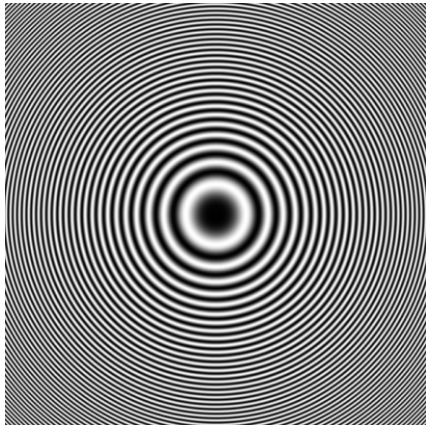
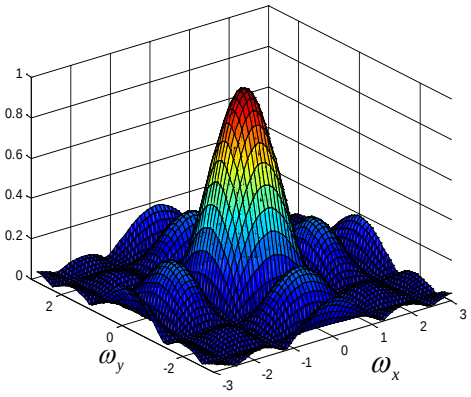
Linear Image Processing and Filtering

- Math of 2-d linear systems
- Separability
- Shift-invariant linear systems, 2-d convolution
- Integral image for fast box-filtering
- 2-d frequency response
- Blurring and sharpening
- Zoneplate
- Image sampling/aliasing
- Wiener filtering
- Nonlinear noise reduction/sharpening





Linear image processing

- Image processing system $S(.)$ is linear, iff superposition principle holds:

$$S\left(\alpha \cdot f\left[x,y\right]+\beta \cdot g\left[x,y\right]\right)=\alpha \cdot S\left(f\left[x,y\right]\right)+\beta \cdot S\left(g\left[x,y\right]\right) \quad \text{for all } \alpha,\beta \in \square$$

- Any linear image processing system can be written as

$$\vec{g}=H\vec{f}$$

Note: matrix H need not be square.

by sorting pixels into a column vector

$$\vec{f}=\left(\begin{array}{cccccccccccc} f\left[0,0\right] & f\left[1,0\right] & \cdots & f\left[N-1,0\right] & f\left[0,1\right] & \cdots & f\left[N-1,1\right] & \cdots & \cdots & f\left[0,L-1\right] & \cdots & f\left[N-1,L-1\right] \end{array} \right)^{T}$$

Impulse response

- Another way to represent any linear image processing scheme

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h[x,\alpha,y,\beta]$$

impulse response

- Input: unit impulse at pixel $[a,b]$

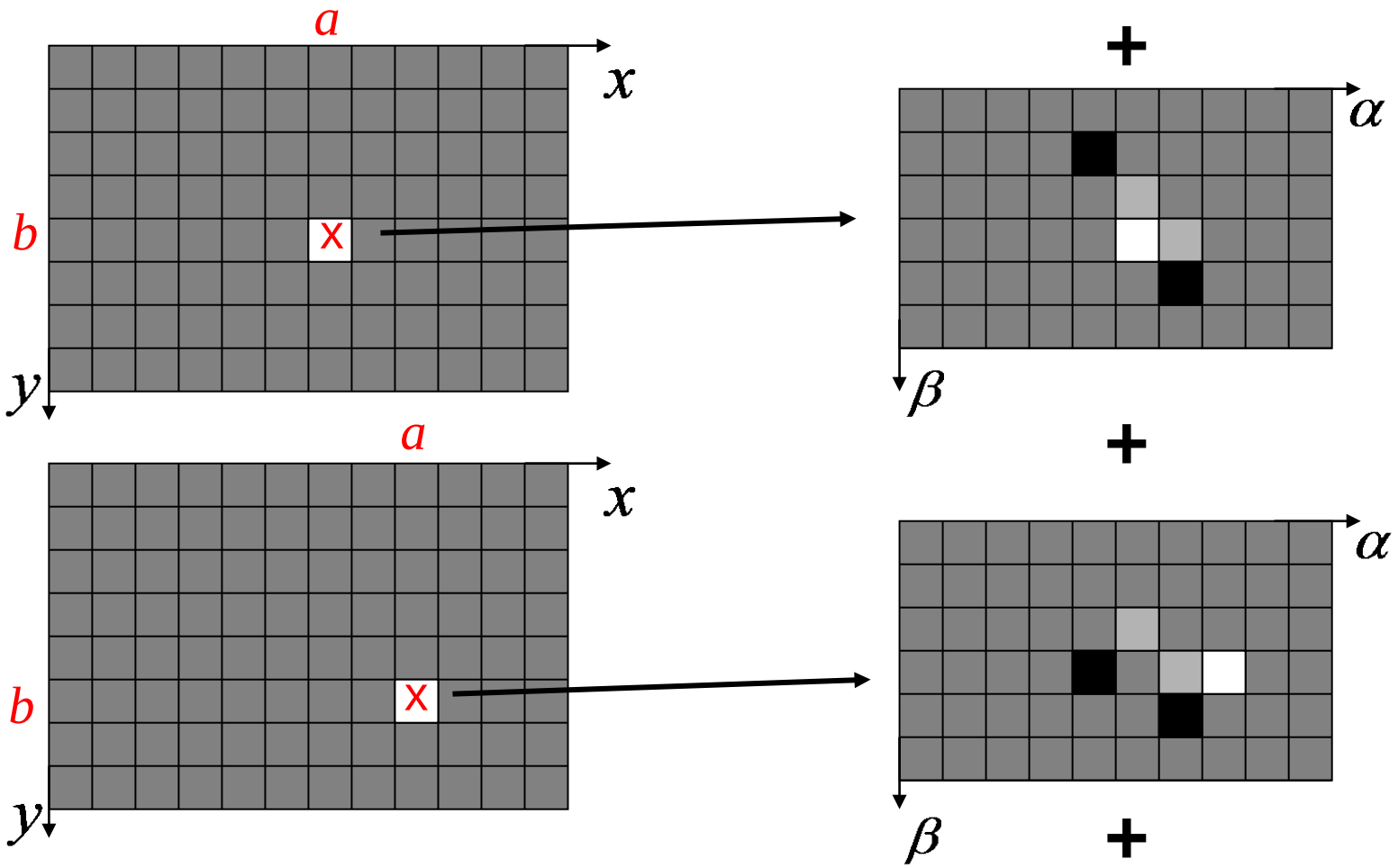
$$f[x,y]=\delta[x-a,y-b]=\begin{cases}1 & x=a\wedge y=b\\ 0 & else\end{cases}$$

