Section 5: Feller Chains

Contents

5.1 Metric Spaces

A metric space is a mathematical object upon which a "distance function" (or metric) can be defined. To be more precise, given a set S, a *metric* is a function $\rho : S \times S \to \mathbb{R}_+$ having the following properties:

i.) $\rho(x, y) = \rho(y, x)$ for all $x, y \in S$;

ii.) $\rho(x, y) = 0$ if and only if $y = x$;

iii.) $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ for all $x, y, z \in S$.

We interpret $\rho(x, y)$ as the distance between x and y. Given a metric space (S, ρ), the notions of convergence and completeness generalize in a natural way.

Definition 5.1.1 We say that an S-valued sequence $(x_n : n \ge 1)$ *converges* in (S, ρ) if there exists $x_\infty \in S$ such that

$$\rho(x_n, x_\infty) \to 0$$

as $n \to \infty$ and write $x_n \to x_\infty$ as $n \to \infty$.

Definition 5.1.2 We say that an S-valued sequence $(x_n : n \ge 1)$ is *Cauchy* if, for each $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for $m, n \ge N$,

$$\rho(x_m, x_n) < \epsilon.$$

Definition 5.1.3 We say that the metric space (S, ρ) is *complete* if every Cauchy sequence is convergent to an element of S.

Example 5.1.1 The Euclidian space $\mathbb{R}^d$ is a metric space when equipped with the metric

$$\rho(x, y) = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}.$$

This is a complete metric space.

Example 5.1.2 The interval $S = (0, \infty)$ is a metric space under either of the following metrics:

$$\rho_1(x, y) = |x - y|;$$
$$\rho_2(x, y) = \left|\log\left(\frac{x}{y}\right)\right|.$$

S is complete under the metric ρ_2 but is not complete under the metric ρ_1. (So, completeness depends on the choice of metric.)

Example 5.1.3 Let S be a discrete space. The discrete metric is given by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

This is a complete metric space.

Example 5.1.4 Let $L^p = \{X : \mathrm{E}|X|^p < \infty\}$ for $1 \leq p < \infty$. Set

$$\rho(X, Y) = \|X - Y\|_p,$$

where

$$\|Z\|_p = (\mathrm{E}|Z|^p)^{\frac{1}{p}}.$$

The Banach space L^p is a complete metric space.

Example 5.1.5 Let $C[0, 1]$ be the space of functions $x : [0, 1] \to \mathbb{R}$ such that $x(\cdot)$ is a continuous function. Put

$$\rho(x, y) = \sup_{0 \leq t \leq 1} |x(t) - y(t)|.$$

The space $C[0, \infty)$ is a metric space when equipped with the metric ρ; ρ is the so-called *uniform metric*. Note that $x_n \to x_\infty$ in (S, ρ) if and only if

$$\sup_{0 \leq t \leq 1} |x_n(t) - x_\infty(t)| \to 0$$

as $n \to \infty$ i.e. $x_n \to x_\infty$ under the uniform metric if and only if x_n converges to x_∞ uniformly on $[0, 1]$. This metric space is complete. (In fact, $C[0, 1]$ is a Banach space under the norm $\|x\| = \sup_{0 \leq t \leq 1} |x|$; a Banach space is a special type of complete metric space.)

Example 5.1.6 Let $C[0, \infty)$ be the space of functions $x : [0, \infty) \to \mathbb{R}$ such that $x(\cdot)$ is a continuous function. Put

$$\rho(x, y) = \int_0^\infty e^{-t} \frac{\sup_{0 \leq u \leq t} |x(u) - y(u)|}{1 + \sup_{0 \leq u \leq t} |x(u) - y(u)|}\, dt.$$

The above function ρ is a metric on $C[0, \infty)$. Here, $x_n \to x_\infty$ in $(C[0, \infty), \rho)$ if and only if for each $t \geq 0$,

$$\sup_{0 \leq s \leq t} |x_n(s) - x_\infty(s)| \to 0$$

as $n \to \infty$ i.e. $x_n \to x_\infty$ under ρ if and only if x_n converges uniformly to x_∞ on each compact time interval $[0, t]$. This notion of convergence is called "uniform convergence on compacts" and one often writes

$$x_n \overset{u.o.c}{\to} x_\infty$$

as $n \to \infty$ to denote this convergence. The metric space $(C[0, \infty), \rho)$ is complete.

