Ballistics

Jenny Hong and Stephen Boyd

EE103
Stanford University

September 6, 2021
Outline

Dynamics

Simulations

Targeting

Robust targeting
Position, velocity, and force

- a projectile moves in 2-dimensional space (for simplicity; real ones move in 3-dimensional space)
- sample position and velocity at times $\tau = 0, h, 2h, \ldots$
- 2-vector $p_t$ is position at time $\tau = th$ for $t = 0, 1, \ldots$
- 2-vector $v_t$ is velocity at time $\tau = th$ for $t = 0, 1, \ldots$
- 2-vector $f_t$ is total force acting on projectile at time $\tau = th$
- 4-vector $x_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$ is projectile state at time $\tau = th$
Force model

\[ f_t = mg - \eta(v_t - w) \]

- 2-vector \( g = (0, -9.8) \) is gravity
- 2-vector \( w \) is wind velocity (assumed constant)
- \( v_t - w \) is relative velocity of projectile through air
- \( \eta \in \mathbb{R} \) is drag coefficient
- \( \eta(v_t - w) \) is drag force
- \( m \) is projectile mass
- ‘ballistic’ means the projectile has no other force acting on it (e.g., thrust or propulsion)
Dynamics

- approximating velocity as constant over time interval
  \[ th \leq \tau \leq (t + 1)h, \]
  \[ p_{t+1} = p_t + hv_t \]

- approximating force as constant over the time interval,
  \[ v_{t+1} = v_t + \left( \frac{h}{m} \right) f_t \]
  \[ = (1 - h\eta/m)v_t + (hg + h\eta w/m) \]

- more compactly: \( x_{t+1} = Ax_t + b \), with

\[
A = \begin{bmatrix}
1 & 0 & h & 0 \\
0 & 1 & 0 & h \\
0 & 0 & 1 - h\eta/m & 0 \\
0 & 0 & 0 & 1 - h\eta/m
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
0 \\
hg_1 + h\eta w_1/m \\
hg_2 + h\eta w_2/m
\end{bmatrix}
\]
Propagating state through time

\[ x_1 = Ax_0 + b \]
\[ x_2 = A(Ax_0 + b) + b = A^2x_0 + Ab + b \]
\[ \vdots \]
\[ x_T = A^T x_0 + (A^{T-1} + \cdots + A + I)b \]
Outline

Dynamics

Simulations

Targeting

Robust targeting
Simulation parameters

- Let’s look at some trajectories, with parameters
  
  \[ m = 5, \quad T = 100, \quad h = 0.1, \quad \eta = 0.05, \quad p_0 = 0 \]

- We’ll use various values of initial velocity \( v_0 \), expressed in terms of
  - Initial speed \( \|v_0\| \)
  - Elevation \( \theta = \tan^{-1}((v_0)_2/(v_0)_1) \)

- We’ll vary the wind velocity \( w \) too
Simulation: with and without wind

- initial speed $||v_0|| = 50$, elevation $\theta = 45^\circ$
- no wind: $w = (0, 0)$
- with wind: $w = (-10, 0)$

(all future simulations include wind)
Simulation: varying elevation

- $\|v_0\| = 50$
- $\theta = 30^\circ, 45^\circ, 80^\circ$
Simulation: varying speed

- $\theta = 50$
- $\|v_0\| = 50, 75, 100$
Outline

Dynamics

Simulations

Targeting

Robust targeting
Targeting problem

Given
- initial position $p_0$
- parameters $h, m, w, \eta$
- flight time $Th$
- desired final position ('target') $p_T$

Find initial velocity $v_0$

Please note
- this is not used for socially positive purposes
- but it is one of the first historical applications
Final state

Final state is

$$x_T = A^T x_0 + (A^{T-1} + \cdots + A + I)b$$

$$= F x_0 + j$$

where

$$F = A^T, \quad j = (A^{T-1} + \cdots + A + I)b$$

- 4 × 4 matrix $F$ maps initial state to final state
- 4-vector $j$ is effect of gravity, wind on final state
Final position

- Final position is
  \[ p_T = F_{11}p_0 + F_{12}v_0 + j_1 \]
  
  \((F_{11} \text{ and } F_{12} \text{ are } 2 \times 2 \text{ subblocks of } F)\)

- Write as \( p_T = Cv_0 + d \), where \( C = F_{12}, \ d = F_{11}p_0 + j_1 \)

- Solving for \( v_0 \) we have (assuming \( C = F_{12} \) is invertible)
  \[ v_0 = C^{-1}(p_T - d) \]

  \( \text{(note that } C \text{ and } d \text{ are known)} \)

- Gives formula for choosing \( v_0 \) (hence, \( \|v_0\| \) and \( \theta \))
Outline

Dynamics

Simulations

Targeting

Robust targeting
Robust ballistics

- suppose we have uncertainty in the wind, drag coefficient, . . .

- uncertainty is modeled as $K$ scenarios (particular values of parameters)
  - each scenario has its own $A^{(j)}, b^{(j)}$
  - hence its own $C^{(j)}, d^{(j)}$

- robust targeting: choose a single $v_0$ to minimize mean-square targeting error

\[ \frac{1}{K} \sum_{j=1}^{K} \| C^{(j)} v_0 + d^{(j)} - p_T \| ^2 \]
Sample simulations

various masses, drag coefficients, and wind, with $T = 100$