- Output: impulse response

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1}\delta[x-a,y-b]\cdot h[x,\alpha,y,\beta]=h[a,\alpha,b,\beta]$$

Interpretation #1: superposition of impulse responses

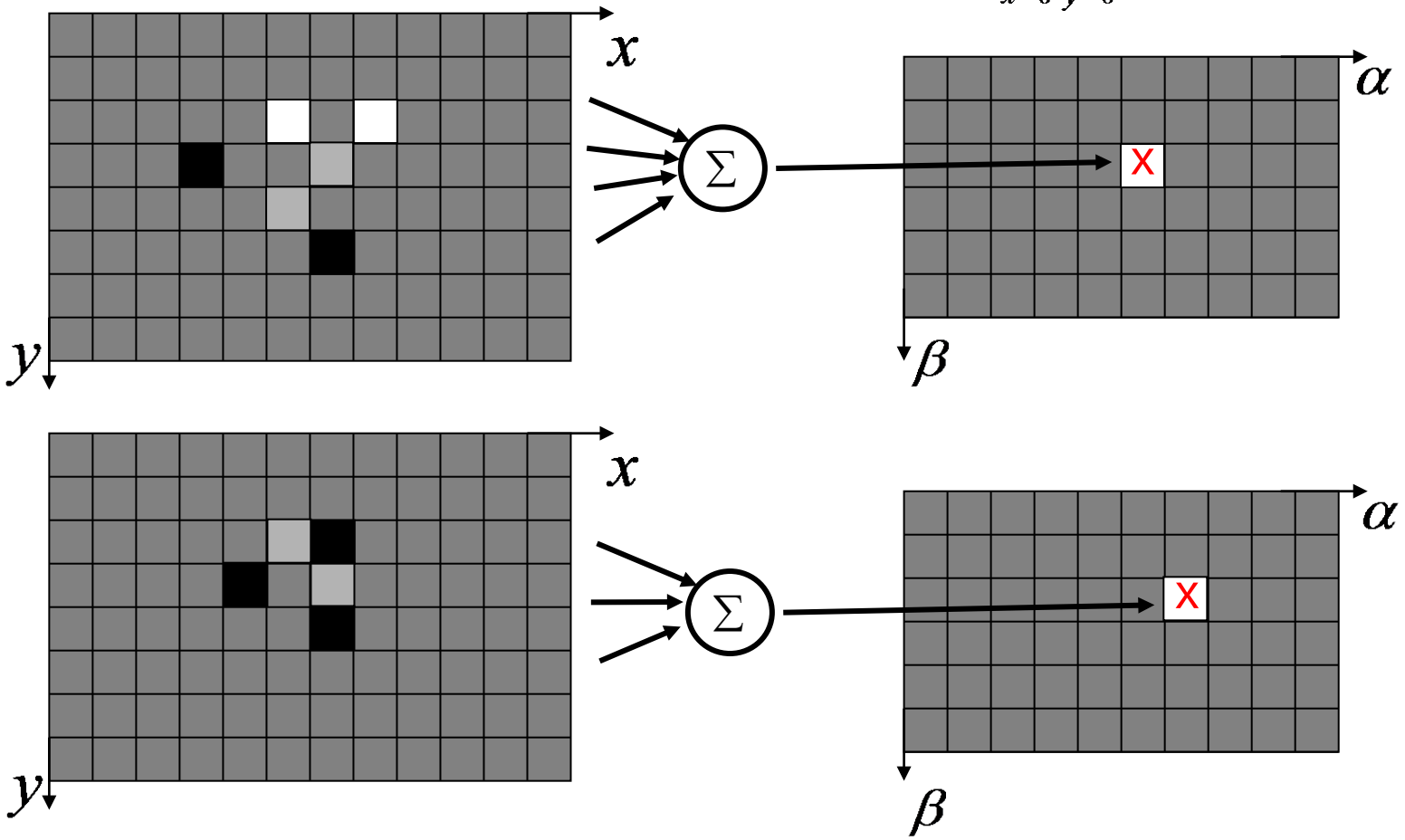
$$\delta[x-a,y-b]f[a,b] \qquad g[\alpha,\beta]=h[a,\alpha,b,\beta]f[a,b]$$



