Eigenimages

- Unitary transforms
- Karhunen-Loève transform and eigenimages
- Sirovich and Kirby method
- Eigenfaces for gender recognition
- Fisher linear discriminant analysis
- Fisherimages and varying illumination
- Fisherfaces vs. eigenfaces

Unitary transforms

- Sort pixels $f[x,y]$ of an image into column vector $\vec{f}$ of length N
- Calculate N transform coefficients

$$\vec{c} = A\vec{f}$$

where $\boldsymbol{A}$ is a matrix of size N**x**N

- The transform A is unitary, iff

$$A^{-1} = \underbrace{A^{*T} \equiv A^{H}}_{\text{Hermitian conjugate}}$$

- If A is real-valued, i.e., $A=A^*$, transform is „orthonormal“

Energy conservation with unitary transforms

- For any unitary transform $\vec{c} = A\vec{f}$ we obtain

$$\left\|\vec{c}\right\|^2 = \vec{c}^H \vec{c} = \vec{f}^H A^H A\vec{f} = \left\|\vec{f}\right\|^2$$

- Interpretation: every unitary transform is simply a rotation of the coordinate system (and, possibly, sign flips)
- Vector length is conserved.
- Energy (mean squared vector length) is conserved.

Energy distribution for unitary transforms

- Energy is conserved, but, in general, unevenly distributed among coefficients.
- Autocorrelation matrix

$$R_{cc} = E\left[\vec{c}\vec{c}^H\right] = E\left[A\vec{f} \cdot \vec{f}^H A^H\right] = AR_{ff}A^H$$

- Diagonal of R_{cc} comprises mean squared values („energies“) of the coefficients c_i

$$E\left[c_i^2\right] = \left[R_{cc}\right]_{i,i} = \left[AR_{ff}A^H\right]_{i,i}$$

- for now: assume R_{ff} is known or can be computed

Eigenmatrix of the autocorrelation matrix

Definition: eigenmatrix $\mathbf{\Phi}$ of autocorrelation matrix R_{ff}

- $\mathbf{\Phi}$ is unitary
- The columns of $\mathbf{\Phi}$ form a set of eigenvectors of R_{ff}, i.e.,

$$R_{ff}\Phi = \Phi\Lambda$$

← Λ is a diagonal matrix of eigenvalues λ_i

$$\Lambda = \begin{pmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_{N-1} \end{pmatrix}$$

- unitary eigenmatrix for auto-correlation matrix always exists
- R_{ff} is symmetric positive (semi-)definite, hence $\lambda_i \geq 0 \quad \text{for all } i$

Karhunen-Loève transform

- Unitary transform with matrix

$$A = \Phi^H$$

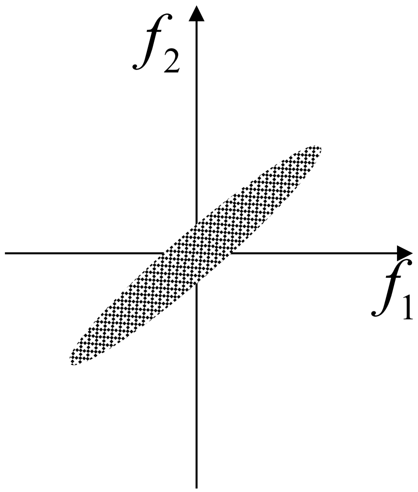
- Transform coefficients are pairwise uncorrelated

$$R_{cc} = A R_{ff} A^H = \Phi^H R_{ff} \Phi = \Phi^H \Phi \Lambda = \Lambda$$

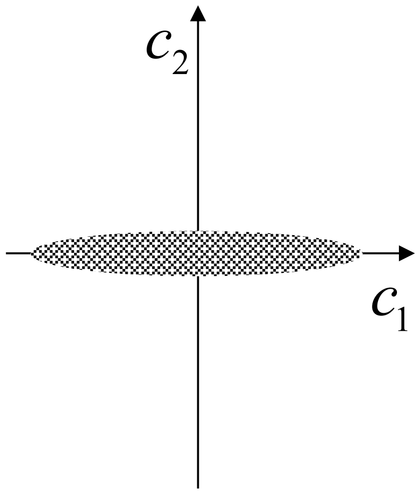
- Columns of Φ are ordered according to decreasing eigenvalues.
- Energy concentration property:
  - No other unitary transform packs as much energy into the first J coefficients.
  - Mean squared approximation error by keeping only first J coefficients is minimized.
  - Holds for any J.

