Section 3: Regenerative Processes

Contents

3.1 Regeneration: The Basic Idea

We describe an approach to studying the long-run behavior of Markov chains that will work for:

- all discrete state space Markov chains (both finite and infinite state space)
- the (significant) majority of continuous state space Markov chains

The idea is to recognize that a discrete state space Markov chain "regenerates" every time it returns to a fixed state $z \in S$. These "z-cycles" are iid (a careful proof of this depends, not surprisingly, on the strong Markov property), so the path evolution of the chain is thereby split up into iid segments. As a consequence, we can exploit the extensive theory for iid sequences to study Markov chains/processes. By studying the chain on the "regenerative time scale", the Markov dependence structure is simplified to the point where iid theory applies.

In order to exploit such an idea, we need to know that infinitely many z-cycles occur. In other words, we will need to determine whether

$$P_x(X_n = z \text{ infinitely often}) = 1$$

for $x \in S$ or not.

Definition 3.1.1 The state z is said to be *recurrent* if $P_z(X_n = z \text{ infinitely often}) = 1$. Otherwise, z is said to be *transient*.

Let

$$N(z) = \sum_{k=0}^{\infty} I(X_k = z)$$

be the total number of visits to z. Observe that the strong Markov property implies that

$$P_z(N(z) \geq k) = P_z(\tau(z) < \infty)^{k-1},$$

so z is recurrent if and only if $P_z(\tau(z) < \infty) = 1$, where $\tau(z) = \inf\{n \geq 1 : X_n = z\}$. (Of course, z is transient if and only if $P_z(\tau(z) < \infty) < 1$.)

It turns out that recurrence/transience is a "class property". We need a definition.

Definition 3.1.2 A Markov chain $X = (X_n : n \geq 0)$ is said to be *irreducible* if, for each $x, y \in S$, there exists $n = n(x, y)$ such that

$$P^n(x, y) > 0.$$

Proposition 3.1.1 Suppose X is irreducible. Either all states are recurrent or all states are transient.

Hence, it makes sense to talk about an irreducible recurrent Markov chain or an irreducible transient Markov chain (as opposed to having to characterize recurrence state-by-state).

3.2 A Necessary and Sufficient Condition for Recurrence

Theorem 3.2.1 Suppose X is irreducible. Then, X is recurrent if and only if there exists $x, y \in S$ such that

$$\sum_{n=0}^{\infty} P^n(x, y) = \infty.$$

Proof: Note that for $x \neq y$

$$\begin{aligned} E_x N(y) &= \sum_{k=1}^{\infty} k P_x(N(y) = k) \\ &= \sum_{k=1}^{\infty} k P_x(\tau(y) < \infty) P_y(\tau(y) < \infty)^{k-1} P_y(\tau(y) = \infty.) \end{aligned}$$

Because X is irreducible, $P_x(\tau(y) < \infty) > 0$. It follows that $E_x N(y) < \infty$ if and only if $P_y(\tau(y) < \infty) < 1$ (i.e. y is transient). But

$$\begin{aligned} E_x N(y) &= E_x \sum_{k=1}^{\infty} I(X_k = y) \\ &= \sum_{k=1}^{\infty} P^k(x, y). \qquad \square \end{aligned}$$

A beautiful example with which to illustrate this result is that of mean-zero random walk.

Definition 3.2.1 A sequence $(X_n : N \geq 0)$ is said to be a *random walk* if

$$X_n = X_0 + Z_1 + \cdots + Z_n,$$

where the Z_i's are iid and independent of X_0. It is a *mean-zero* random walk if $EZ_i = 0$.

Definition 3.2.2 A *nearest neighbor* random walk on $\mathbb{Z}$ is a random walk for which $X_0 \in \mathbb{Z}$ and $Z_i \in \{-1, 1\}$. A nearest neighbor random walk on $\mathbb{Z}^d$ is a sequence $(\mathbf{X}_n : n \geq 0)$, where $\mathbf{X}_n = (X_n(1), \ldots, X_n(d))$ and $X(i) \triangleq (X_n(i) : n \geq 0)$ is a collection of independent one-dimensional nearest neighbor random walks. The random walk is said to be *symmetric* if $P(Z_i = 1) = P(Z_i = -1) = 1/2$.

Use of Stirling's formula establishes that:

- Nearest neighbor symmetric random walk is recurrent in $d = 1, 2$.
- Nearest neighbor symmetric random walk is transient in $d \geq 3$.