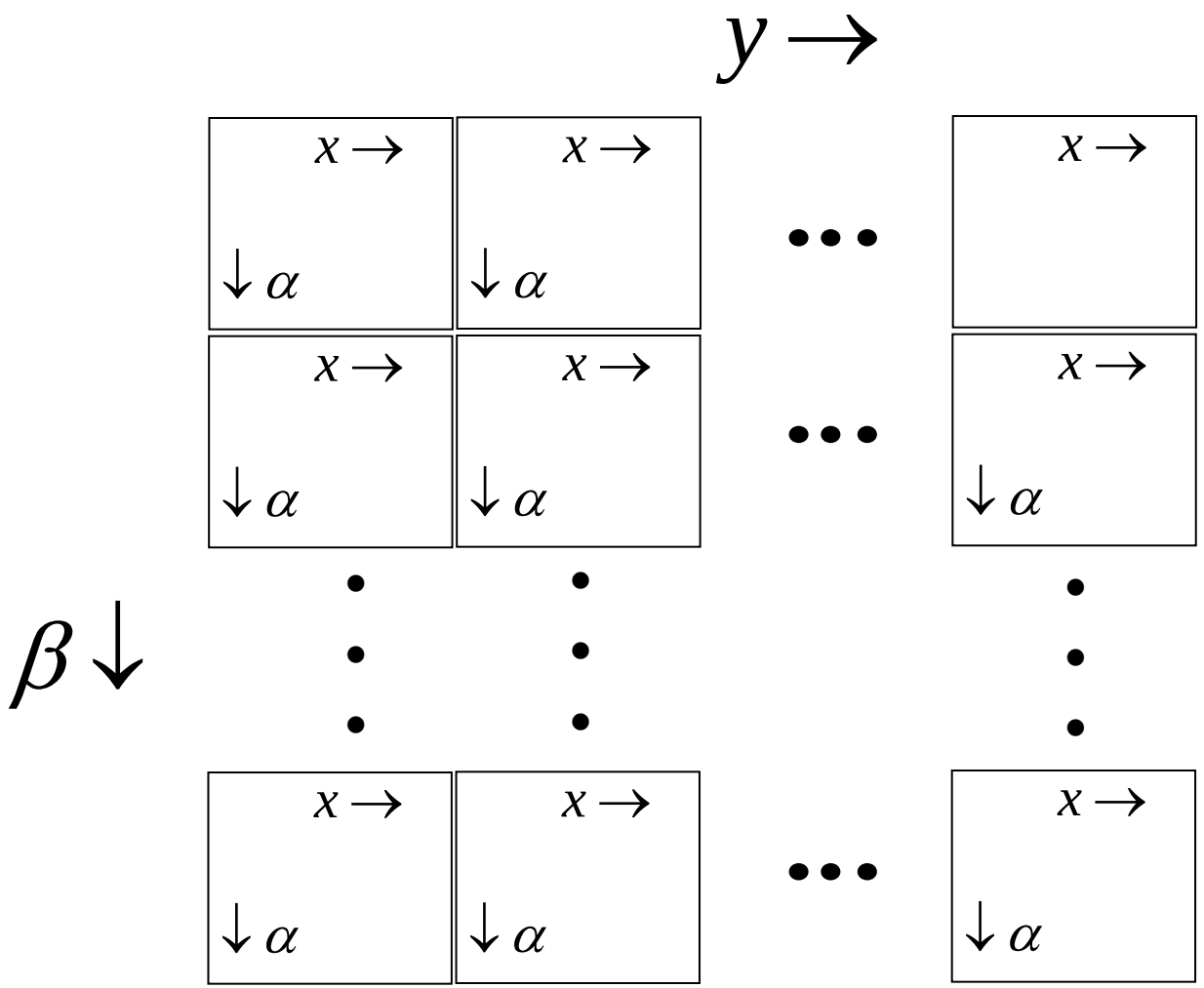
Interpretation #2: linear combination of input values

$$f[x,y]\cdot h[x,\alpha,y,\beta]$$

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h[x,\alpha,y,\beta]$$



Relationship of H and $h[x,\langle,y,®]$



Separable linear image processing

- Impulse response is separable in $[x,\alpha]$ and $[y,\beta]$, i.e., can be written as

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_x[x,\alpha]h_y[y,\beta]$$

- Processing can be carried out
  - row by row, then column by column

$$g[\alpha,\beta]=\sum_{y=0}^{L-1} h_y[y,\beta]\sum_{x=0}^{N-1} f[x,y]\cdot h_x[x,\alpha]$$

  - column by column, then row by row

$$g[\alpha,\beta]=\sum_{x=0}^{N-1} h_x[x,\alpha]\sum_{y=0}^{L-1} f[x,y]\cdot h_y[y,\beta]$$

Separable linear image processing (cont.)