Illustration of energy concentration

Strongly correlated samples,
equal energies



$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



<u>After KLT:</u>
uncorrelated samples,
most of the energy in
first coefficient

Basis images and eigenimages

- For any transform, the inverse transform

$$\vec{f} = A^{-1}\vec{c}$$

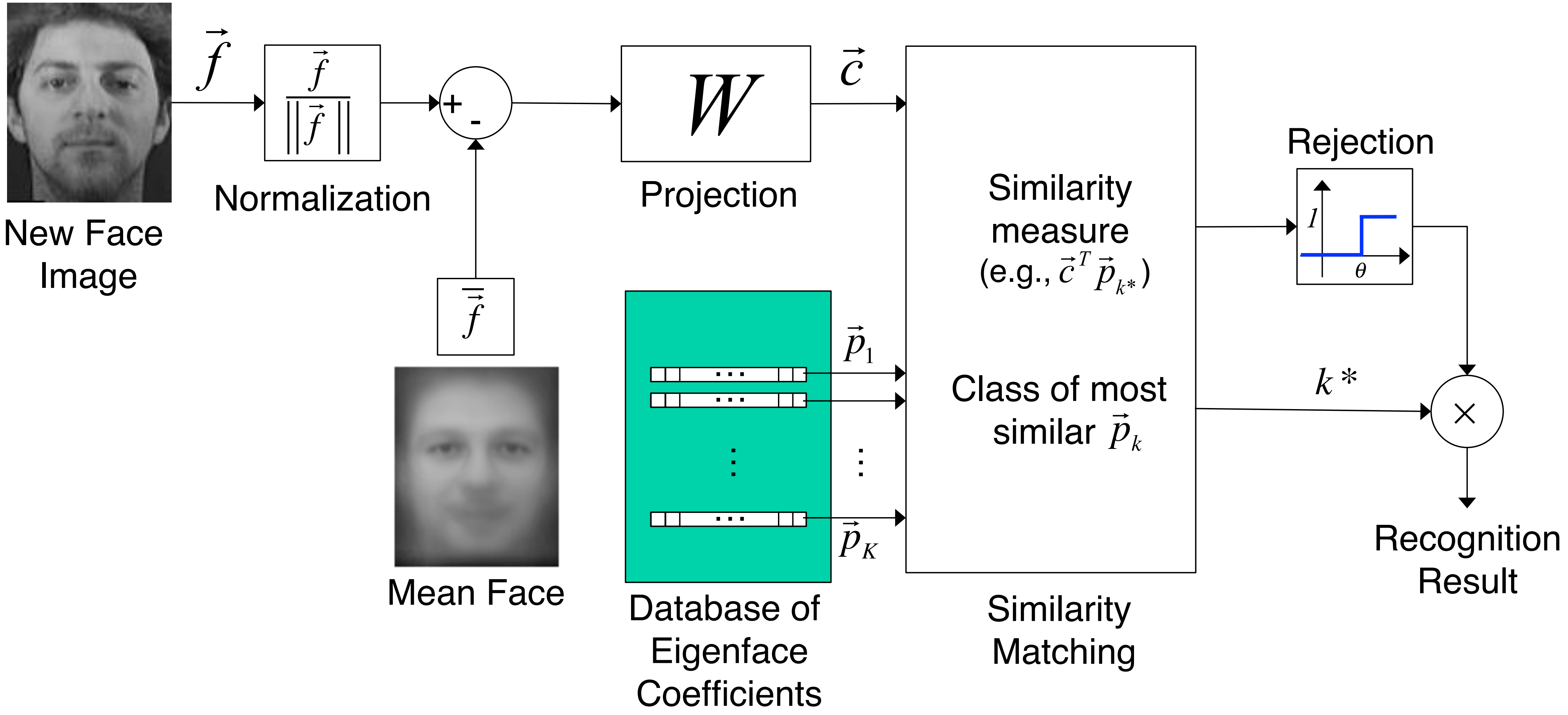
  can be interpreted in terms of the superposition of columns of A^{-1} („basis images“)
- For the KL transform, the basis images are the eigenvectors of the autocorrelation matrix R_{ff} and are called „eigenimages.“
- If energy concentration works well, only a limited number of eigenimages is needed to approximate a set of images with small error. These eigenimages span an optimal linear subspace of dimensionality J.

Eigenimages for recognition

- To recognize complex patterns (e.g., faces), large portions of an image have to be considered
- High dimensionality of "image space" means high computational burden for many recognition techniques

  Example: nearest-neigbor search requires pairwise comparison with every image in a database
- Transform $\vec{c} = W\vec{f}$ can reduce dimensionality from N to J by representing the image by J coefficients
- Idea: tailor a KLT to a specific set of training images
  representative of the recognition task
  to preserve the salient features

Eigenimages for recognition



Computing eigenimages from a training set

- How to obtain NxN covariance matrix?
  - Use training set $\vec{\Gamma}_1, \vec{\Gamma}_2, \ldots, \vec{\Gamma}_{L+1}$ (each column vector represents one image)
  - Let $\vec{\mu}$ be the mean image of all $L+1$ training images
  - Define training set matrix $S = \left(\vec{\Gamma}_1 - \vec{\mu}, \vec{\Gamma}_2 - \vec{\mu}, \vec{\Gamma}_3 - \vec{\mu}, \ldots, \vec{\Gamma}_L - \vec{\mu} \right)$,

  and calculate scatter matrix $R = \sum_{l=1}^{L} \left(\vec{\Gamma}_l - \vec{\mu} \right) \left(\vec{\Gamma}_l - \vec{\mu} \right)^H = SS^H$

  Problem 1: Training set size should be $L+1 >> N$

  If $L < N$, scatter matrix R is rank-deficient

  Problem 2: Finding eigenvectors of an NxN matrix.

- Can we find a small set of the most important eigenimages from a small training set $L << N$?

Sirovich and Kirby algorithm

- Instead of eigenvectors of SS^H, consider the eigenvectors of $S^H S$, i.e.,

$$S^H S\vec{v}_i = \lambda_i \vec{v}_i$$

- Premultiply both sides by S

$$SS^H S\vec{v}_i = \lambda_i S\vec{v}_i$$

- By inspection, we find that $S\vec{v}_i$ are eigenvectors of SS^H

Sirovich and Kirby Algorithm (for $L << N$)

- Compute the LxL matrix $S^H S$
- Compute L eigenvectors $\vec{v}_i$ of $S^H S$
- Compute eigenimages corresponding to the $L_0 \leq L$ largest eigenvalues as a linear combination of training images $S\vec{v}_i$

L. Sirovich and M. Kirby, "Low-dimensional procedure for the characterization of human faces," Journal of the Optical Society of America A, 4(3), pp. 519-524, 1987.

Example: eigenfaces

- The first 8 eigenfaces obtained from a training set of 100 male and 100 female training images