This result generalizes to much more general mean-zero increment distributions; see, for example, K. L. Chung's *A Course in Probability Theory.* This is one of the major results of 20th century probability.

Exercise 3.2.1 Suppose that X is a random walk on $\mathbb{Z}^d$ for which $EZ \neq 0$. Prove that $(X_n : n \geq 0)$ is transient.

3.3 Branching Chains

The "branching chain" model arises in many applications (biological population modeling, nuclear reactions, etc.). The simplest version is a Markov chain $X = (X_n : n \geq 0)$ on $\mathbb{Z}_+$ with dynamics described by:

$$X_{n+1} = \sum_{i=1}^{X_n} Z_{n+1,i}$$

where $(Z_{ni} : n \geq 1, i \geq 1)$ is a collection of iid $\mathbb{Z}_+$-valued rv's independent of X_0. A key quantity of interest is the probability of extinction

$$u^*(x) = P_x(T < \infty),$$

where $T = \inf\{n \geq 0 : X_n = 0\}$. Clearly, if $P(Z = 0) > 0$, $P_x(T < \infty) > 0$. Because each individual's progeny evolve independently,

$$\begin{aligned} u^*(1) &= \sum_{x=0}^{\infty} P(Z = x) u^*(1)^x \\ &= \varphi(u^*(1)), \end{aligned} \tag{3.3.1}$$

where $\varphi(y) = Ey^Z$ for $y \geq 0$. Assume that $EZ < \infty$, and note that (3.3.1) asserts that $u^*(1)$ is a root of $\varphi(x) = x$. Of course, $\mathbf{1}$ is a root. But this non-linear equation may have multiple roots.

Observe that $\varphi(\cdot)$ is non-decreasing, convex, with $\varphi(0) = P(Z = 0) > 0$ and $\varphi'(1) = EZ$. If:

- $\varphi'(1) \leq 1$: unique root of $\varphi(x) = x$ on $[0, 1]$ occurs at $x = 1$. So, $P_x(T < \infty) = 1$.
- $\varphi'(1) > 1$: unique root $\tilde{z}$ of $\varphi(x) = x$ on $(0, 1)$, with second root $= 1$.

Note that both $(\tilde{z}^x : x \geq 0)$ and $(1^x : x \geq 0)$ are solutions of

$$u(x) = P(Z = 0)^x + \sum_{y=1} P(X_n = y | X_0 = 0) u(y); \tag{3.3.2}$$

$(u^*(x) : x \geq 0)$ is the minimal non-negative solution of (3.3.2). So, $u^*(x) = \tilde{z}^x$ (since $\tilde{z} < 1$) when $EZ > 1$.

Exercise 3.3.1 a) Prove that $P_y(X_n = y \text{ infinitely often}) = 0$ for $y \geq 1$.

b) Prove that $X_n \to \infty$ a.s. on $\{T = \infty\}$.

3.4 Regenerative Processes

We now formalize the notion of a regenerative process. Given an S-valued stochastic process $X = (X(t) : t \geq 0)$ with an associated increasing sequence of finite-valued random times $(T(n) : n \geq 0)$, define the "cycles" $(W_n : n \geq 0)$ via

$$W_n(t) = \begin{cases} X(T(n-1)+t) : & 0 \leq t < \tau_n \\ \Delta; & t \geq \tau_n, \end{cases}$$

where $T(-1) = 0$, $\tau_n = T(n) - T(n-1)$, and $\Delta \notin S$.

Definition 3.4.1 We say that X is *regenerative* (with respect to $(T(n) : n \geq 0)$) if:

- $W_0, W_1, \ldots$ are independent
- $W_1, W_2, \ldots$ are identically distributed.

We say that X is *non-delayed* if $T(0) = 0$; otherwise, X is delayed. We say that X is *positive recurrent* if $E\tau_1 < \infty$.

Let $N(t) = \max\{n \geq -1 : T(n) \leq t\}$ be the index of the last cycle completed prior to time t.

Remark 3.4.1 When $T(0) = 0$, $N(t)$ is a *renewal counting process.*

Proposition 3.4.1 $N(t)/t \to 1/E\tau_1$ a.s. as $t \to \infty$.

This is the strong law of large numbers for renewal counting processes.