- If the digital input and output images are written as a matrices $\mathbf{f}$ and $\mathbf{g}$, we can conveniently write

$$\mathbf{g} = \mathbf{H}_y^T \cdot \mathbf{f} \cdot \mathbf{H}_x$$

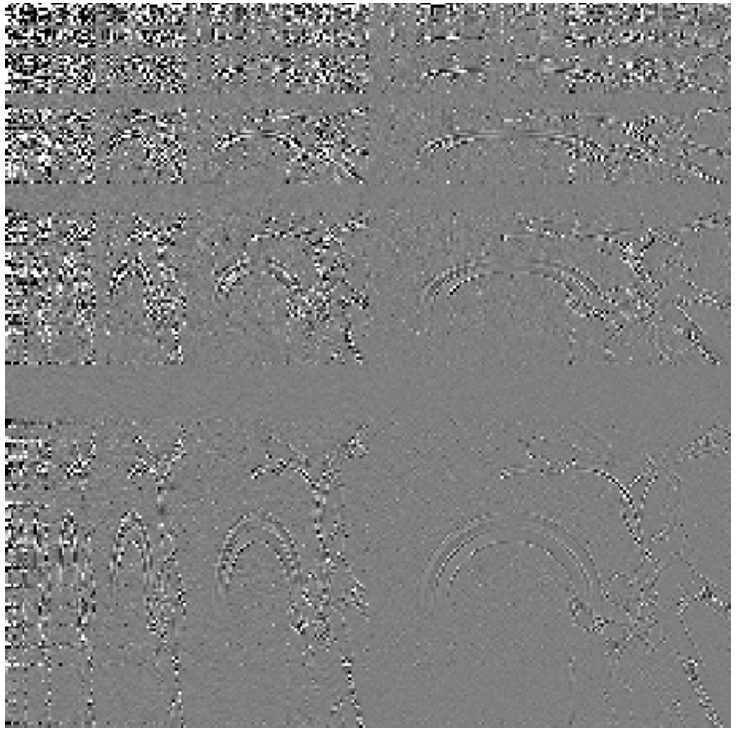
$$\mathbf{H}_y = \begin{bmatrix} h_y[0,0] & h_y[0,1] & \cdots & h_y[0,L_g-1] \\ h_y[1,0] & h_y[1,1] & \cdots & h_y[1,L_g-1] \\ \vdots & \vdots & & \vdots \\ h_y[L-1,0] & h_y[L-1,1] & \cdots & h_y[L-1,L_g-1] \end{bmatrix}$$

$$\mathbf{H}_x = \begin{bmatrix} h_x[0,0] & h_x[0,1] & \cdots & h_x[0,N_g-1] \\ h_x[1,0] & h_x[1,1] & \cdots & h_x[1,N_g-1] \\ \vdots & \vdots & & \vdots \\ h_x[N-1,0] & h_x[N-1,1] & \cdots & h_x[N-1,N_g-1] \end{bmatrix}$$

- Output image $\mathbf{g}$ has size L_g x N_g
- If the operator does not change image size, $\mathbf{H}_x$ and $\mathbf{H}_y$ are square matrices

Example: Separable Haar transform

Original *Bike*
256x256



256x256
Haar transform
$\mathbf{H_x = H_y = Hr_{256}}$

Haar transform

- Haar transform matrix for sizes $N=2,4,8$

$$\mathbf{Hr}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{Hr}_4 = \frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}$$

$$\mathbf{Hr}_8 = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{pmatrix}$$

- Can be computed by taking sums and differences
- Fast algorithms by recursively applying $\mathbf{Hr}_2$

Example: Subsampling

- Image subsampling 2:1 horizontally and vertically
- Small input image of size 8x8, output image size 4x4

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example: Subsampling (cont.)

- A somewhat better technique for 2:1 image size reduction

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Side by side comparison

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Another example: filtering

- Each pixel is replaced by the average of two horizontally (vertically) neighboring pixels

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 0.5 & 0 & & \cdots & & & & 0 \\ 0.5 & 0.5 & & & & & & \\ 0 & 0.5 & 0.5 & & & & & \\ & & 0.5 & 0.5 & \ddots & & & \vdots \\ & & & 0.5 & 0.5 & & & \\ \vdots & & & \ddots & 0.5 & 0.5 & & \\ & & & & & 0.5 & 0.5 & 0 \\ 0 & & & \cdots & & 0 & 0.5 & 0.5 \end{bmatrix}$$

- Shift-invariant operation (except for image boundary)

Shift-invariant systems and Toeplitz matrices

- For a separable, shift-invariant, linear system

$$h_x[x,\alpha] = h_{siv/x}[\alpha - x] \quad \text{and} \quad h_y[y,\beta] = h_{siv/y}[\beta - y]$$

- Matrices $\mathbf{H_x}$ and $\mathbf{H_y}$ are square, and Toeplitz matrices, e.g.,

$$\mathbf{H_x} = \begin{bmatrix} h_{siv/x}[0] & h_{siv/x}[1] & \cdots & h_{siv/x}[N-1] \\ h_{siv/x}[-1] & h_{siv/x}[0] & \cdots & h_{siv/x}[N-2] \\ \vdots & \vdots & & \vdots \\ h_{siv/x}[1-N] & h_{siv/x}[2-N] & \cdots & h_{siv/x}[0] \end{bmatrix}$$