Mean Face

Eigenface 1

Eigenface 2

Eigenface 3

Eigenface 4

Eigenface 5

Eigenface 6

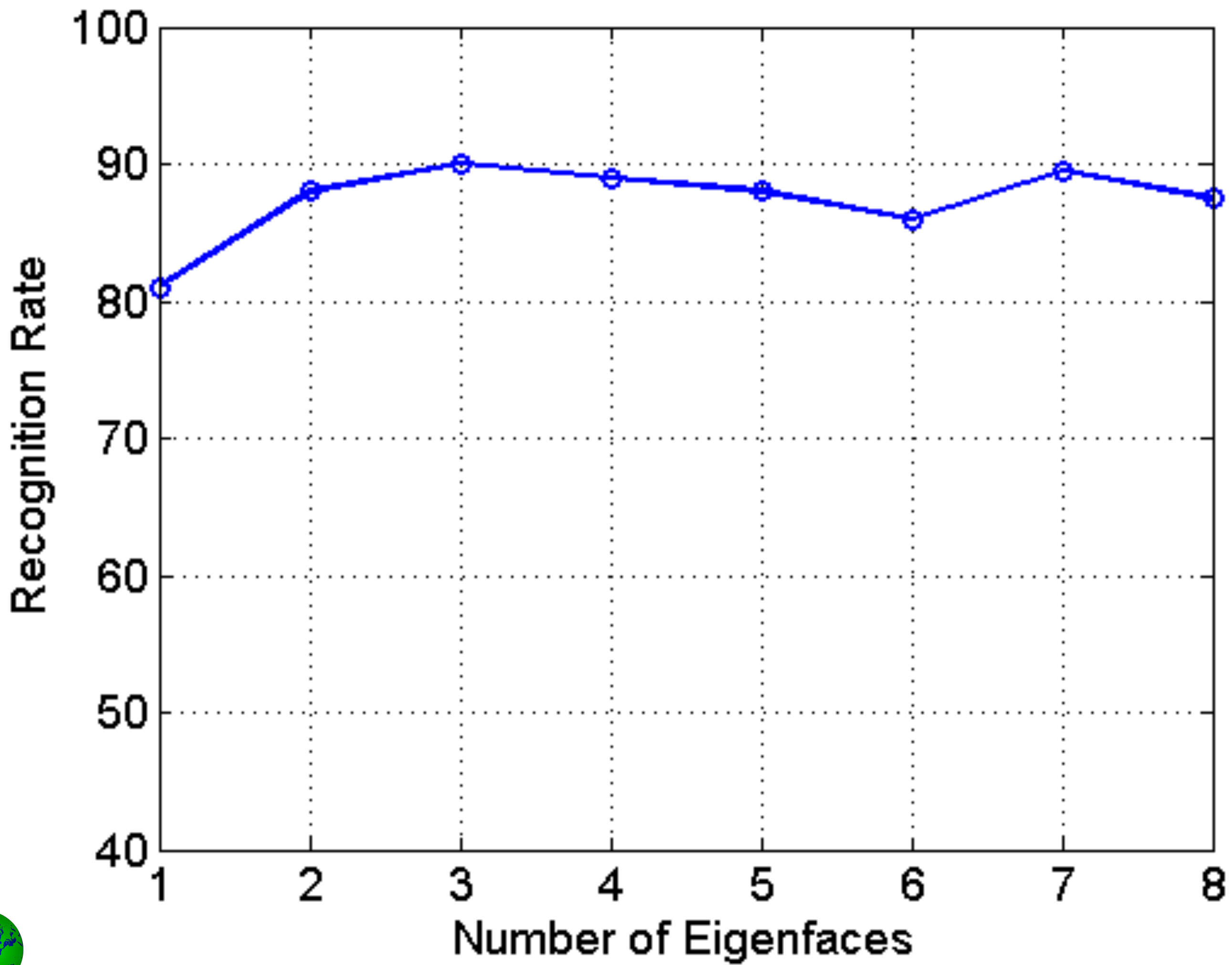
Eigenface 7

Eigenface 8

- Can be used to generate faces by adjusting 8 coefficients.
- Can be used for face recognition by nearest-neighbor search in 8-d „face space.“

Gender recognition using eigenfaces

Nearest neighbor search "face space"



Female face samples

Male face samples

Fisher linear discriminant analysis

- Eigenimage method maximizes "scatter" within the linear subspace over the entire image set – regardless of classification task

$$W_{opt} = \arg\max_{W} \left(\det\left(W R W^{H} \right) \right)$$

- Fisher linear discriminant analysis (1936): maximize between-class scatter, while minimizing within-class scatter

$$W_{opt} = \arg\max_{W} \left(\frac{\det\left(W R_B W^{H} \right)}{\det\left(W R_W W^{H} \right)} \right)$$

$$R_B = \sum_{i=1}^{c} N_i \left(\vec{\mu}_i - \vec{\mu} \right) \left(\vec{\mu}_i - \vec{\mu} \right)^{H}$$

Samples in class i

Mean in class i

$$R_W = \sum_{i=1}^{c} \sum_{\vec{\Gamma}_l \in Class(i)} \left(\vec{\Gamma}_l - \vec{\mu}_i \right) \left(\vec{\Gamma}_l - \vec{\mu}_i \right)^{H}$$

Fisher linear discriminant analysis (cont.)

- Solution: Generalized eigenvectors $\overrightarrow{w_i}$ corresponding to the J largest eigenvalues $\{\lambda_i \mid i = 1,2,...,J\}$, i.e.

$$R_B \overrightarrow{w_i} = \lambda_i R_W \overrightarrow{w_i} \ , \quad i = 1,2,...,J$$

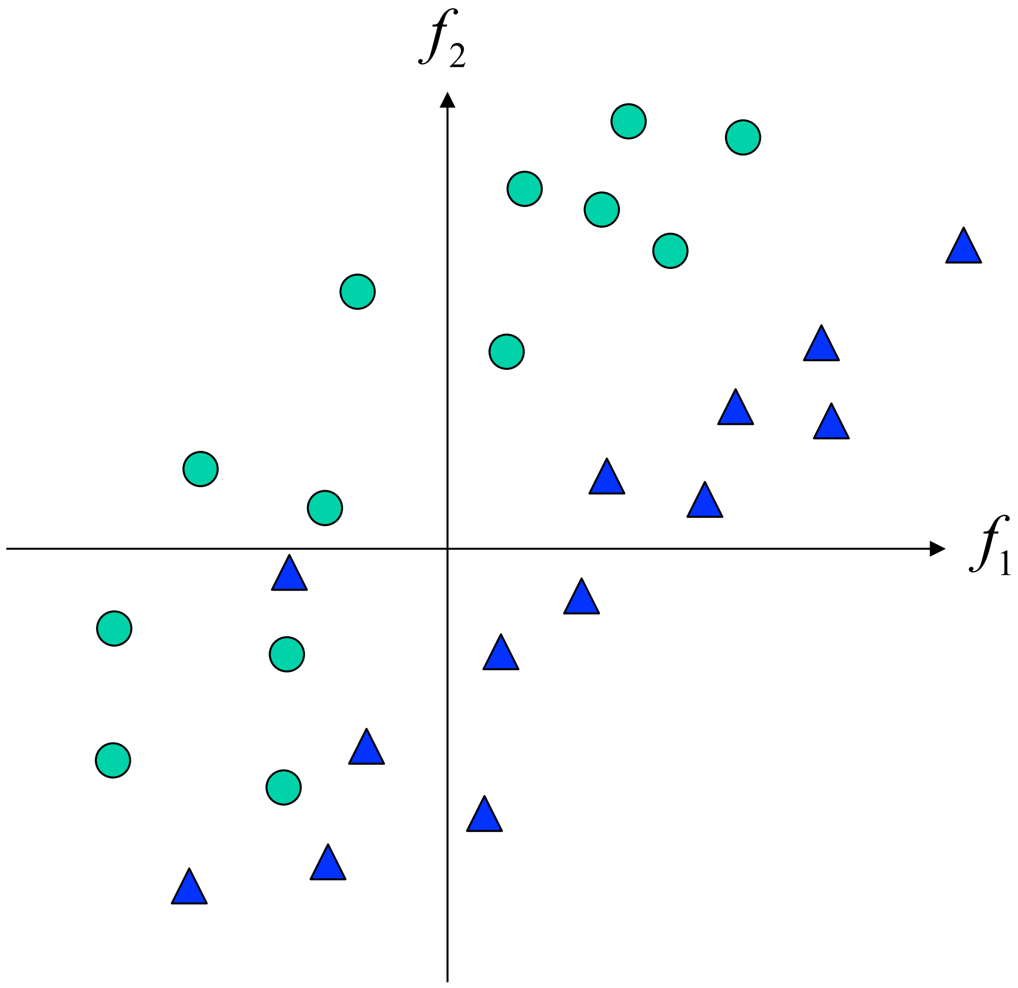
- → solve eigen-problem on this: $$\left(R_W^{-1} R_B\right) \overrightarrow{w_i} = \lambda_i \overrightarrow{w_i} \ , \quad i = 1,2,...,J$$