Every point in $\mathbb{R}^d$ can be approximated by a point having rational coordinates. Of course, the set of points having rational coordinates is a countable subset of $\mathbb{R}^d$. We will need a generalization of this concept to the metric space context.

Definition 5.1.4 A metric space S equipped with metric ρ is said to be *separable* if there exists a countable subset $A = (x_n : n \geq 1)$ of S such that for each $x \in S$ and $\epsilon > 0$, $\{y \in A : \rho(x, y) < \epsilon\} \neq \emptyset$.

Each of the metric spaces introduced above is separable.

Finally, in order to rigorously state our desired result on existence of stationary distribution, we will need the notion of "compactness".

Definition 5.1.5 A subset $K \subset S$ is *compact* (in (S, ρ)) if each sequence $(x_n : x_n \in K, n \geq 1)$ has a further subsequence $(x_{n_k} : k \geq 1)$ that converges to a limit $x_\infty \in K$.

5.2 Weak Convergence on Metric Spaces

We wish to extend the notion of weak convergence from the setting of real-valued random variables to random variables taking values in a separable metric space.

Definition 5.2.1 We say that a sequence $(X_n : n \geq 1)$ of S-valued random variables *converges weakly* to an S-valued random variable X_∞ (and write $X_n \Rightarrow X_\infty$ as $n \to \infty$) if there exists a probability space supporting a sequence $(X_n^* : 1 \leq n \leq \infty)$ such that:

i.) $X_n^* \stackrel{\mathcal{D}}{=} X_n$ for $1 \leq n \leq \infty$;

ii.) $\rho(X_n^*, X_\infty^*) \to 0$ a.s. as $n \to \infty$.

As in the setting of real-valued random variables, we may equivalently write that $P_n \Rightarrow P_\infty$ as $n \to \infty$, where $P_n(\cdot) \triangleq P(X_n \in \cdot)$ for $1 \leq n \leq \infty$.

Note that the rv $\rho(X_n^*, X_\infty^*)$ is real-valued, so one reduces weak convergence on S to a.s. convergence of real-valued rv's.

As in the setting of weak convergence of real-valued rv's, weak convergence can be re-formulated.

Definition 5.2.2 Given two metric spaces (S_1, ρ_1) and (S_2, ρ_2), we say that $g : S_1 \to S_2$ is *continuous* if for each sequence $x_n \to x_\infty$ in (S_1, ρ_1) as $n \to \infty$, $g(x_n) \to g(x_\infty)$ in (S_2, ρ_2) as $n \to \infty$.

Proposition 5.2.1 The sequence $(X_n : n \geq 1)$ of S-valued random variables converges weakly to X_∞ as $n \to \infty$ if and only if

$$\mathrm{E}g(X_n) \to \mathrm{E}g(X_\infty)$$

for each bounded and continuous function $g : S \to \mathbb{R}$.

See p.99-110 of *Markov Processes: Characterization and Convergence* (1986) by S. Ethier and T. Kurtz for a proof of this result.

One nice property of weak convergence is that it is inherited under continuous mappings.

Proposition 5.2.2 Let (S_1, ρ_1) and (S_2, ρ_2) be two metric spaces. If $(X_n : n \geq 1)$ is an S-valued sequence such that $X_n \Rightarrow X_\infty$ as $n \to \infty$ and if $g : S_1 \to S_2$ is continuous, then $g(X_n) \Rightarrow g(X_\infty)$ as $n \to \infty$.

To verify this, we need only show that

$$\mathrm{E}f(g(X_n)) \to \mathrm{E}f(g(X_\infty))$$

as $n \to \infty$ for all bounded continuous $f : S_2 \to \mathbb{R}$. But $f \circ g$ is bounded and continuous, so we are done.

The following extension is also very useful. We start with a definition.