- Operation is a 2-d separable convolution („filtering“)

$$g[\alpha,\beta] = \sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y] \cdot h_{siv/x}[\alpha - x] h_{siv/y}[\beta - y]$$

Non-separable 2-d convolution

- Convolution kernel of linear shift-invariant system („filter“) can also be non-separable

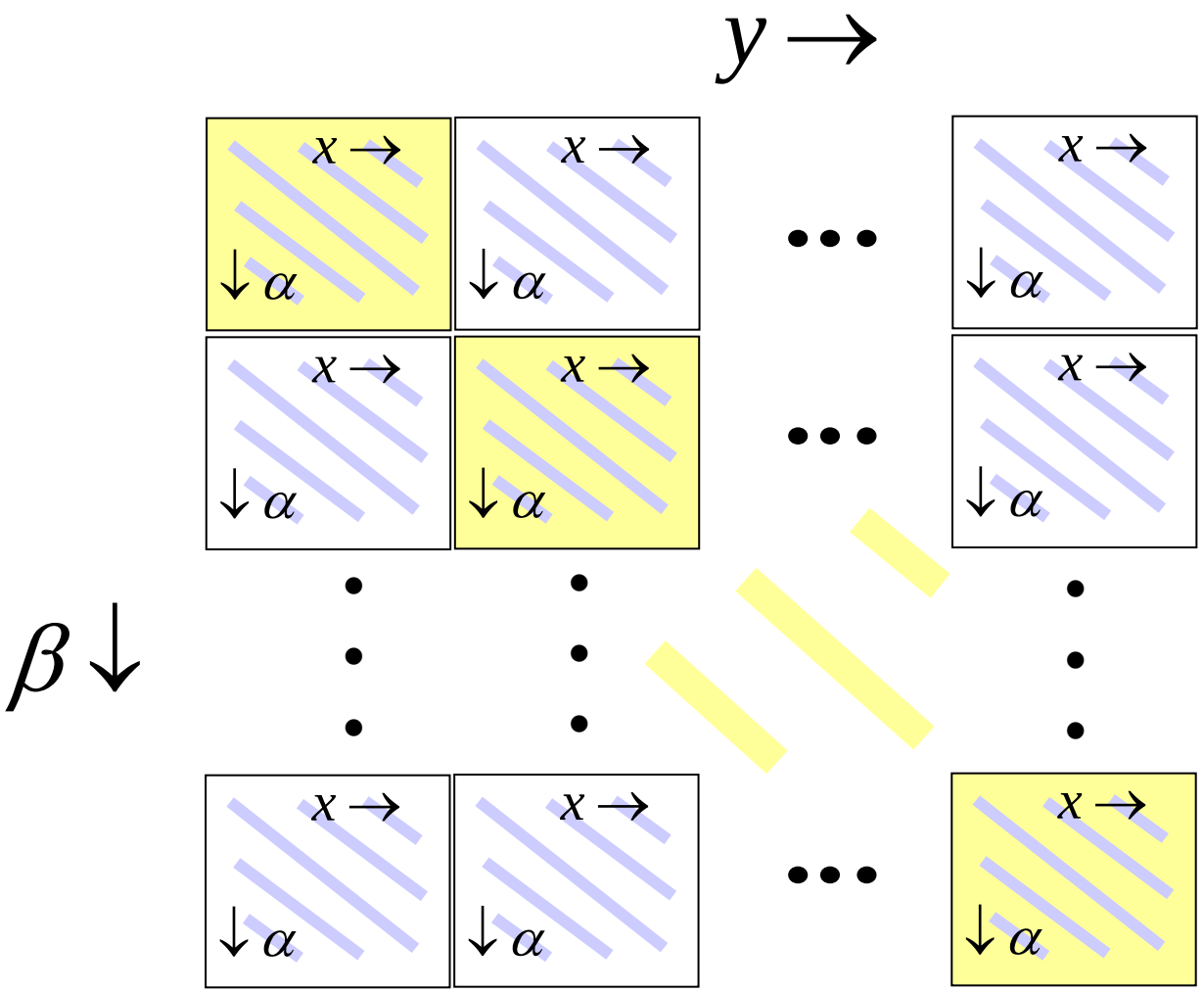
$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_{siv}[\alpha-x,\beta-y]$$

- Viewed as a matrix operation . . .