- Problem: within-class scatter matrix R_w at most of rank L-1 (for L images total in all classes combined), hence usually singular.
- Apply KLT first to reduce dimensionality of feature space to L-1 (or less), proceed with Fisher LDA in lower-dimensional space

Eigenimages vs. Fisherimages

2-d example:

Goal: project samples on a 1-d subspace, then perform classification.

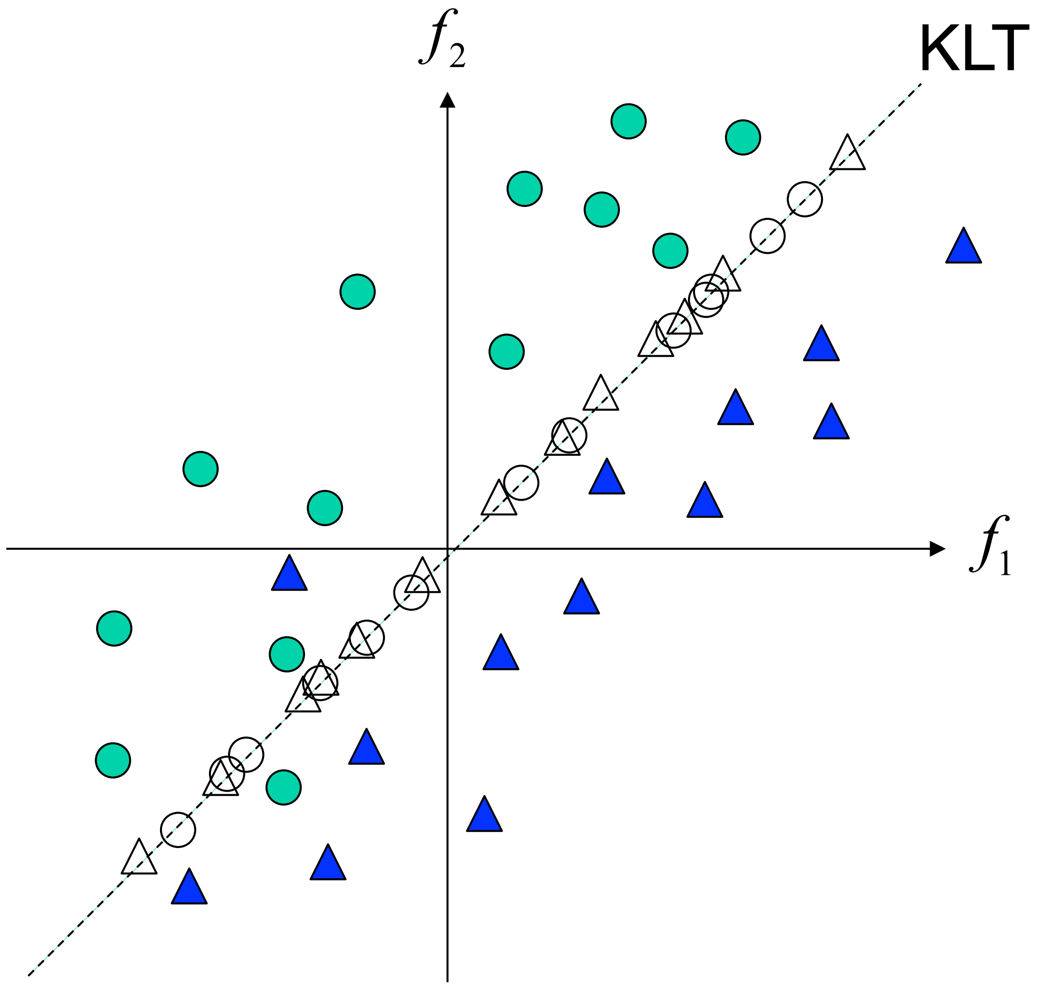


Eigenimages vs. Fisherimages

<u>2-d example:</u>

Goal: project samples on a 1-d subspace, then perform classification.

The KLT preserves maximum energy, but the 2 classes are no longer distinguishable.