Theorem 3.4.1 (Strong Law for Regenerative Processes) Let X be regenerative with respect to $(T(n) : n \geq 0)$, and let $f : S \to \mathbb{R}_+$. If:

- $\displaystyle\int_0^{T(0)} |f(X(s))|ds < \infty$ a.s.
- $\displaystyle E\int_{T(0)}^{T(1)} f(X(s))ds$ and $E\tau_1$ are not simultaneously infinite

then

$$\frac{1}{t}\int_0^t f(X(s))ds \to \frac{EY_1(f)}{E\tau_1} \quad \text{a.s.}$$

as $t \to \infty$, where

$$Y_i(f) \triangleq \int_{T(i-1)}^{T(i)} f(X(s))ds.$$

Outline of Proof:

- $N(t) \to \infty$ a.s. (since $T(n) < \infty$ a.s. for $n \geq 0$)
- Use "path-by-path" argument based on squeeze

$$\frac{\sum_{i=0}^{N(t)} Y_i(f)}{\sum_{i=0}^{N(t)+1} \tau_i} \leq \frac{1}{t} \int_0^t f(X(s))ds \leq \frac{\sum_{i=0}^{N(t)+1} Y_i(f)}{\sum_{i=0}^{N(t)} \tau_i}.$$

Proposition 3.4.2 $EN(t)/t \to 1/E\tau_1$ as $t \to \infty$.

This is the so-called "elementary renewal theorem". The key to the proof is the Wald identity.

Wald Identity: Let $(Z_n : n \geq 1)$ be a sequence of independent non-negative rv's for which $EZ_i = \mu_i$ for $i \geq 1$. If T is a stopping time adapted to $(Z_n : n \geq 1)$, then

$$E \sum_{i=1}^{T} Z_i = E \sum_{i=1}^{T} \mu_i.$$

Proof of Wald's Identity:

$$\begin{aligned}
E \sum_{i=1}^{T} Z_i &= E \sum_{i=1}^{\infty} Z_i I(T \geq i) \\
&= \sum_{i=1}^{\infty} EZ_i I(T \geq i) \quad \text{(by Fubini)} \\
&= \sum_{i=1}^{\infty} EZ_i (1 - I(T \leq i-1)) \\
&= \sum_{i=1}^{\infty} EZ_i (1 - q_{i-1}(Z_1, \ldots, Z_{i-1})) \quad \text{(since } T \text{ is a stopping time)} \\
&= \sum_{i=1}^{\infty} EZ_i E(1 - q_{i-1}(Z_1 \ldots, Z_{i-1})) \quad \text{(independence)} \\
&= \sum_{i=1}^{\infty} \mu_i EI(T \geq i) \\
&= E \sum_{i=1}^{\infty} \mu_i I(T \geq i) \quad \text{(by Fubini)} \\
&= E \sum_{i=1}^{T} \mu_i.
\end{aligned}$$

Corollary: If the Z_i's are also identically distributed, $E \sum_{i=1}^{T} Z_i = ET \cdot EZ_1$.

Outline of proof of Proposition 3.4.2

- $N(t)+1$ is a stopping time adapted to $(\tau_i : i \geq 0)$. (Note that $N(t)$ is *not* a stopping time)
- Wald's identity implies that

$$E \sum_{i=0}^{N(t)+1} \tau_i = E\tau_0 + E(N(t)+1) \cdot E\tau_1$$

- Since $t \leq \sum_{i=0}^{N(t)+1} \tau_i$, $\liminf_{t\to\infty} t^{-1} EN(t) \geq 1/E\tau_1$ when $E\tau_0 < \infty$.
- For "lim sup", "truncate" the τ_i's to the $(\tau_i \wedge n)$'s and let $N_n(t)+1 = \min\{k \geq 0 : \sum_{i=n}^{k} \tau_i \wedge n \geq t\}$.
- Note that $N(\cdot) \leq N_n(\cdot)$ and $\sum_{i=0}^{N_n(t)+1} \tau_i \wedge n \leq t+n$
- Apply Wald's identity and divide by t:

$$\limsup_{t\to\infty} \frac{1}{t} EN_n(t) \leq \frac{1}{E\tau_1 \wedge n}$$

- So, $\limsup_{t\to\infty} t^{-1} EN(t) \leq 1/E\tau_1 \wedge n$.
- Send $n \to \infty$ and apply the Monotone Convergence Theorem (MCT).
- Deal separately with $E\tau_0 = \infty$.

Theorem 3.4.2 Let X be regenerative with respect to $(T(n) : n \geq 0)$ and let $f : S \to \mathbb{R}_+$. If $E(Y_0(f) + Y_1(f)) < \infty$, then

$$\frac{1}{t} \int_0^t Ef(X(s))ds \to \frac{EY_1(f)}{E\tau_1}$$

as $t \to \infty$.