Definition 5.2.3 Let (S_1, ρ_1) and (S_2, ρ_2) be two metric spaces and suppose $g : S_1 \to S_2$. We say that g is *continuous at x* if whenever $x_n \to x$ as $n \to \infty$, $g(x_n) \to g(x)$ as $n \to \infty$

Proposition 5.2.3 Let (S_1, ρ_1) and (S_2, ρ_2) be two metric spaces. Given $g : S_1 \to S_2$, let $D_g = S_1 - \{x \in S_1 : g \text{ is continuous at } x\}$ be the set of discontinuities of g. If $(X_n : n \geq 1)$ is a sequence of S_1-valued random variables for which $X_n \Rightarrow X_\infty$, then $g(X_n) \Rightarrow g(X_\infty)$ as $n \to \infty$, provided that $P(X_\infty \in D_g) = 0$.

For a proof, see p.29-31 of *Weak Convergence of Probability Measures* by P. Billingsley (1968).

Remark 5.2.1 Proposition 5.2.3 is sometimes called the "Extended Continuous Mapping Principle".

As noted earlier, asserting that $X_n \Rightarrow X_\infty$ as $n \to \infty$ is equivalent to requiring that $P_n \Rightarrow P_\infty$ as $n \to \infty$. But $P_n \in \mathscr{P}(S)$, the set of probability distributions (or measures, if you prefer that terminology) on S. The set $\mathscr{P}(S)$ can be viewed as a metric space. In particular, one can define a metric $\widetilde{\rho}$ on $\mathscr{P}(S)$ (called the Prohorov metric); see p. 96 of *Markov Processes: Characterization and Convergence* (1986) by S. Ethier and T. Kurtz for the precise definition.

Proposition 5.2.4 Suppose that (S, ρ) is a complete separable metric space. Then, $(\mathscr{P}(S), \widetilde{\rho})$ is a complete separable metric space under the Prohorov metric $\widetilde{\rho}$.

See p.96-102 of *Markov Processes: Characterization and Convergence*(1986) by S. Ethier and T. Kurtz for the proof.

Weak convergence turns out to be equivalent to demanding that P_n converges to P_∞ in the Prohorov metric.

Theorem 5.2.1 Suppose that (S, ρ) is a complete separable metric space. A sequence $(P_n : 1 \leq n \leq \infty)$ of probabilities on S satisfies $P_n \Rightarrow P_\infty$ as $n \to \infty$ if and only if $\widetilde{\rho}(P_n, P_\infty) \to 0$ as $n \to \infty$.

See p.107-108 of *Markov Processes: Characterization and Convergence* (1986) by S. Ethier and T. Kurtz for the proof.

5.3 Relative Compactness and Tightness

Let (S, ρ) be a metric space.

Definition 5.3.1 A set $A \subseteq S$ is said to be *relatively compact* if for each sequence $(x_n : x_n \in A, n \geq 1)$, there exists a subsequence $(x_{n_k} : k \geq 1)$ and $x_\infty \in S$ such that $x_{n_k} \to x_\infty$ as $k \to$.

Example 5.3.1 If $S = \mathbb{R}$ and $\rho(x, y) = |x - y|$, a set A is relatively compact if and only if it is bounded (i.e. $-\infty < \inf\{x : x \in A\} \leq \sup\{x : x \in A\} < \infty$).

A key step in generalizing arguments to the setting of a general state space is to identify conditions under which the analog of the sequence $(\overline{\mu}_n : n \geq 1)$ is relatively compact (so that convergent subsequences can be extracted).

Definition 5.3.2 A family of probabilities $(P_\lambda : \lambda \in \Lambda)$ on S is said to be *tight* if, for each $\epsilon > 0$, there exists a compact set $K = K(\epsilon)$ such that

$$P_\lambda(K) \geq 1 - \epsilon$$

for $\lambda \in \Lambda$. A family of S-valued random variables $(X_\lambda : \lambda \in \Lambda)$ is said to be *tight* if $(P_\lambda : \lambda \in \Lambda)$ is tight, where $(P_\lambda(\cdot) \triangleq P(X_\lambda \in \cdot)$.

Exercise 5.3.1 Let $(X_\lambda : \lambda \in \Lambda)$ be a family of real-valued random variables. Prove that $(X_\lambda : \lambda \in \Lambda)$ is tight if there exists $p > 0$ such that

$$\sup_{\lambda \in \Lambda} \mathrm{E}|X_\lambda|^p < \infty.$$

Theorem 5.3.1 (Prohorov's Theorem) Suppose that S is complete and separable. Then, a family $(P_\lambda : \lambda \in \Lambda)$ of probabilities on S is relatively compact in $(\mathscr{P}(S), \widetilde{\rho})$ if and only if $(P_\lambda : \lambda \in \Lambda)$ is tight.

