* = \sum_{n=0}^{\infty} F^{(n)} * b.$$

Example 9.1.2 Let $X = (X(t) : t \geq 0)$ be a non-delayed regenerative process and let $a^*(t) = \mathrm{P}(t - T(N(t)) \leq x)$ be the distribution of the "current age" process $t - T(N(t)$. Here, a^* satisfies

$$a = b + F * a, \tag{9.1.2}$$

where

$$b(t) = \mathrm{I}(t \leq x),$$

$$F(t) = \mathrm{P}(\tau_1 \le t).$$

The function a^* is the minimal non-negative solution of (9.1.2) and is given by

$$a^* = \sum_{n=0}^{\infty} F^{(n)} * b.$$

Example 9.1.3 Let $X = (X(t) : t \ge 0)$ be a non-delayed S-valued regenerative process and let $T = \inf\{t \ge 0 : X(t) \in A\}$ be the first hitting time if some subset $A \subseteq S$. Put $a^*(t) = \mathrm{P}(T > t)$. Then, a^* satisfies

$$a = b + G * a, \tag{9.1.3}$$

where

$$b(t) = \mathrm{P}(T \wedge \tau_1 > t),$$
$$G(t) = \mathrm{P}(\tau_1 \le t, T > \tau_1).$$

The function a^* is the minimal non-negative solution of (9.1.3) and is given by

$$a^* = \sum_{n=0}^{\infty} G^{(n)} * b.$$

Note that G is not a probability distribution function here (since $G(\infty) = \mathrm{P}(T < \tau_1) < 1$ in general).

Definition 9.1.2 A renewal equation is said to be a *proper renewal equation* if G is a probability distribution function; otherwise, the renewal equation is said to be an *improper renewal equation*. In particular, if $G(0) = 0$ and $G(\infty) < 1$, the improper renewal equation is said to be *defective*; if $G(0) = 0$ and $G(\infty) > 1$, the improper renewal equation is said to be *excessive*.

Improper renewal equations can frequently be transformed into proper renewal equations via use of the following trick. Note that if a is given by

$$a^* = \sum_{n=0}^{\infty} G^{(n)} * b,$$

then

$$\widetilde{a}(t) = \sum_{n=0}^{\infty} (\widetilde{G}^{(n)} * \widetilde{b})(t),$$

where

$$\widetilde{a}(t) = e^{\gamma t} a(t)$$
$$\widetilde{b}(t) = e^{\gamma t} b(t)$$
$$\widetilde{G}(dt) = e^{\gamma t} G(dt),$$

and $\widetilde{a}$ satisfies the renewal equation

$$\widetilde{a} = \widetilde{b} + \widetilde{G} * \widetilde{a}. \tag{9.1.4}$$

So if we can find γ^* such that

$$\widetilde{G}(dt) = e^{\gamma^* t} G(dt)$$

is a probability distribution function, (9.1.4) becomes a proper renewal equation.

Exercise 9.1.1 a.) Prove that if a renewal equation is excessive, there always exists a unique γ^* such that $\widetilde{G}$ is a probability distribution function.

b.) Prove that if a renewal equation is defective, there need not exist a γ^* such that $\widetilde{G}$ is a probability distribution function.

9.2 Solving the Renewal Equation

If F is the distribution of an exponential rv with parameter $\lambda > 0$, then

$$F^{(n)}(\mathrm{d}t) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} \exp(-\lambda t)\mathrm{d}t$$

so

$$\sum_{n=0}^{\infty} F^{(n)}(\mathrm{d}t) = \delta_0(dt) + \lambda \mathrm{d}t.$$

Hence,

$$\sum_{n=0}^{\infty} (F^{(n)} * b)(t) = b(t) + \lambda \int_0^t b(s)\,\mathrm{d}s.$$

So, the renewal equation can be explicitly solved when F is exponential.

It can also be solved in "closed form" in a limited number of other cases. Let

$$\widetilde{a}(\gamma) = \int_0^\infty e^{-\gamma t} a(t)\,\mathrm{d}t,$$

$$\widetilde{b}(\gamma) = \int_0^\infty e^{-\gamma t} b(t)\,\mathrm{d}t,$$

$$\widetilde{G}(\gamma) = \int_0^\infty e^{-\gamma t} G(\mathrm{d}t),$$

and note that $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{G}$ are the *Laplace transforms* of a, b, and G, respectively. If a satisfies

$$a = b + G * a,$$

it follows that

$$\widetilde{a} = \widetilde{b} + \widetilde{G}\widetilde{a},$$

so that

$$\widetilde{a} = \frac{\widetilde{b}}{1 - \widetilde{G}}.$$

If $\widetilde{b}$ and $\widetilde{G}$ can be computed in closed form, then a can potentially be computed in closed form by calculating the inverse Laplace transform corresponding to $\widetilde{b}(1-\widetilde{G})^{-1}$. For example, this can generally be done when $\widetilde{G}$ has a rational Laplace transform (i.e. is the ratio of two polynomials in γ).

9.3 Asymptotic Behavior of the Solution of the Renewal Equation

Typically, the solution to the renewal equation can not be computed in closed form. In such settings, one usually must be satisfied with computing the limiting behavior of

$$\sum_{n=0}^{\infty}(F^{(n)} * b)(t)$$

when t tends to infinity.