Outline of proof of Theorem 3.4.2

- $\frac{1}{t} \int_0^t f(X(s))ds \leq \frac{1}{t} \sum_{i=0}^{N(t)+1} Y_i(f)$
- Same type of argument as in Theorem 3.4.1 proves that

$$t^{-1} \sum_{i=0}^{N(t)+1} Y_i(f) \to \frac{EY_1(f)}{E\tau_1} \quad \text{a.s.}$$

  as $t \to \infty$
- Wald's identity proves that

$$t^{-1} E \sum_{i=0}^{N(t)+1} Y_i(f) = t^{-1} EY_0(f) + t^{-1} E(N(t)+1) EY_1(f)$$

- Elementary renewal theorem shows that $t^{-1}E\sum_{i=0}^{N(t)+1} Y_i(f) \to EY_1(f)/E\tau_1$ as $t \to \infty$, so $(t^{-1}\sum_{i=0}^{N(t)+1} Y_i(f) : t \geq 0)$ is uniformly integrable
- So, $(t^{-1}\int_0^t f(X(s))ds : t \geq 0)$ is uniformly integrable
- So, $t^{-1}E\int_0^t f(X(s))ds \to EY_1(f)/E\tau_1$

3.5 Positive Recurrent Discrete Time Markov Chains

Assume $X = (X_n : n \geq 0)$ is irreducible and recurrent. For $y \in S$, let

$$f(x) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{else} \end{cases}$$

Note that

$$\begin{aligned} P_x(Y_0(f) = k) &= P_x(\sum_{i=0}^{\tau(z)-1} I(X_i = y) = k) \\ &= P_x(\tau(y) < \tau(z))P_y(\tau(y) < \tau(z))^{k-1}P_y(\tau(z) < \tau(y)) \end{aligned}$$

so $E_xY_0(f) < \infty$, $E_xY_1(f) < \infty$. Hence, Theorem 3.4.1 implies that

$$n^{-1}\sum_{i=0}^{n-1} I(X_i = y) \to \frac{E_z\sum_{i=0}^{\tau(z)-1} I(X_i = y)}{E_z\tau(z)} \quad \text{a.s.}$$

as $n \to \infty$, and Theorem 3.4.2 implies that

$$n^{-1}\sum_{i=0}^{n-1} P^i(x, y) \to \frac{E_z\sum_{i=0}^{\tau(z)-1} I(X_i = y)}{E_z\tau(z)} \tag{3.5.1}$$

as $n \to \infty$. Note that:

- right-hand side of (3.5.1) is independent of z (since left-hand side is)
- limit is independent of x
- if $E_z\tau(z) = \infty$, limit is zero (i.e., "null")
- if $E_z\tau(z) < \infty$, limit is positive

Definition 3.5.1 A recurrent state $z \in S$ is said to be *positive recurrent* if $E_z\tau(z) < \infty$ and *null recurrent* otherwise.

Since the right-hand side of (3.5.1) is independent of z, positive recurrence/null recurrence is a class property (i.e. either holds for all states in S or no states in S). When X is positive recurrent, set

$$\pi(y) = \frac{E_z\sum_{i=0}^{\tau(z)-1} I(X_i = y)}{E_z\tau(z)} \tag{3.5.2}$$

- $\pi = (\pi(y) : y \in S)$ is a probability mass function on S
- For $f : S \to \mathbb{R}_+$,
$$\frac{EY_1(f)}{E\tau_1} = \sum_y \pi(y) f(y).$$
- If $E_z \sum_{j=0}^{\tau(z)-1} |f(X_j)| < \infty$, then $E_x \sum_{j=0}^{\tau(z)-1} |f(X_j)| < \infty$ for all $x \in S$ (so $E_x Y_1(|f|) < \infty$ and $E_x Y_0(|F|) < \infty$ for each $x \in S$); Theorem 3.4.2 then implies that
$$\frac{1}{n}\sum_{j=0}^{n-1} E_x f(X_j) \to \sum_y \pi(y) f(y)$$
as $n \to \infty$

Exercise 3.5.1 Suppose X is positive recurrent and $\sum_y \pi(y)|f(y)| < \infty$. Show, by example, that
$$\frac{1}{n}\sum_{j=0}^{n-1} E_\mu f(X_j) \nrightarrow \sum_y \pi(y) f(y)$$
as $n \to \infty$, if $\mu = (\mu(x) : x \in S)$ does not have finite support.