A complete proof of this result can be found on p.103-107 of *Markov Processes: Characterization and Convergence* (1986) by S. Ethier and T. Kurtz.

Because of this result's importance, we outline its proof when $S = \mathbb{R}$, so that we are dealing with real-valued random variables. Suppose that $(P_\lambda : \lambda \in \Lambda)$ is tight and consider a sequence $(P_n : n \geq 1) \subseteq (P_\lambda : \lambda \in \Lambda)$. Set

$$F_n(x) = P_n((-\infty, x]).$$

It suffices to show that we can find a subsequence $(n_k : k \geq 1)$ and a distribution function F_∞ such that

$$F_{n_k}(x) \to F_\infty(x)$$

at all continuity points of $F_\infty(\cdot)$. Since the F_n's are non-decreasing, it is enough to prove that it holds for $x \in \mathbb{Q}$ (the rational numbers). Note that $(F_n(x_j) : n \geq 1)$ is a bounded sequence for each $x_j \in \mathbb{Q}$, so we can find a convergent subsequence n'_k (typically depending on x_j) and a limit $G(x_j)$ such that $F_{n'_k}(x_j) \to G(x_j)$ as $k \to \infty$.

A beautiful argument due to Cantor (the so-called "Cantor diagonalization argument") proves that one can find one common subsequence $(n''_k : k \geq 1)$ and a non-decreasing $G(\cdot)$ such that

$$F_{n''_k}(x_j) \to G(x_j)$$

as $k \to \infty$, simultaneously at each $x_j \in \mathbb{Q}$ at which $G(\cdot)$ is continuous. Hence, we have established that

$$F_{n''_k}(x) \to G(x)$$

as $k \to \infty$, at each x at which $G(\cdot)$ is continuous. (The fact that one can always find such a subsequence $(n''_k : k \geq 1)$ and non-decreasing G is what is called Helly's Theorem in probability.)

But the sequence $(P_n : n \geq 1)$ is tight, so for each $\epsilon > 0$, there exists an interval $[-K, K]$ such that

$$F_n(K) - F_n(-K) \geq 1 - \epsilon$$

It follows from that

$$G(K) - G(-K) \geq 1 - \epsilon.$$

Since ϵ was arbitrary, it follows that G is a probability distribution function. So, $(P_\lambda : \lambda \in \Lambda)$ is relatively compact.

The converse is left as an exercise.

Exercise 5.3.2 Prove that if $(P_\lambda : \lambda \in \Lambda)$ is a relatively compact family of probability distributions on $\mathbb{R}$, then $(P_\lambda : \lambda \in \Lambda)$ is tight.

5.4 Existence of Stationary Distributions for Markov Chains on Compact State Space

Let S be a metric space that is compact under ρ; such a compact metric space is automatically complete and separable (see p.72-77 of *Metric Spaces* (1968) by Copson for details.) We follow the same argument as we did for finite state chains. For an arbitrary probability μ on S (i.e. $\mu \in \mathscr{P}(S)$, set

$$\begin{aligned}
\overline{\mu}_n(\cdot) &\triangleq \frac{1}{n}\sum_{j=0}^{n-1}(\mu P^j)(\cdot) \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\int_S \mu(dx)P_x(X_j \in \cdot) \\
&= \mathrm{E}_\mu \frac{1}{n}\sum_{j=0}^{n-1} I(X_j \in \cdot).
\end{aligned}$$

Since S is compact, Prohorov's Theorem guarantees that $(\overline{\mu}_n : n \geq 1)$ is relatively compact. So, there exists a subsequence $(\overline{\mu}_{n_k} : k \geq 1)$ and a probability on S (call it $\overline{\mu}_\infty$ such that

$$\overline{\mu}_{n_k} \Rightarrow \overline{\mu}_\infty$$

as $k \to \infty$. We would now like to conclude that the limiting probability $\overline{\mu}_\infty$ is a stationary distribution, in the sense that

$$\overline{\mu}_\infty(\cdot) = \int_S \overline{\mu}_\infty(dx)P_x(X_1 \in \cdot);$$

i.e.

$$\overline{\mu}_\infty = \overline{\mu}_\infty P.$$

Unfortunately, the following example shows that such a limiting probability $\overline{\mu}_\infty$ need not be a stationary distribution.