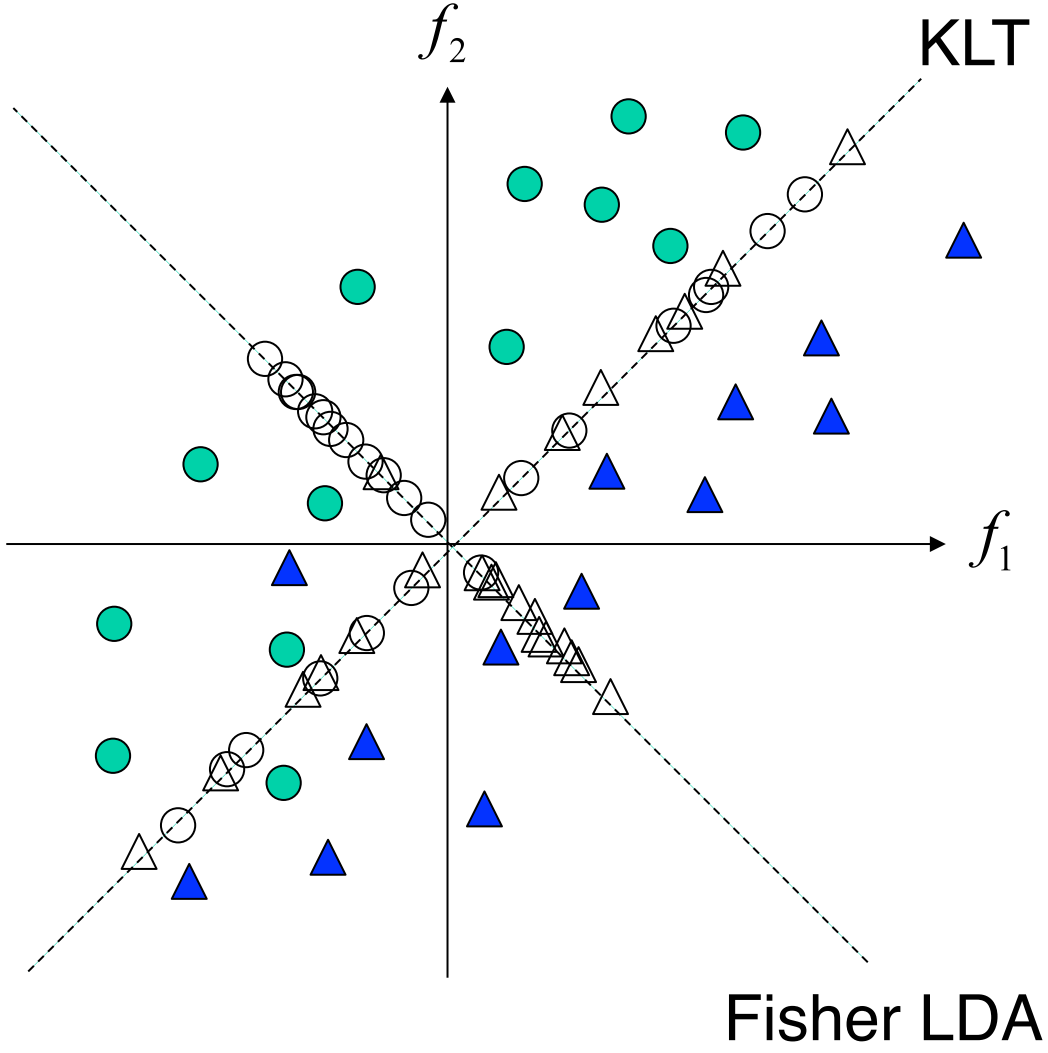
Eigenimages vs. Fisherimages

2-d example:

Goal: project samples on a 1-d subspace, then perform classification.

The KLT preserves maximum energy, but the 2 classes are no longer distinguishable.

Fisher LDA separates the classes by choosing a better 1-d subspace.

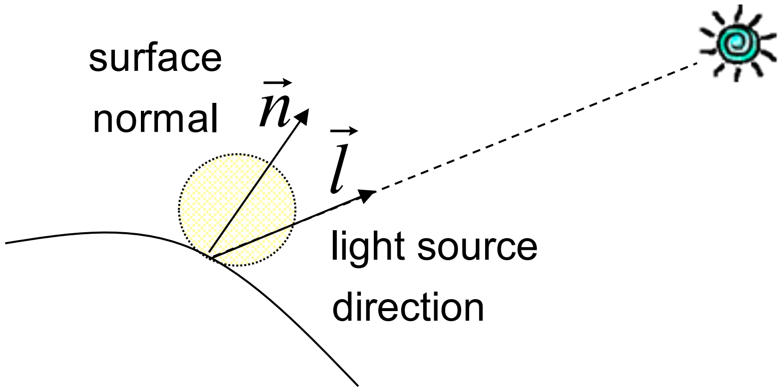


Fisherimages and varying illlumination

Differences due to varying illumination can be much larger than differences among faces!

Fisherimages and varying illlumination

- All images of same Lambertian surface with different illumination (without shadows) lie in a 3d linear subspace
- Single point source at infinity



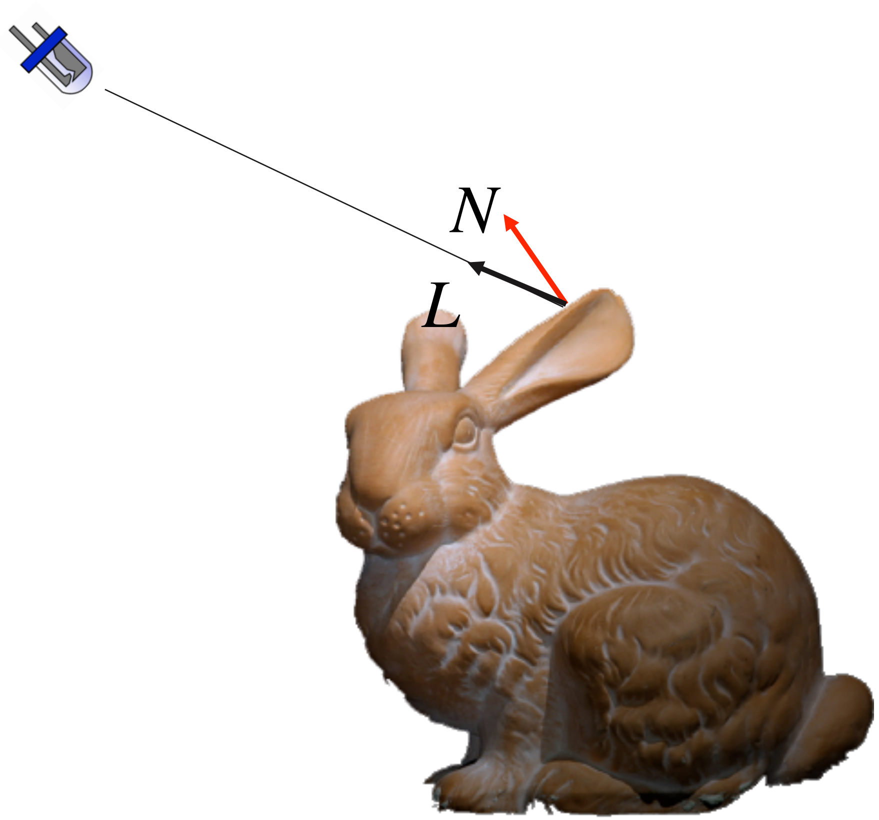
Light source intensity

$$f(x,y) = a(x,y)\left(\vec{l}^{\,T}\vec{n}(x,y)\right)L$$

Surface albedo

- Superposition of arbitrary number of point sources at infinity still in same 3d linear subspace, due to linear superposition of each contribution to image
- Fisherimages can eliminate within-class scatter