For π as defined via (3.5.2), π satisfies the linear system $\pi = \pi P$. The proof is similar to that used in the compactness argument for Chapter 1; (3.5.1) implies that
$$\pi(y) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} P^i(x,y)$$
$$= \lim_{n\to\infty} \sum_z n^{-1} \sum_{i=0}^{n-1} P^i(x,z) P(z,y).$$
To bring the limit inside the sum over z: For $\epsilon > 0$, select a finite set $K \subseteq S$ such that
$$\sum_{y \in K^c} \pi(y) < \epsilon.$$
Then, select n_0 so that for $n \geq n_0$,
$$|n^{-1} \sum_{i=0}^{n-1} P^i(x,z) - \pi(z)| < \epsilon/|K|$$
for $z \in K$. It follows that
$$\sum_{z \in K^c} n^{-1} \sum_{i=0}^{n-1} P^i(x,z) P(z,y) \leq 2\epsilon$$
for $n \geq n_0$. On the other hand,
$$\sum_{z \in K} n^{-1} \sum_{i=0}^{n-1} P^i(x,z) P(z,y) \to \sum_{z \in K} \pi(z) P(z,y).$$
Since ϵ was arbitrary, $\pi(y) = \sum_z \pi(z) P(z,y)$.

Exercise 3.5.2 a) Prove that X is positive recurrent if and only if $\pi = \pi P$ has a solution π within the class of probability mass functions.

b) Prove that if X is positive recurrent, the solution π to $\pi = \pi P$ is unique within the class of probability mass functions.

c) Give an example that shows that $\pi = \pi P$ may have multiple linearly independent solutions when π is not restricted to be a probability mass function. (Note that your example must be irreducible and positive recurrent.)

We are now ready to state a theorem that summarizes the above discussion.

Theorem 3.5.1 Let $X = (X_n : n \geq 0)$ be an irreducible Markov chain taking values in S.

a) If X is null recurrent and $f : S \to \mathbb{R}_+$ is such that $EY_1(f) < \infty$, then

$$\frac{1}{n}\sum_{j=0}^{n-1} f(X_j) \to 0 \quad \text{a.s.}$$

as $n \to \infty$. If, in addition, $E_x Y_0(f) < \infty$, then

$$\frac{1}{n}\sum_{j=0}^{n-1} E_x f(X_j) \to 0$$

as $n \to \infty$.

b) X is positive recurrent if and only if there exists a probability mass function solution to $\pi = \pi P$; the solution π is unique within the class of probability mass functions.

c) If X is positive recurrent, then for each $f : S \to \mathbb{R}_+$,

$$\frac{1}{n}\sum_{j=0}^{n-1} f(X_j) \to \sum_y \pi(y) f(y) \quad P_x \text{ a.s.} \tag{3.5.3}$$

as $n \to \infty$ and

$$\frac{1}{n}\sum_{j=0}^{n-1} E_x f(X_j) \to \sum_y \pi(y) f(y) \tag{3.5.4}$$

as $n \to \infty$. If $f : S \to \mathbb{R}$ and $\sum_y \pi(y)|f(y)| < \infty$, then (3.5.3) and (3.5.4) continue to hold.

Remark 3.5.1 One implication of (3.5.2) is that for a positive recurrent chain, $\pi(z) = 1/E_z\tau(z)$.

Remark 3.5.2 Suppose that X is null recurrent. Consider

$$\eta(y) = E_z \sum_{j=0}^{\tau(z)-1} I(X_j = y).$$

It turns out that $\eta = (\eta(y) : y \in S)$ is an "invariant measure", in the sense that η satisfies $\eta = \eta P$; see Resnick, p. 118. (Of course, η can not be summable, since otherwise X would be positive recurrent.) Does this invariant measure have a probability interpretation?

The answer is: "Yes!" Suppose that at time $n = 0$, we independently assign to each $y \in S$ a Poisson number of particles to that state having mean $\eta(y)$. Let each of the particles (across all states) independently evolve according to the transition dynamics of the Markov chain. Then, the distribution of the particles at time 1 (and all future times) will be identical to the distribution at time $n = 0$.

Remark 3.5.3 We have seen that all recurrent Markov chains have "invariant measures". A transient irreducible Markov chain may either have an "invariant measure" η (i.e. a positive solution η to $\eta = \eta P$ or not. In particular, existence of an invariant measure of X does not imply that X is recurrent.