Example 5.4.1 Suppose that $S = [0, 1]$ and

$$X_{n+1} = \begin{cases} \frac{1}{2}X_n & \text{if } X_n > 0; \\ 1 & \text{if } X_n = 0. \end{cases}$$

If $\mu(dx) = \delta_1(dx)$ (i.e. μ is a unit mass at 1), then

$$\mu P^k = \delta_{2^{-k}}$$

and

$$\overline{\mu}_n = \frac{1}{n}\sum_{j=0}^{n-1}(\mu P^j) \Rightarrow \delta_0$$

as $n \to \infty$, so $\overline{\mu}_\infty = \delta_0$. But $\overline{\mu}_\infty$ is not a stationary distribution, since $\overline{\mu}_\infty P = \delta_1$.

The difficulty in the above example is that X has "discontinuous dynamics" at 0.

Definition 5.4.1 A Markov chain $X = (X_n : n \geq 1)$ taking values in a complete separable metric space S is said to be a *Feller chain* if whenever $x_n \to x \in S$, then

$$P(x_n, \cdot) \Rightarrow P(x, \cdot)$$

as $n \to \infty$.

Recall that if $g : S \to \mathbb{R}$, then Pg is the function defined via

$$(Pg)(x) = \mathrm{E}_x g(X_1)$$

for $x \in S$. Let bC be the set of bounded continuous functions $g : S \to \mathbb{R}$.

Proposition 5.4.1 A Markov chain taking values in a complete separable metric space S is Feller if and only if $P : bC \to bC$.

To prove this, we need to show that if $g \in bC$, then $Pg \in bC$ precisely when the chain is Feller. But

$$\int_S g(y)P(x_n, dy) \to \int_S g(y)P(x, dy)$$

for each $g \in bC$. Of course, this says that

$$(Pg)(x_n) \to (Pg)(x)$$

as $n \to \infty$, so Pg is continuous. Since Pg is trivially bounded, it follows that $Pg \in bC$.

We are now ready to state our main theorem.

Theorem 5.4.1 Let $X = (X_n : n \geq 1)$ be a Feller chain taking values on a metric space (S, ρ) that is compact. Then, X possesses a stationary distribution.

To prove this theorem, note that there exists a probability $\overline{\mu}_\infty$ such that

$$\overline{\mu}_{n_k} \Rightarrow \overline{\mu}_\infty$$

as $k \to \infty$. Hence, for each $g \in bC$, we can conclude that

$$\int_S \overline{\mu}_{n_k}(dx)g(x) \to \int_S g(x)\overline{\mu}_\infty(dx)$$

as $k \to \infty$. Because X is Feller, $Pg \in bC$, so

$$\int_S \overline{\mu}_{n_k}(dx)(Pg)(x) \to \int_S (Pg)(x)\overline{\mu}_\infty(dx).$$

But

$$\begin{aligned}
\int_S \overline{\mu}_{n_k}(dx)(Pg)(x) &= \frac{1}{n_k}\sum_{j=0}^{n_k-1}\int_S (\mu P^j)(dx)(Pg)(x) \\
&= \frac{1}{n_k}\sum_{j=0}^{n_k-1}\int_S P_\mu(X_j \in dx)\mathrm{E}_x g(X_1) \\
&= \frac{1}{n_k}\sum_{j=1}^{n_k}\mathrm{E}_\mu g(X_j) \\
&= \frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathrm{E}_\mu g(X_j) + \frac{1}{n_k}\mathrm{E}_\mu g(X_{n_k}) - \frac{1}{n_k}\mathrm{E}_\mu g(X_0) \\
&= \frac{1}{n_k}\sum_{j=0}^{n_k-1}\int_S (\mu P^j)(dx)g(x) + \frac{1}{n_k}\mathrm{E}_\mu g(X_{n_k}) - \frac{1}{n_k}\mathrm{E}_\mu g(X_0) \\
&= \int_S \overline{\mu}_{n_k}(dx)g(x) + \frac{1}{n_k}\mathrm{E}_\mu g(X_{n_k}) - \frac{1}{n_k}\mathrm{E}_\mu g(X_0) \\
&\to \int_S \overline{\mu}_\infty(dx)g(x)
\end{aligned}$$

as $k \to \infty$. It follows that

$$\int_S \overline{\mu}_\infty(dx)g(x) = \int_S (\overline{\mu}_\infty P)(dx)g(x),$$

invoking the proposition below.