Side Note: Photometric Stereo



observed intensity

normalized lighting direction

$$I = \rho L \cdot N$$

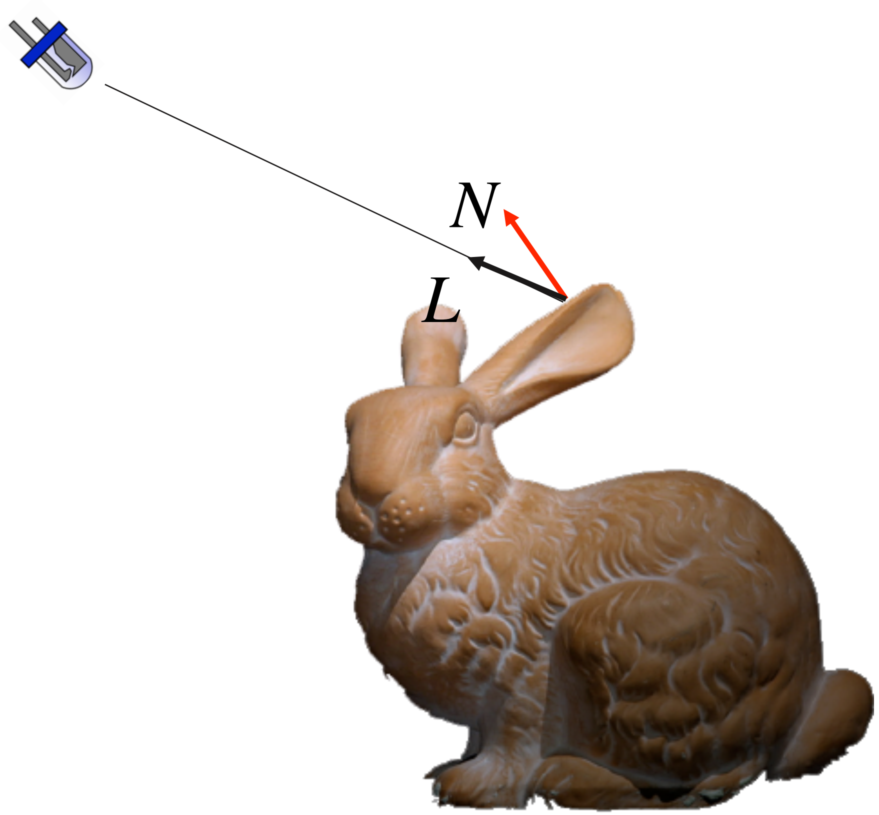
albedo (constant)

normalized surface normal

- diffuse (Lambertian) surfaces are viewpoint independent

[Woodham 1980]

Side Note: Photometric Stereo

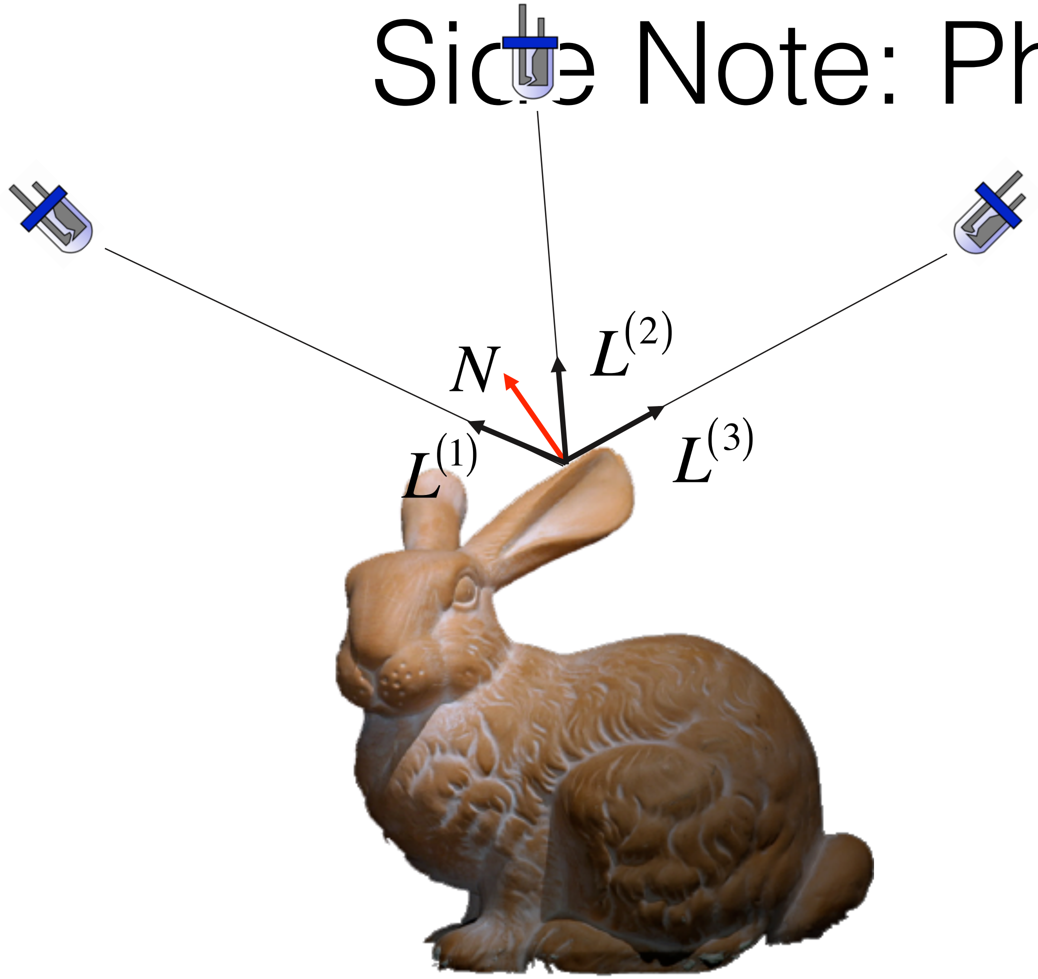


$$I = \rho \begin{bmatrix} L_x & L_y & L_z \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}$$

