Linear-Quadratic Control

Jenny Hong     Nicholas Moehle     Stephen Boyd

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Stanford University

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Outline

Linear dynamical system

Control

Variations

Examples

Linear quadratic regulator
Linear dynamical system

\[ x_{t+1} = Ax_t + Bu_t, \quad t = 1, 2, \ldots \]

- \( n \)-vector \( x_t \) is state at time \( t \)
- \( m \)-vector \( u_t \) is input at time \( t \)
- \( n \times n \) matrix \( A \) is dynamics matrix
- \( n \times m \) matrix \( B \) is input matrix
- sequence \( x_1, x_2, \ldots \) is called state trajectory
Simulation

- given \( x_1, u_1, u_2, \ldots \) find \( x_2, x_3, \ldots \)
- can be done by recursion: for \( t = 1, 2, \ldots \),

\[
x_{t+1} = Ax_t + Bu_t
\]
Vehicle example

consider a vehicle moving in a plane:

- sample position and velocity at times $\tau = 0, h, 2h, \ldots$
- 2-vectors $p_t$ and $v_t$ are position and velocity at time $ht$
- 2-vector $u_t$ gives applied force on the vehicle time $ht$
- friction force is $-\eta v_t$
- vehicle has mass $m$
- for small $h$,

$$m \frac{v_{t+1} - v_t}{h} \approx -\eta v_t + u_t, \quad \frac{p_{t+1} - p_t}{h} \approx v_t$$

- we use approximate state update

$$v_{t+1} = (1 - h\eta/m)v_t + (h/m)u_t, \quad p_{t+1} = p_t + hv_t$$
vehicle state is 4-vector \( x_t = (p_t, v_t) \)

dynamics recursion is

\[
x_{t+1} = Ax_t + Bu_t,
\]

where

\[
A = \begin{bmatrix}
1 & 0 & h & 0 \\
0 & 1 & 0 & h \\
0 & 0 & 1 - h\eta/m & 0 \\
0 & 0 & 0 & 1 - h\eta/m
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
h/m & 0 \\
0 & h/m
\end{bmatrix}
\]
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- $x_1$ is given
- choose $u_1, u_2, \ldots, u_{T-1}$ to achieve some goals, e.g.,
  - terminal state should have some fixed value: $x_T = x^{\text{des}}$
  - $u_1, u_2, \ldots, u_{T-1}$ should be small, say measured as
    \[
    \|u_1\|^2 + \cdots + \|u_{T-1}\|^2
    \]
    (sometimes called ‘energy’)
- many control problems are linearly constrained least-squares problems
Minimum-energy state transfer

- given initial state $x_1$ and desired final state $x_{\text{des}}$
- choose $u_1, \ldots, u_{T-1}$ to minimize ‘energy’

\[
\begin{align*}
\text{minimize} & \quad \|u_1\|^2 + \cdots + \|u_{T-1}\|^2 \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\
& \quad x_T = x_{\text{des}}
\end{align*}
\]

variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$

- roughly speaking: find minimum energy inputs that steer the state to given target state over $T$ periods
State transfer example

vehicle model with $T = 100$, $x_1 = (10, 10, 10, -5)$, $x^{des} = 0$
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Output tracking

- $y_t = Cx_t$ is output (e.g., position)
- $y_t$ should follow a desired trajectory, i.e., sum square tracking error
  \[ \| y_2 - y_2^\text{des} \|^2 + \cdots + \| y_T - y_T^\text{des} \|^2 \]
  should be small
- the output tracking problem is
  \[
  \begin{align*}
  \text{minimize} & \quad \sum_{t=2}^{T} \| y_t - y_t^\text{des} \|^2 + \rho \sum_{t=1}^{T-1} \| u_t \|^2 \\
  \text{subject to} & \quad x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T - 1 \\
  & \quad y_t = Cx_t, \quad t = 1, \ldots, T - 1
  \end{align*}
  \]
  variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}, y_2, \ldots, y_T$
- parameter $\rho > 0$ trades off control ‘energy' and tracking error
vehicle model with $T = 100$, $\rho = 0.1$, $x_1 = 0$, $y_t = p_t$ (position tracking)
Waypoints

- using output, can specify _waypoints_
- specify output (position) \( w^{(k)} \) at time \( t_k \) at \( K \) total places

\[
\begin{align*}
\text{minimize} & \quad \|u_1\|^2 + \cdots + \|u_{T-1}\|^2 \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T - 1 \\
& \quad Cx_{t_k} = w^{(k)}, \quad k = 1, \ldots, K
\end{align*}
\]

variables are \( x_2, \ldots, x_T, u_1, \ldots, u_{T-1} \)
Waypoints example

- Vehicle model
  - $T = 100$, $x_1 = (10, 10, 20, 0)$, $x_{\text{des}} = 0$
  - $K = 4$, $t_1 = 10$, $t_2 = 30$, $t_3 = 40$, $t_4 = 80$
  - $w^{(1)} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$, $w^{(2)} = \begin{bmatrix} 1.5 \\ 4 \end{bmatrix}$, $w^{(3)} = \begin{bmatrix} 4.5 \\ 1 \end{bmatrix}$, $w^{(4)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Waypoints example
Variations
Rendezvous