$$\vec{g}=H\vec{f}$$

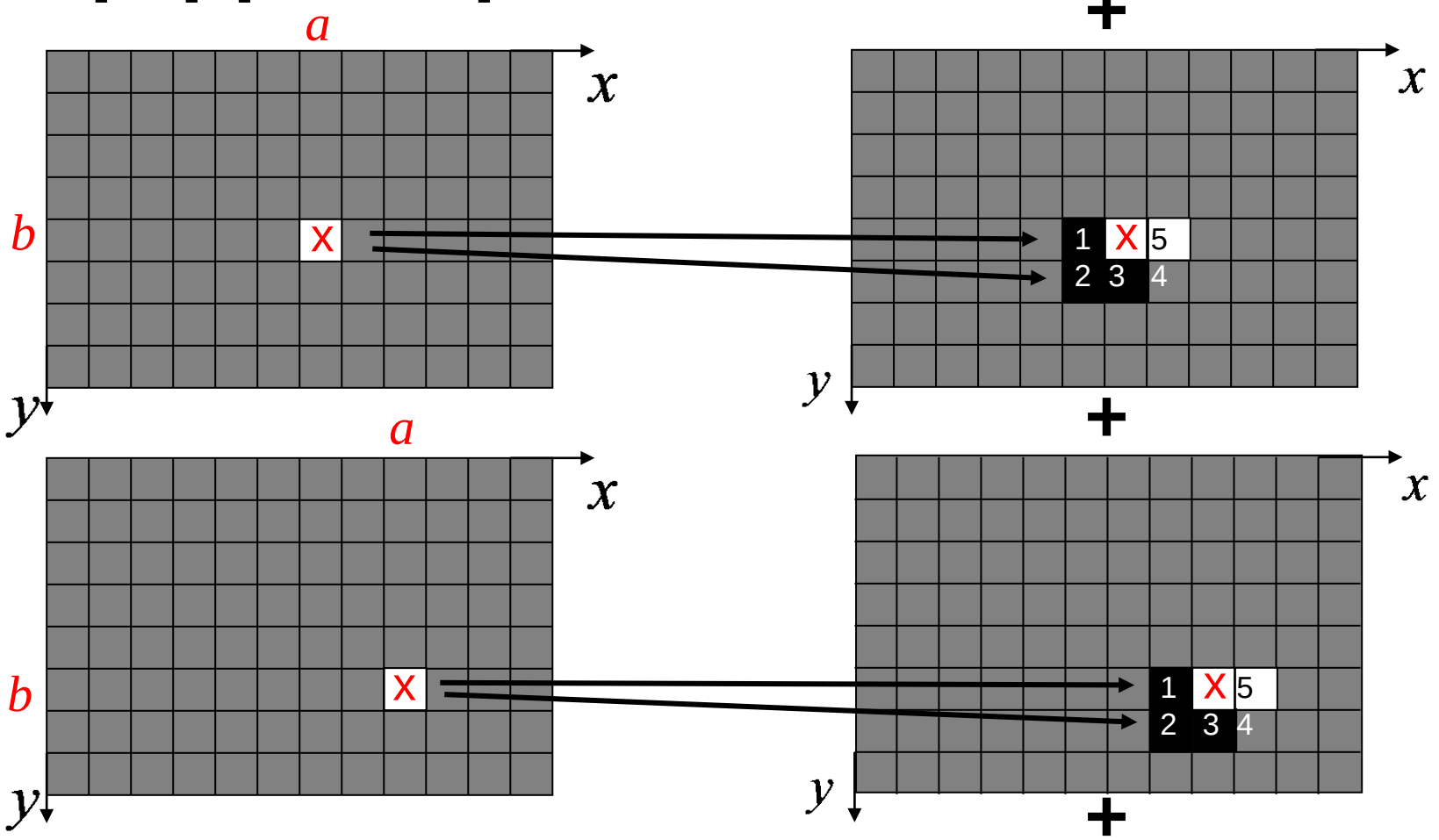
. . . $\boldsymbol{H}$ is a block Toeplitz matrix