- diffuse (Lambertian) surfaces are viewpoint independent

[Woodham 1980]

Side Note: Photometric Stereo



$$\begin{bmatrix} I^{(1)} \\ I^{(2)} \\ I^{(3)} \end{bmatrix} = \rho \begin{bmatrix} L_x^{(1)} & L_y^{(1)} & L_z^{(1)} \\ L_x^{(2)} & L_y^{(2)} & L_z^{(2)} \\ L_x^{(3)} & L_y^{(3)} & L_z^{(3)} \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}$$

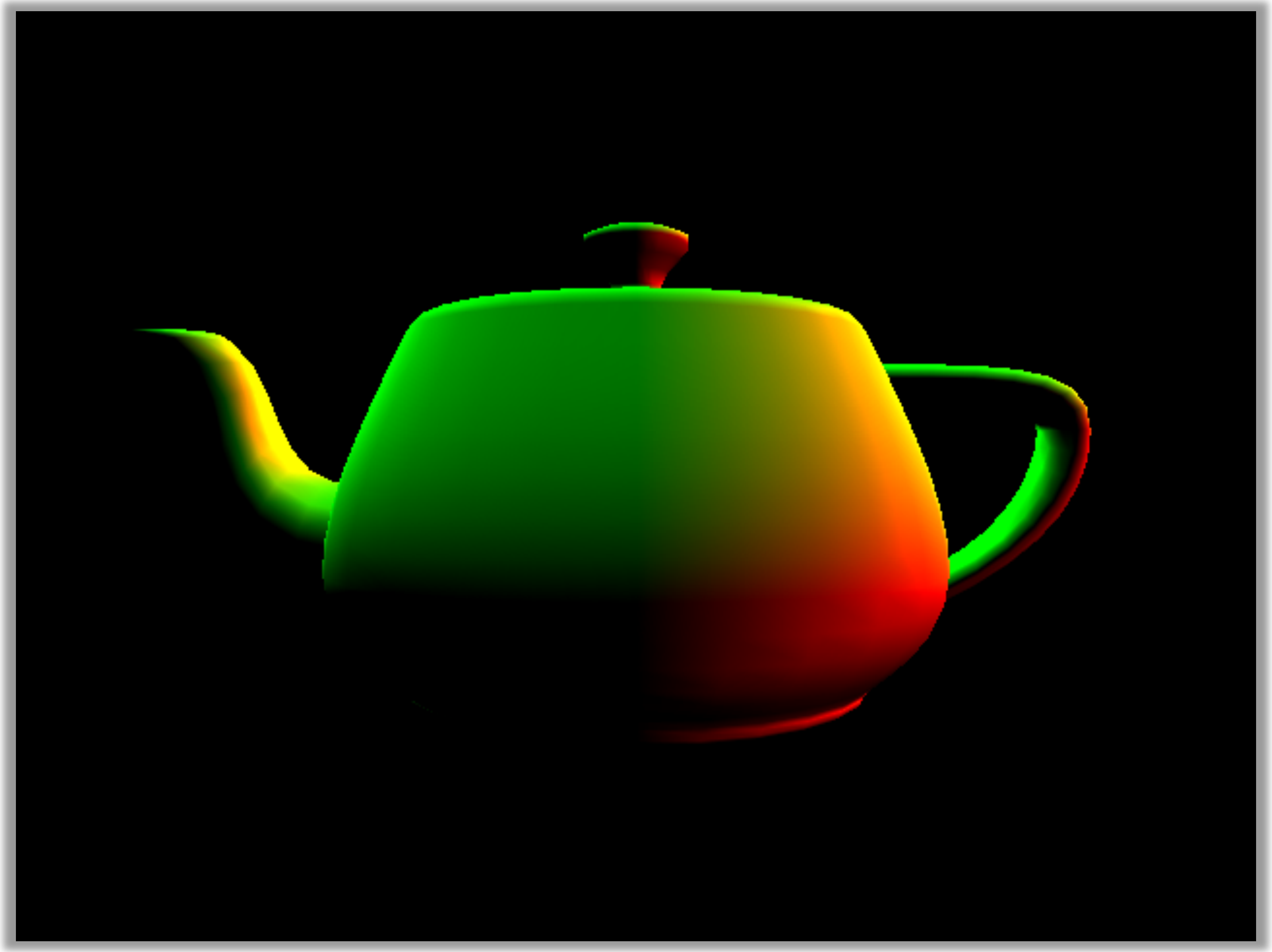
$= I$ $\qquad$ $= L$

- diffuse (Lambertian) surfaces are viewpoint independent
- assume albedo is constant, invert matrix $\quad N = L^{-1} I$

[Woodham 1980]

Side Note: Photometric Stereo

input



output: recovered normals

[Woodham 1980]

