Population Dynamics

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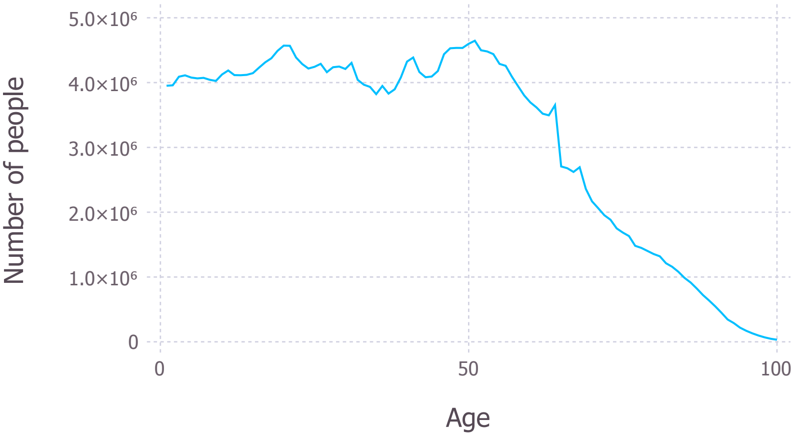
Outline

Population distribution

- $x_t \in \mathbf{R}^{100}$ gives population distribution in year $t = 1, \ldots, T$
- $(x_t)_i$ is the number of people with age $i - 1$ in year t
  (say, on January 1)
- total population in year t: $\mathbf{1}^T x_t$
- number of people age 70 or older in year t: $(0_{70}, \mathbf{1}_{30})^T x_t$

Population distribution of the U.S.

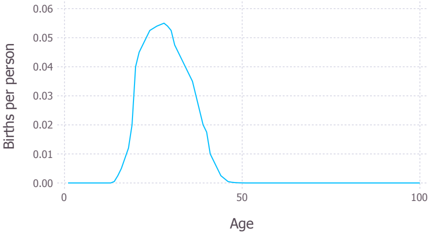
(from 2010 census)

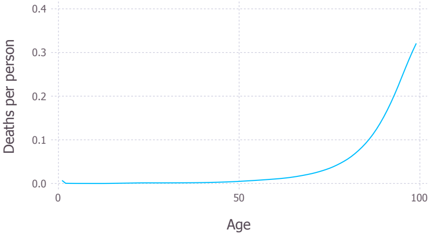


Birth and death rates

- birth rate $b \in \mathbf{R}^{100}$, death (or mortality) rate $d \in \mathbf{R}^{100}$
- b_i is the number of births per person with age $i-1$
- d_i is the portion of those aged $i-1$ who will die this year (we'll take $d_{100} = 1$)
- b and d can vary with time, but we'll assume they are constant

Birth and death rates in the U.S.





Outline

Population distribution

Dynamics

Dynamics

- let's find next year's population distribution x_{t+1} (ignoring immigration; we'll add that later)
- number of 0-year-olds next year is total births this year:

$$(x_{t+1})_1 = b^T x_t$$

- number of i-year-olds next year is number of $(i-1)$-year-olds this year, minus those who die:

$$(x_{t+1})_{i+1} = (1-d_i)(x_t)_i, \quad i = 1, \ldots, 99$$

- $x_{t+1} = Ax_t$, where

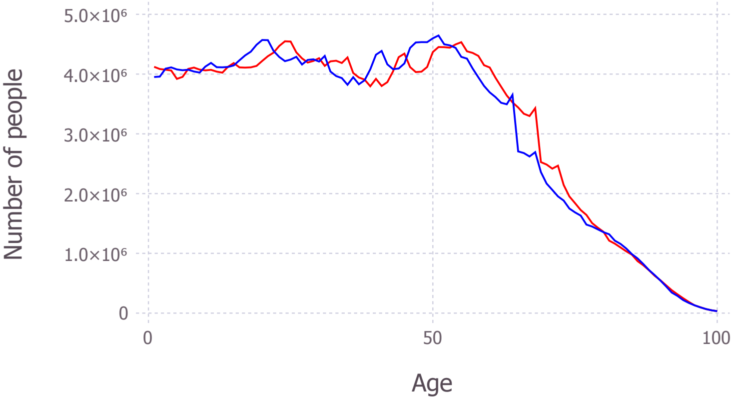
$$A = \begin{bmatrix} b_1 & b_2 & \cdots & b_{99} & b_{100} \\ 1-d_1 & & & \cdots & 0 \\ 0 & 1-d_2 & & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & 1-d_{99} & 0 \end{bmatrix}$$

Dynamics

- to predict distribution s years in the future: $x_{t+s} = A^s x_t$
- A^s propagates current population distribution s years forward
- to predict total population s years in future: $\mathbf{1}^T x_{t+s} = \mathbf{1}^T A^s x_t$
- what do the entries of the row vector $\mathbf{1}^T A^{10}$ mean?