Structure of H for non-separable convolution



Convolution: superposition of impulse responses

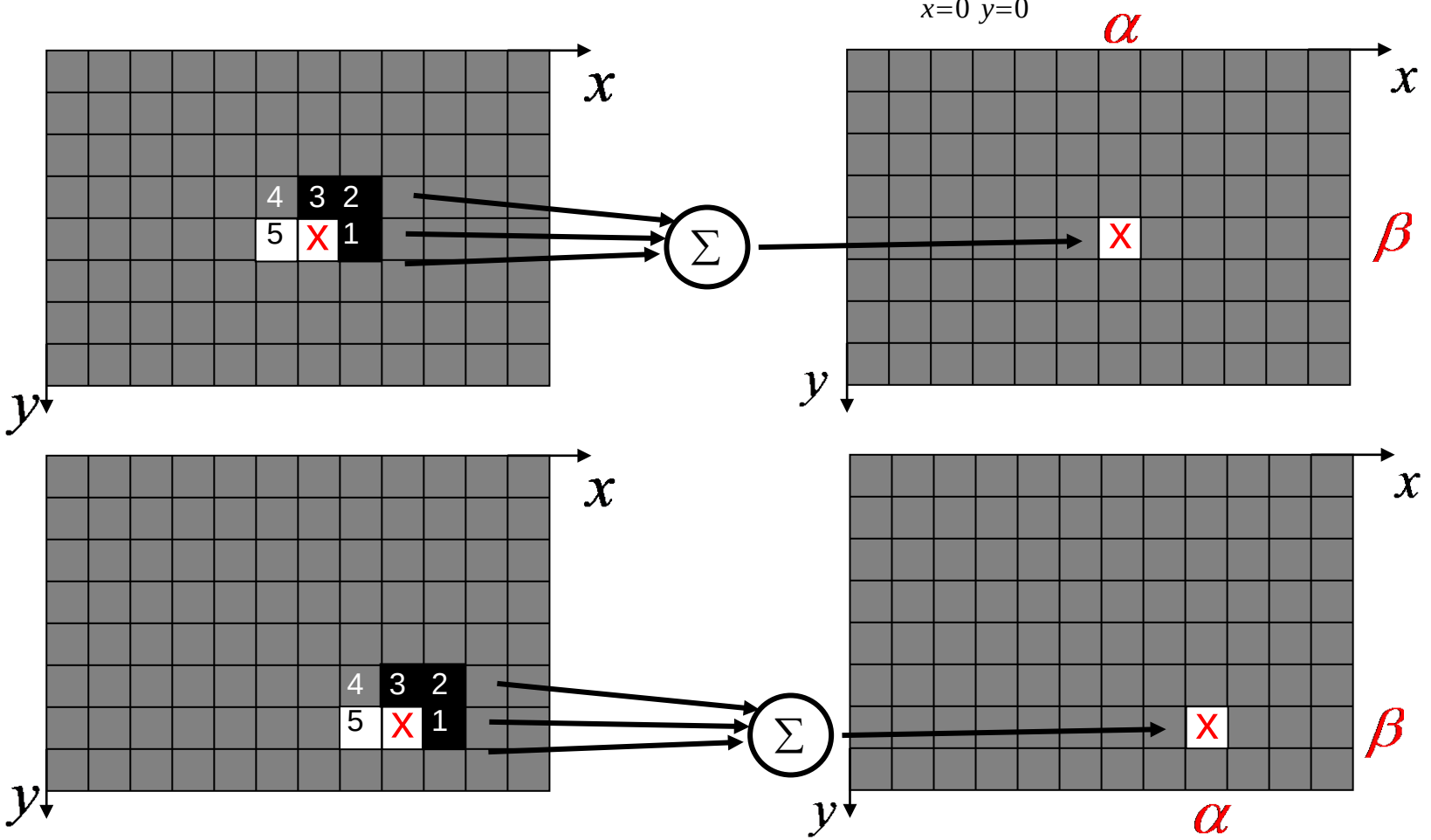
$$f[a,b]\delta[x-a,y-b] \qquad g[x,y]=h_{siv}[x-a,y-b]f[a,b]$$



Convolution: linear combination of neighboring pixel values

$$f[x,y]\cdot h_{siv}[\alpha-x,\beta-y]$$

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_{siv}[\alpha-x,\beta-y]$$



Convolution examples

Original
Bike

Bike blurred by convolution

Impulse response „box filter“

$$\frac{1}{25}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & [1] & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Convolution examples

Original
Bike

Bike blurred horizontally

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 & 1 & [1] & 1 & 1 \end{pmatrix}$$

Convolution examples

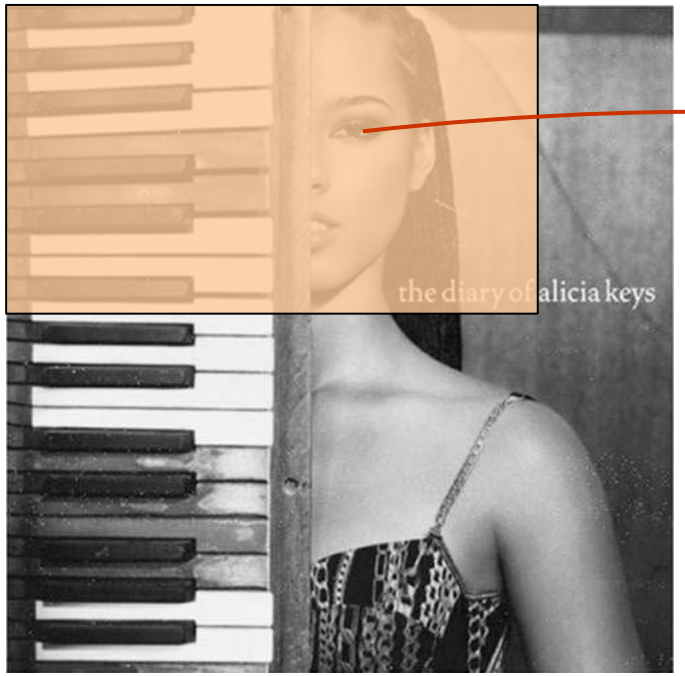
Original
Bike

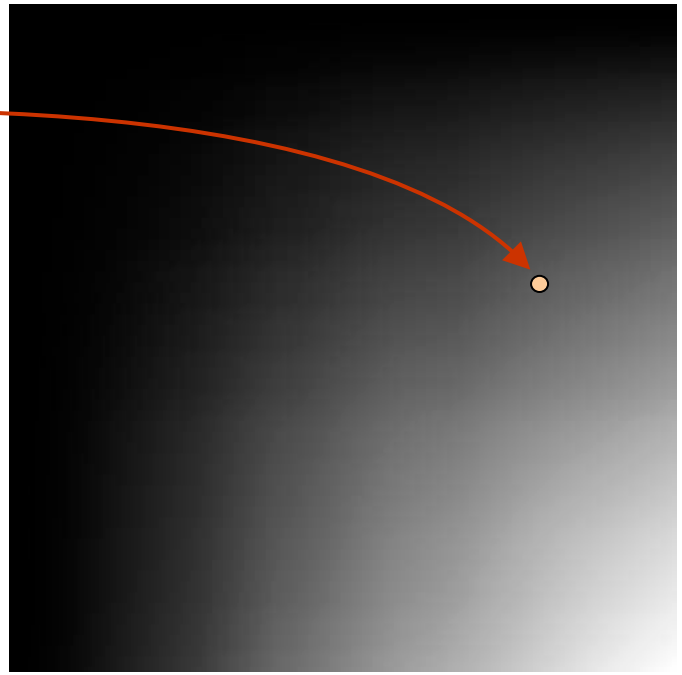
Bike blurred vertically

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ [1] \\ 1 \\ 1 \end{pmatrix}$$