Definition 9.3.1 The function

$$U(t) = \sum_{n=0}^{\infty} F^{(n)}(t)$$

is called the *renewal function* and the corresponding distribution/measure is called the *renewal measure*. If F has a density f,

$$u(t) = \sum_{n=1}^{\infty} f^{(n)}(t)$$

is called the *renewal density*. If F is the distribution function of an integer-valued rv τ_1, then

$$u_n = \sum_{k=1}^{n} p_n^{(k)}$$

(where $p_n^{(k)} = P(\tau_1 + \cdots + \tau_k = n)$) is called the *renewal mass function*.

We focus first on the case where F is the distribution function of an integer-valued rv τ_1. Note that

$$\begin{aligned} u_n &= \sum_{k=1}^{\infty} P(\tau_1 + \cdots + \tau_k = n) \\ &= \mathrm{E}\sum_{k=1}^{\infty} \mathrm{I}(T(k) = n) \\ &= \mathrm{EI}(\text{regeneration at time } n). \end{aligned}$$

It turns out that the asymptotic behavior of u_n as $n \to \infty$ can be analyzed via consideration of an appropriate Markov chain $X = (X_n : n \geq 0)$. In particular, let

$$X_n = \inf\{T(k) - n : T(k) \geq n\}$$

be the residual life process corresponding to the sequence of event times $T(1), T(2), \ldots$. Note that

$$u_n = P_0(X_n = 0),$$

where $X = (X_n : n \geq 0)$ is an $\mathbb{Z}_+$-valued Markov chain with transition probabilities given by

$$P(i, i-1) = 1$$

for $i \geq 1$, and

$$P(0, i) = P(\tau_1 = i + 1).$$

Note that the state 0 is always recurrent for X. Furthermore, X is positive recurrent if and only if $\mathrm{E}\tau_1 < \infty$, in which case π is given by

$$\pi(i) = \frac{\sum_{j>i} P(\tau_1 < j)}{\mathrm{E}\tau_1} = \frac{P(\tau_1 > i)}{\mathrm{E}\tau_1}.$$

Finally, X is aperiodic if and only if

$$\gcd\{k \geq 1 : P(\tau_1 = k) > 0\} = 1.$$

If X is aperiodic and $\mathrm{E}\tau_1 < \infty$, it follows that

$$u_n = P_0(X_n = 0) \to \pi(0) = \frac{1}{\mathrm{E}\tau_1}$$

as $n \to \infty$. It follows that if

$$\sum_{k=0}^{\infty} |b_k| < \infty, \tag{9.3.1}$$

then the Dominated Convergence Theorem implies that

$$\sum_{k=0}^{\infty} (F^{(k)} * b)(n) = \sum_{k=0}^{n} u_{n-k} b_k \to \frac{1}{\mathrm{E}\tau_1} \sum_{k=0}^{\infty} b_k$$

as $n \to \infty$.

Theorem 9.3.1 (Discrete Renewal Theorem) Suppose $\mathrm{E}\tau_1 < \infty$ and

$$\gcd\{k \geq 1 : P(\tau_1 = k) > 0\} = 1.$$

If (9.3.1) holds, then

$$\sum_{k=0}^{n} u_{n-k} b_k \to \frac{1}{\mathrm{E}\tau_1} \sum_{k=0}^{\infty} b_k$$

as $n \to \infty$. If $\mathrm{E}\tau_1 = \infty$ and (9.3.1) holds, then

$$\sum_{k=0}^{n} u_{n-k} b_k \to 0$$

as $n \to \infty$.

Let $X = (X_n : n \geq 0)$ be a S-valued non-delayed regenerative sequence and suppose $f : S \to \mathbb{R}$ is a bounded function (i.e. $\sup\{|f(x)| : x \in S\}$). As noted earlier,

$$a_n^* = \mathrm{E} f(X_n)$$

satisfies a renewal equation, namely

$$a_n^* = b_n + \int_{[0,n]} a^*(t-s) P(\tau_1 \in \mathrm{d}s) = b_n + \sum_{j=1}^{n} a_{n-j}^* p_j,$$

so

$$a_n^* = \sum_{j=0}^{n} u_{n-j} b_j,$$

where

$$b_j = \mathrm{E} f(X_j) \mathrm{I}(\tau_1 > j).$$

The discrete renewal theorem guarantees that if τ_1 is aperiodic and has finite mean, then

$$a_n^* \to \frac{\sum_{j=0}^{\infty} b_j}{\mathrm{E}\tau_1}$$

as $n \to \infty$. In other words,

$$\mathrm{E} f(X_n) \to \frac{\sum_{j=0}^{\infty} \mathrm{E} f(X_j) \mathrm{I}(\tau_1 > j)}{\mathrm{E}\tau_1} = \frac{\mathrm{E} \sum_{j=0}^{\tau_1 - 1} f(X_j)}{\mathrm{E}\tau_1}$$

as $n \to \infty$.

Exercise 9.3.1 Prove that if $X = (X_n : n \geq 0)$ is a delayed S-valued regenerative sequence with $\mathrm{E}\tau_1 < \infty$ and $f : S \to \mathbb{R}$ is a bounded function, then

$$\mathrm{E} f(X_n) \to \frac{\mathrm{E} \sum_{j=0}^{\tau_1 - 1} f(X_j)}{\mathrm{E}\tau_1}$$

as $n \to \infty$, provided that τ_1 is aperiodic.

All the above discussion for discrete-time renewal theory extends to the continuous-time setting.