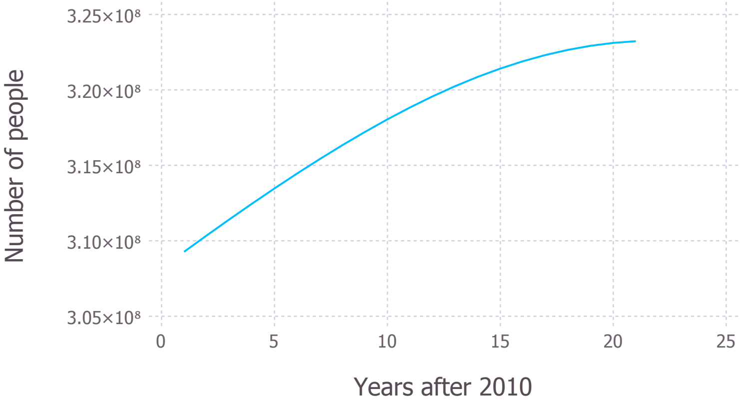
Predicting future population distributions

predicting U.S. 2015 distribution from 2010 (ignoring immigration)



Predicting population growth

predicted population growth (ignoring immigration)



Initial population distributions

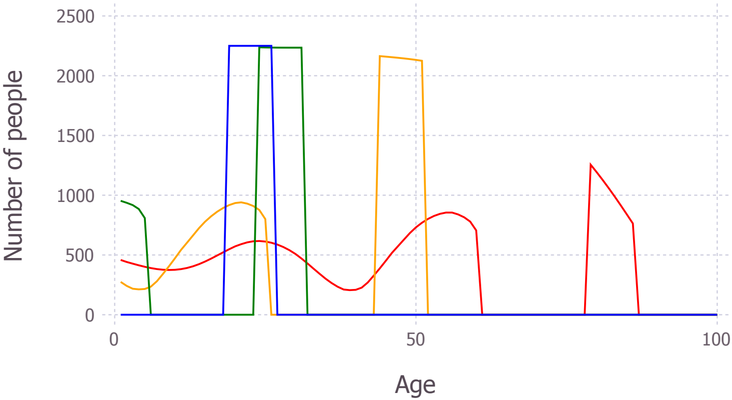
- what if we changed x_0?
- instead of U.S. Census data, let's use a "college nation"

$$(x_0)_i = \begin{cases} 2200 & i = 19, 20, \ldots, 27 \\ 0 & \text{otherwise} \end{cases}$$

(approximate population distribution of Stanford students)

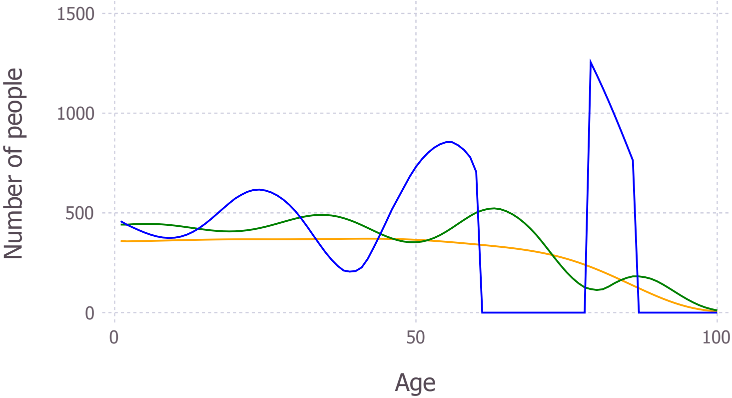
Predicting future population distributions