Proposition 5.4.2 Suppose that $\nu_1, \nu_2 \in \mathscr{P}(S)$, where S is a complete separable metric space. If

$$\int_S \nu_1(dx)g(x) = \int_S \nu_2(dx)g(x)$$

for all $g \in bC$, then $\nu_1 = \nu_2$.

See p.9 of of *Weak Convergence of Probability Measures* by P. Billingsley (1968) for a proof.

5.5 Examples

We wish to illustrate the application of Theorem 5.4.1 with an of example.

Example 5.5.1 Suppose that $X = (X_n : n \geq 0)$ is a Markov chain that obeys the stochastic recursion

$$X_{n+1} = \rho X_n + Z_{n+1},$$

where $(Z_n : n \geq 1)$ is iid. Assume that $|\rho| < 1$ and that the Z_i's are bounded rv's (so that there exists $c < \infty$ such that $P(|Z_n| \leq c) = 1$). We may then view X as evolving in a compact state space S.

In particular, we claim that there exists $m < \infty$ such that if $X_0 \in S = [-m, m]$, then $X_1 \in S$. To identify m, suppose that $|X_0| \leq m$. Then,

$$|X_1| \leq |\rho| \cdot |X_0| + c \leq m|\rho| + c$$

We require that $|X_1| \leq m$, and hence

$$m|\rho| + c \leq m.$$

In other words,

$$m \geq \frac{c}{1 - |\rho|}.$$

So put $S = [-m, m]$, where $m = c(1 - |\rho|)^{-1}$. The interval $S = [-m, m]$ is compact under the Euclidian norm $\rho(x, y) = |x - y|$.

To show that X has a stationary distribution π, we show that X is Feller. Suppose that $g \in bC$. Observe that

$$(Pg)(x) = \mathrm{E}_x g(X_1) = \mathrm{E}_x g(\rho X_0 + Z_1) = \mathrm{E} g(\rho x + Z_1).$$

Clearly,

$$g(\rho x_n + Z_1) \to g(\rho x + Z_1) \quad \text{a.s.}$$

as $n \to \infty$, and so the Bounded Convergence Theorem yields

$$\mathrm{E} g(\rho x_n + Z_1) \to \mathrm{E} g(\rho x + Z_1)$$

as $n \to \infty$, proving that $Pg \in bC$. So, X is Feller.

Exercise 5.5.1 Consider $\rho = 1/2$ and $Z_n \sim \textit{Bernoulli}(1/2)$. Prove that the uniform distribution on $[0, 2]$ is a stationary distribution for X.

Exercise 5.5.2 Suppose that $X = (X_n : n \geq 0)$ is a $\mathbb{R}^d$-valued Markov chain satisfying

$$X_{n+1} = F X_n + Z_{n+1},$$

where $(Z_n : n \geq 1)$ is an iid sequence of bounded $\mathbb{R}^d$-valued random variables. If $|||F|||_e < 1$, prove that X has a stationary distribution π.

Exercise 5.5.3 Suppose that $X = (X_n : n \geq 0)$ is a real-valued Markov chain obeying the recursion

$$X_{n+1} + a X_{n+1}^b = X_n + Z_{n+1},$$

where $(Z_n : n \geq 1)$ is an iid sequence of positive bounded rv's and $a, b > 0$. Prove that X has a stationary distribution π.

Exercise 5.5.4 Prove that the Markov chain $((X_n, \nu_n) : n \geq 0)$ that arises in the context of the filter discussed in Section 5 has a stationary distribution.

Exercise 5.5.5 a.) Suppose that $X = (X_n : n \geq 0)$ is a real-valued Markov chain satisfying

$$X_{n+1} = |X_n - Z_{n+1}|,$$

where $(Z_n : n \geq 1)$ is an iid sequence of bounded positive rv's. Prove that X has a stationary distribution π.

b.) Suppose that, in addition, Z has a density $f_Z(\cdot)$. Prove that if π is a probability having a density $p(x)$that is proportional to $P(Z > x)$, then π is a stationary distribution of X.