- we control two vehicles with dynamics

\[ x_{t+1} = Ax_t + Bu_t, \quad z_{t+1} = Az_t + Bv_t \]

- final relative state constraint \( x_T = z_T \)

- formulate as state transfer problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{t=1}^{T-1} (\|u_t\|^2 + \|v_t\|^2) \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T - 1, \\
& \quad z_{t+1} = Az_t + Bv_t, \quad t = 1, \ldots, T - 1, \\
& \quad x_T = z_T
\end{align*}
\]

variables are \( x_2, \ldots, x_T, u_1, \ldots, u_{T-1}, z_2, \ldots, z_T, v_1, \ldots, v_{T-1} \)
Rendezvous example

\[ x_1 = (0, 0, 0, -5), \quad z_1 = (10, 10, 5, 0) \]
Formation

- generalize rendezvous example to several vehicles
- final position for each vehicle defined relative to others (e.g., relative to a 'leader')
- leader has a final velocity constraint
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Linear quadratic regulator
minimize energy while driving state to the origin:

\[
\begin{align*}
\text{minimize} & \quad \sum_{t=2}^{T} \|x_t\|^2 + \rho \sum_{t=1}^{T-1} \|u_t\|^2 \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T - 1
\end{align*}
\]

variables are \(x_2, \ldots, x_T, u_1, \ldots, u_{T-1}\)

\(\sum_{t=2}^{T} \|x_t\|^2\) is (sum square) regulation

\(x = 0\) is some desired (equilibrium, target) state

parameter \(\rho > 0\) trades off regulation versus control ‘energy’
LQR problem is a linearly constrained least-squares problem:

\[
\begin{align*}
\text{minimize} & \quad \|Fz\|^2 \\
\text{subject to} & \quad Gz = d
\end{align*}
\]

- variable \( z \) is \((x_2, \ldots, x_T, u_1, \ldots, u_{T-1})\)
- \( F, G \) depend on \( A, B, \rho; \) \( d \) depends (linearly) on \( x_1 \)
- solution is \( \hat{z} = Hd \) for some \( H \)
- optimal first input \( \hat{u}_1 \) is a linear function of \( x_1 \), i.e.,

\[
\hat{u}_1 = Kx_1
\]

for some \( m \times n \) matrix \( K \) (called LQR gain matrix)

- finding \( K \) involves taking correct ‘slice’ of inverse KKT matrix
- entries of \( K \) depend on horizon \( T \), and converge as \( T \) grows large

Linear quadratic regulator
State feedback control

- find $K$ for LQR problem (with large $T$)
- for each $t$,
  - measure state $x_t$
  - implement control $u_t = Kx_t$
- with $u_t = Kx_t$ is called state feedback control policy
- combine with (‘open-loop dynamics’) $x_{t+1} = Ax_t + Bu_t$ to get closed-loop dynamics

$$x_{t+1} = (A + BK)x_t$$

- we can simulate open- and closed-loop dynamics to compare
Example: longitudinal flight control

Variables are (small) deviations from operating point or trim conditions;

State is $x_t = (w_t, v_t, \theta_t, q_t)$:
- $w_t$: velocity of aircraft along body axis
- $v_t$: velocity of aircraft perpendicular to body axis (down is positive)
- $\theta_t$: angle between body axis and horizontal (up is positive)
- $q_t = \dot{\theta}_t$: angular velocity of aircraft (pitch rate)

Input is $u_t = (e_t, f_t)$:
- $e_t$: elevator angle ($e_t > 0$ is down)
- $f_t$: thrust

Linear quadratic regulator
Linearized dynamics

for 747, level flight, 40000 ft, 774 ft/sec, dynamics are
\[ x_{t+1} = Ax_t + Bu_t, \]
where
\[
A = \begin{bmatrix}
0.99 & 0.03 & -0.02 & -0.32 \\
0.01 & 0.47 & 4.7 & 0.00 \\
0.02 & -0.06 & 0.40 & -0.00 \\
0.01 & -0.04 & 0.72 & 0.99 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0.01 & 0.99 \\
-3.44 & 1.66 \\
-0.83 & 0.44 \\
-0.47 & 0.25 \\
\end{bmatrix}
\]

▶ units: ft, sec, crad (\(= 0.01 \text{rad} \approx 0.57^\circ\))
▶ discretization is 1 sec
gain matrix $K$ converged for $T \approx 30$
LQR for 747 model

LQR gain matrix (for $T = 100$, $\rho = 100$) is:

$$K = \begin{bmatrix}
-0.038 & 0.021 & 0.319 & -0.270 \\
-0.061 & -0.004 & -0.120 & 0.007
\end{bmatrix}$$

e.g., $K_{14} = -0.27$ is gain from pitch rate $(x_t)_4$ to elevator angle $(u_t)_1$
$u_t = 0$ ('open loop')
747 simulation

\[ u_t = K x_t \text{ (‘closed loop’)} \]