predict s years into the future

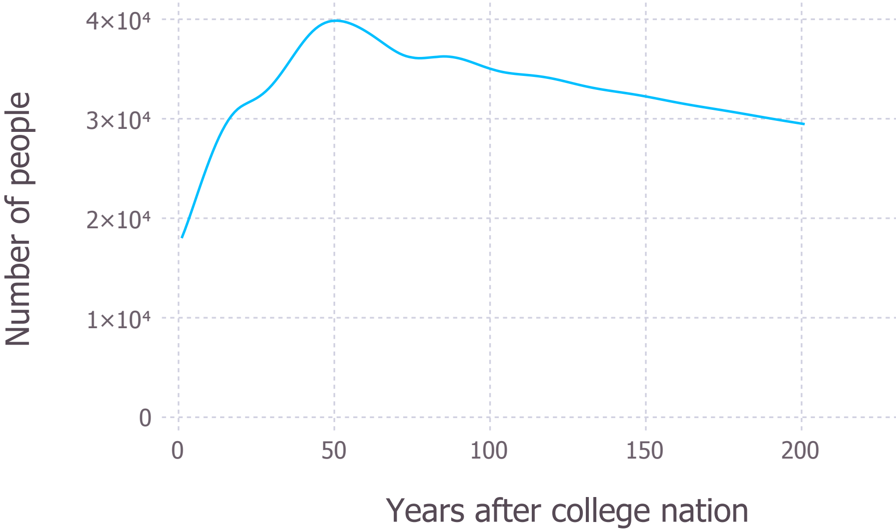


Predicting future population distributions

predict a little farther into the future



Population growth



Immigration

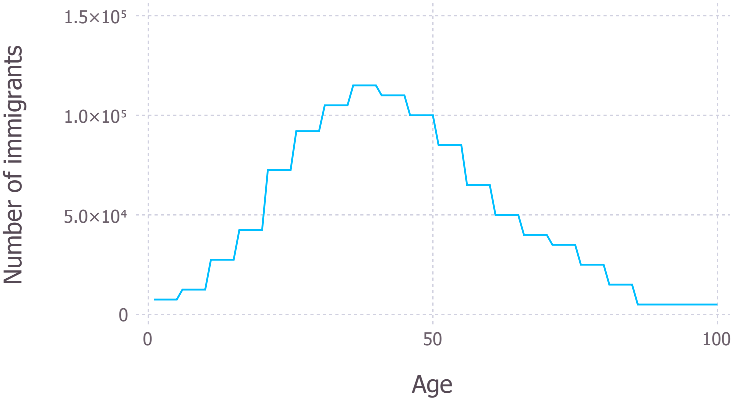
- $u \in \mathbf{R}^{100}$ is immigration: u_i is the net immigration of $(i-1)$-year-olds
- dynamics with immigration: $x_{t+1} = Ax_t + u$
- to propagate distribution forward s years:

$$\begin{aligned} x_{t+1} &= Ax_t + u \\ x_{t+2} &= A(Ax_t + u) + u = A^2 x_t + Au + u \\ &\vdots \\ x_{t+s} &= A^s x_t + A^{s-1} u + \cdots + Au + u \end{aligned}$$

- $(A^{s-1} + \cdots + A + I)u$ is population distribution at $t+s$ due to immigration over years $t, \ldots, s-1$

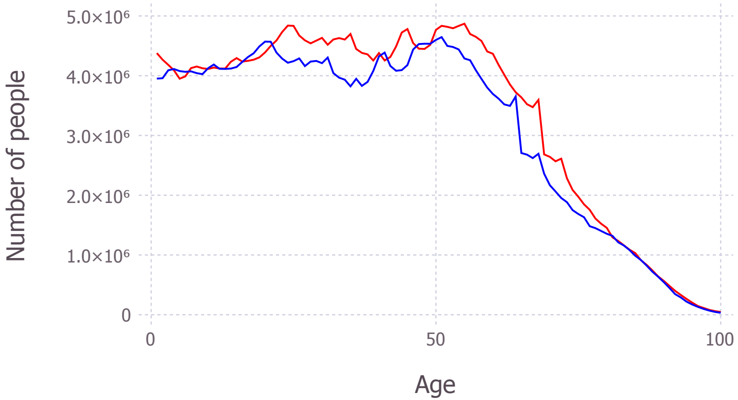
Immigration into the U.S.

piecewise constant approximation



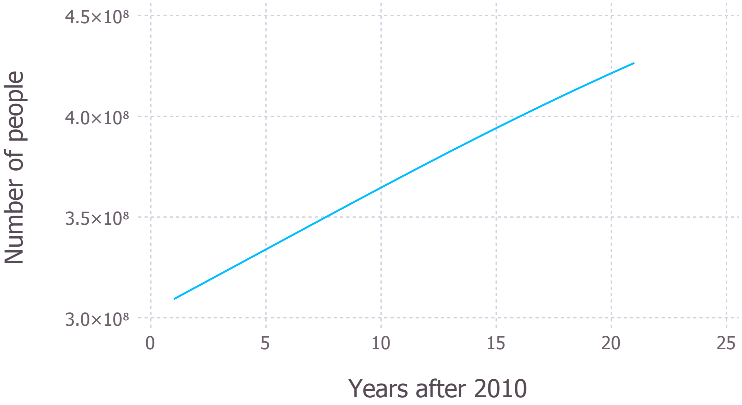
Predicting future population distributions

predicting U.S. 2015 distribution from 2010



Population growth with immigration

- we can plot $\mathbf{1}^T x_t$ for $t = 1, \ldots, 1000$ with immigration



Inferring immigration

- given x_1, x_T, b, and d, infer (constant) immigration vector u
- we have $x_T = A^{T-1}x_1 + A^{T-2}u + \cdots + u$ and so

$$(A^{T-1} + \cdots + A + I)u = x_T - A^{T-1}x_1$$

  (what does righthand side mean?)
- so immigration is

$$u = (A^{T-2} + \cdots + A + I)^{-1}(x_T - A^{T-1}x_1)$$