Ballistics

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Outline

Dynamics

Simulations

Targeting

Robust targeting
Position, velocity, and force

- a projectile moves in 2-dimensional space (for simplicity; real ones move in 3-dimensional space)

- sample position and velocity at times $\tau = 0, h, 2h, \ldots$

- 2-vector $p_t$ is position at time $\tau = th$ for $t = 0, 1, \ldots$

- 2-vector $v_t$ is velocity at time $\tau = th$ for $t = 0, 1, \ldots$

- 2-vector $f_t$ is total force acting on projectile at time $\tau = th$

- 4-vector $x_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$ is projectile state at time $\tau = th$
Force model

\[ f_t = mg - \eta(v_t - w) \]

- 2-vector \( g = (0, -9.8) \) is gravity
- 2-vector \( w \) is wind velocity (assumed constant)
- \( v_t - w \) is relative velocity of projectile through air
- \( \eta \in \mathbb{R} \) is drag coefficient
- \( \eta(v_t - w) \) is drag force
- \( m \) is projectile mass
- ‘ballistic’ means the projectile has no other force acting on it (e.g., thrust or propulsion)
Dynamics

- approximating velocity as constant over time interval
  \[ th \leq \tau \leq (t + 1)h, \]
  \[ p_{t+1} = p_t + hv_t \]

- approximating force as constant over the time interval,
  \[ v_{t+1} = v_t + (h/m)f_t \]
  \[ = (1 - h\eta/m)v_t + (hg + h\eta w/m) \]

- more compactly: \[ x_{t+1} = Ax_t + b, \] with
  \[
  A = \begin{bmatrix}
  1 & 0 & h & 0 \\
  0 & 1 & 0 & h \\
  0 & 0 & 1 - h\eta/m & 0 \\
  0 & 0 & 0 & 1 - h\eta/m
  \end{bmatrix}, \quad b = \begin{bmatrix}
  0 \\
  0 \\
  hg_1 + h\eta w_1/m \\
  hg_2 + h\eta w_2/m
  \end{bmatrix}
  \]
to propagate forward $T$ time steps

\[
\begin{align*}
x_1 &= Ax_0 + b \\
x_2 &= A(Ax_0 + b) + b = A^2x_0 + Ab + b \\
&\vdots \\
x_T &= A^T x_0 + (A^{T-1} + \cdots + A + I)b
\end{align*}
\]
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Simulation parameters

- let’s look at some trajectories, with parameters

  \[ m = 5, \quad T = 100, \quad h = 0.1, \quad \eta = 0.05, \quad p_0 = 0 \]

- we’ll use various values of initial velocity \( v_0 \), expressed in terms of
  - initial speed \( \| v_0 \| \)
  - elevation \( \theta = \tan^{-1}\left(\frac{(v_0)_2}{(v_0)_1}\right) \)

- we’ll vary the wind velocity \( w \) too
Simulation: with and without wind

- initial speed $\|v_0\| = 50$, elevation $\theta = 45^\circ$
- no wind: $w = (0, 0)$
- with wind: $w = (-10, 0)$

(all future simulations include wind)
Simulation: varying elevation

- $\|v_0\| = 50$
- $\theta = 30^\circ, 45^\circ, 80^\circ$
Simulation: varying speed

- $\theta = 50$
- $\|v_0\| = 50, 75, 100$
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Targeting problem

- given
  - initial position $p_0$
  - parameters $h, m, w, \eta$
  - flight time $Th$
  - desired final position ('target') $p_T$
- find initial velocity $v_0$

- please note
  - this is not used for socially positive purposes
  - but it is one of the first historical applications
Final state

final state is

\[ x_T = A^T x_0 + (A^{T-1} + \cdots + A + I)b \]
\[ = F x_0 + j \]

where

\[ F = A^T, \quad j = (A^{T-1} + \cdots + A + I)b \]

- 4 × 4 matrix \( F \) maps initial state to final state
- 4-vector \( j \) is effect of gravity, wind on final state
Final position

- final position is
  \[ p_T = F_{11}p_0 + F_{12}v_0 + j_1 \]
  \((F_{11} \text{ and } F_{12} \text{ are } 2 \times 2 \text{ subblocks of } F)\)

- write as \( p_T = Cv_0 + d \), where \( C = F_{12}, d = F_{11}p_0 + j_1 \)

- solving for \( v_0 \) we have (assuming \( C = F_{12} \) is invertible)
  \[ v_0 = C^{-1}(p_T - d) \]
  (note that \( C \) and \( d \) are known)

- gives formula for choosing \( v_0 \) (hence, \( \|v_0\| \) and \( \theta \))
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Robust ballistics

- Suppose we have uncertainty in the wind, drag coefficient, ...

- Uncertainty is modeled as $K$ scenarios (particular values of parameters)
  - Each scenario has its own $A^{(j)}, b^{(j)}$
  - Hence its own $C^{(j)}, d^{(j)}$

- Robust targetting: Choose a single $v_0$ to minimize mean-square targetting error

$$\frac{1}{K} \sum_{j=1}^{K} \| C^{(j)} v_0 + d^{(j)} - p_T \|^2$$
Sample simulations

various masses, drag coefficients, and wind, with $T = 100$