Engr108: Introduction to Matrix Methods J. Duchi
June 2023

Midterm Examination

This is a 90 minute in-class midterm examination.

You may use one piece of paper (double-sided), but otherwise this examination is closed book. While you may ask us questions if you find a question confusing, we've tried pretty hard to make the exam unambiguous and clear, so we're unlikely to say much.

Please respect the honor code.

All problems have equal weight. Some are (quite) straightforward. Others, not so much.

Some problems involve applications. But you do not need to know *anything* about the problem area to solve the problem; the problem statement contains everything you need.

The problems do *not* appear in order of increasing difficulty.

Name: ____

Stanford ID #: ____

M.1 Let x and y be n-vectors. Under each of the following conditions, identify whether x and y either (i) form an acute angle ($< 90°$), (ii) form an obtuse angle ($> 90°$), or (iii) are orthogonal (at a $90°$ angle). (You may simply write one of "acute", "obtuse", or "orthogonal".)

(a) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

(b) $\|x + y\|^2 < \|x\|^2 + \|y\|^2$

(c) $\|x + y\|^2 > \|x\|^2 + \|y\|^2$

M.2 In k means, we maintain cluster representatives $\{z_1, \dots, z_k\}$, each an n-vector, and at each iteration find the cluster representative z_i nearest a point x. Suppose that $k = 2$ and that we have cluster representatives $z_1, z_2 \in \mathbf{R}^n$, where $\|z_1\| = \|z_2\|$. (Note: it is important for *both* parts below that $\|z_1\| = \|z_2\|$.)

(a) Show that a point x is assigned to cluster 1, that is, $\|x - z_1\| < \|x - z_2\|$, if and only if $(z_1 - z_2)^T x > 0$.

(b) In the case $n = 2$, draw a picture of your result. Your picture should contain the points z_1 and z_2, the vector $z_1 - z_2$, and a line such that any point x falling on one side of the line will be assigned to cluster 1, and any point x falling on the other side of the line will be assigned to cluster 2. Label the sides of the line.

M.3 Suppose the n-vectors $u_1, \ldots, u_k$ are orthonormal, and let x be another n-vector. You observe that

$$(u_1^T x)^2 + (u_2^T x)^2 + \cdots + (u_k^T x)^2 = \|x\|^2 .$$

What can you conclude about the vector x? Justify your answer.

M.4 Let x be an n-vector and $u_1, \ldots, u_k$ be orthonormal n-vectors. The *projection of x onto the span of* $u_1, \ldots, u_k$ is

$$\widehat{x} = (u_1^T x)u_1 + \cdots + (u_k^T x)u_k \in \mathbf{R}^n.$$

Let u be an n-vector with norm $\|u\| = 1$, orthogonal to $u_1, \ldots, u_k$, and

$$\widetilde{x} = \widehat{x} + (u^T x)u = (u_1^T x)u_1 + \cdots + (u_k^T x)u_k + (u^T x)u$$

be the projection of x onto the span of $u_1, \ldots, u_k, u$. Show $\widetilde{x}$ is closer to x than $\widehat{x}$ in that

$$\|\widetilde{x} - x\|^2 \leq \|\widehat{x} - x\|^2.$$

M.5 Let M be the square block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $B \in \mathbf{R}^{m \times n}$ and $C \in \mathbf{R}^{n \times m}$. Let x be an m-vector and y be an n-vector, and $z = (x, y)$.

(a) What dimensions does the matrix A have?

(b) What dimensions does the matrix D have?

(c) Write the matrix-vector product Mz, that is,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

as a single vector. Your formula should involve all of A, B, C, D, x, and y.

(d) If M is symmetric, is $A = A^T$?

(e) If M is symmetric, give a formula for C in terms of B, or explain why you cannot.