Fisherface trained to recognize gender

Female face samples

Male face samples

Mean image

$\vec{\mu}$

Female mean

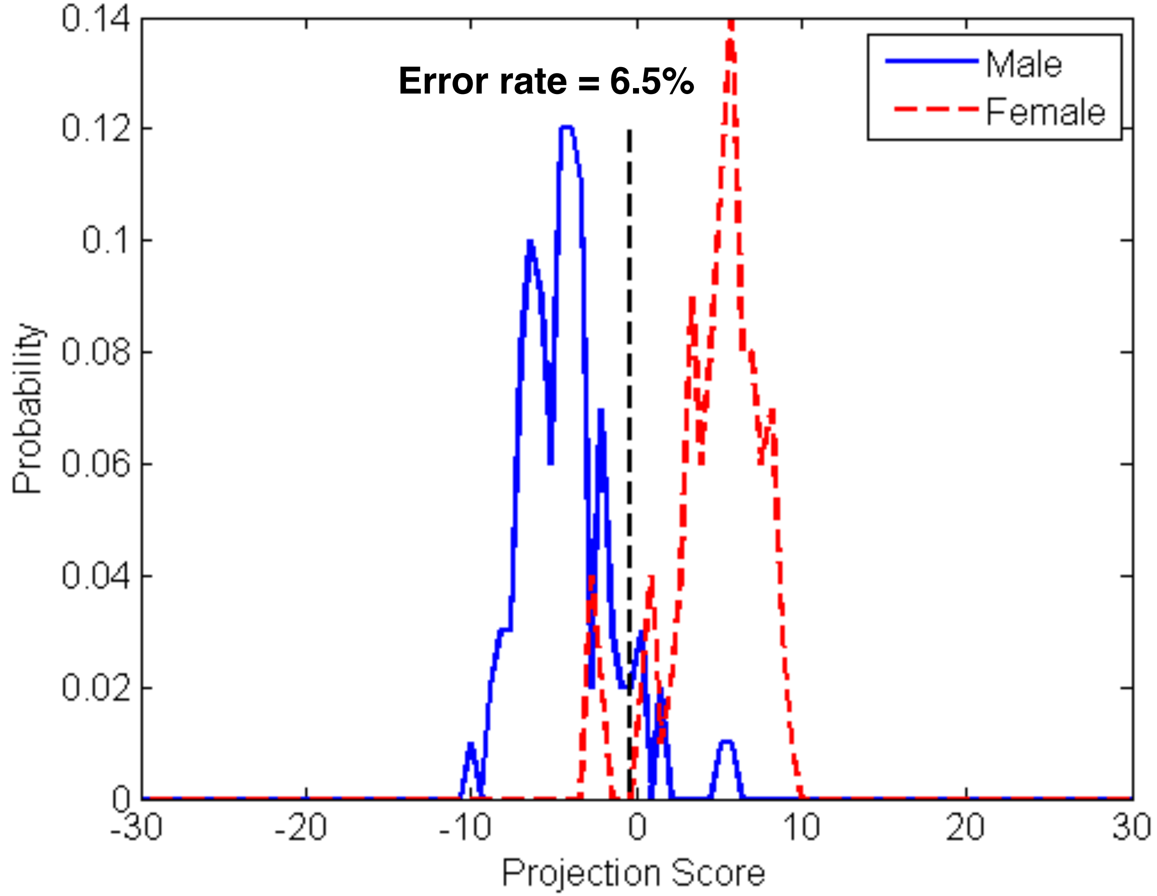
$\vec{\mu}_1$

Male mean

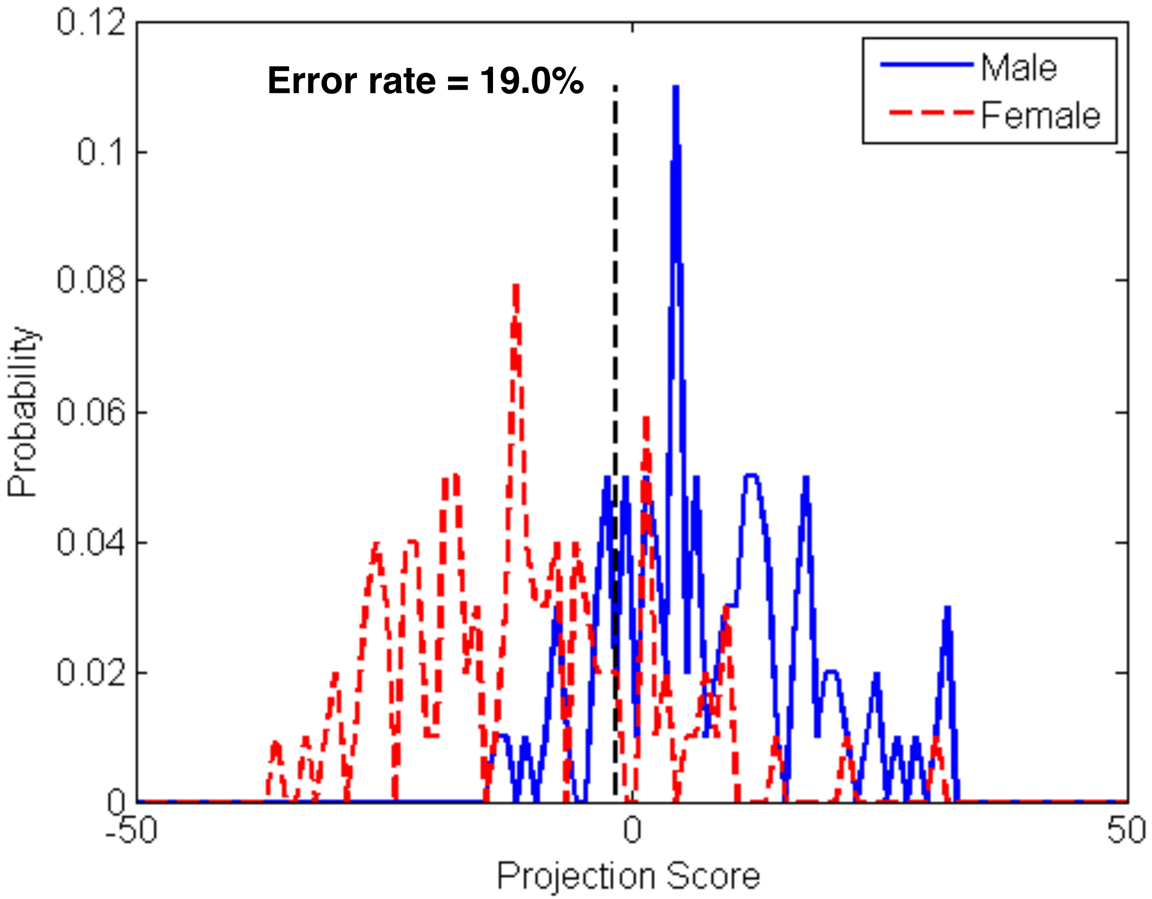
$\vec{\mu}_2$

Fisherface

Gender recognition using 1st Fisherface



Gender recognition using 1st eigenface



Person identification with Fisherfaces and eigenfaces