Integral image



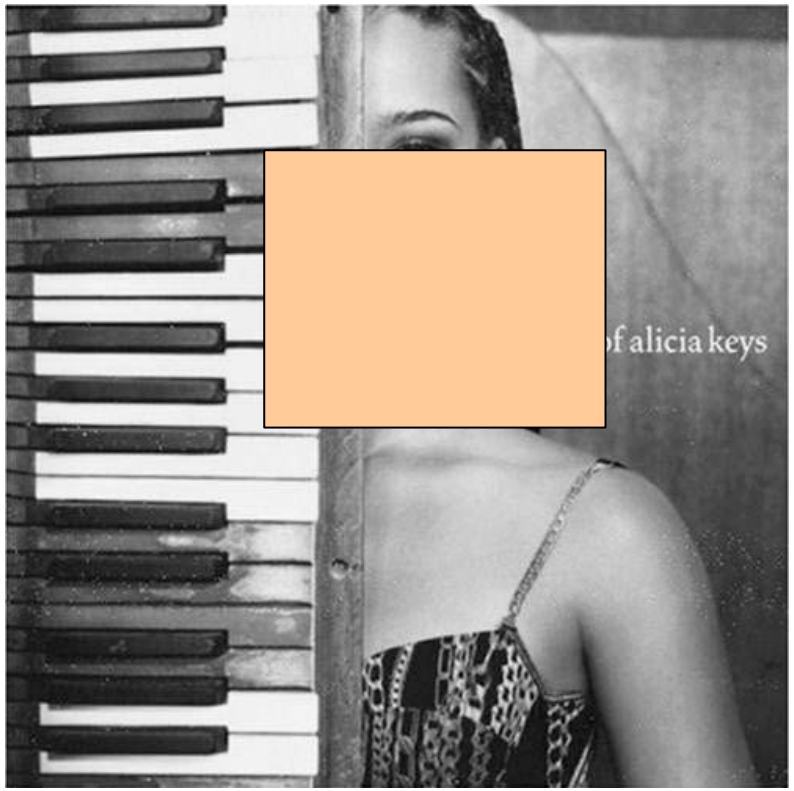


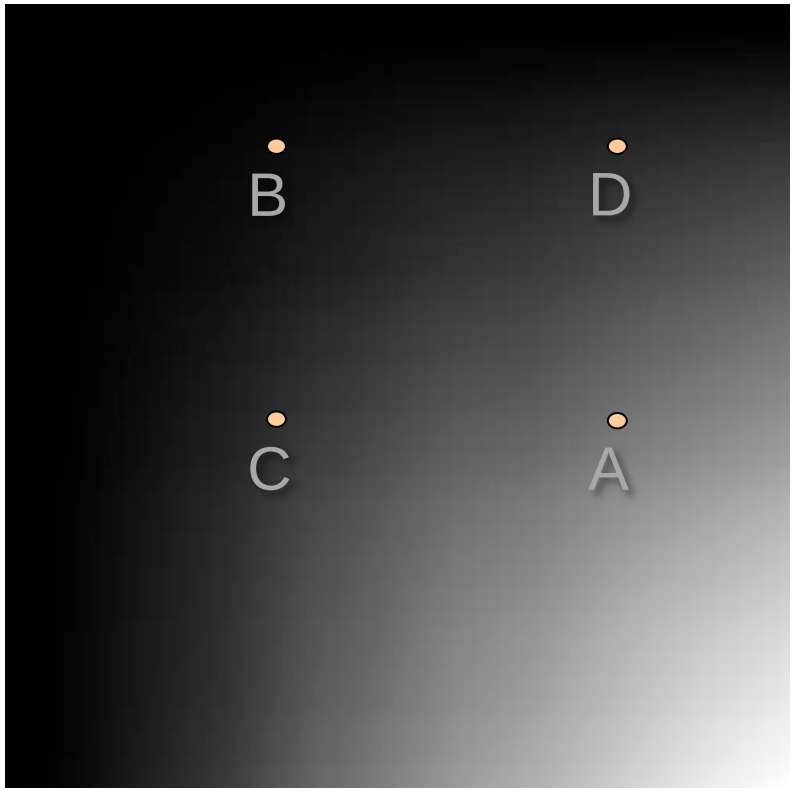
$$j[x,y] = j[x,y-1] + f[x,y]$$
$$i[x,y] = i[x-1,y] + j[x,y]$$

$$f[x,y]$$

$$i[x,y] = \sum_{u=0}^{x} \sum_{v=0}^{y} f[u,v]$$

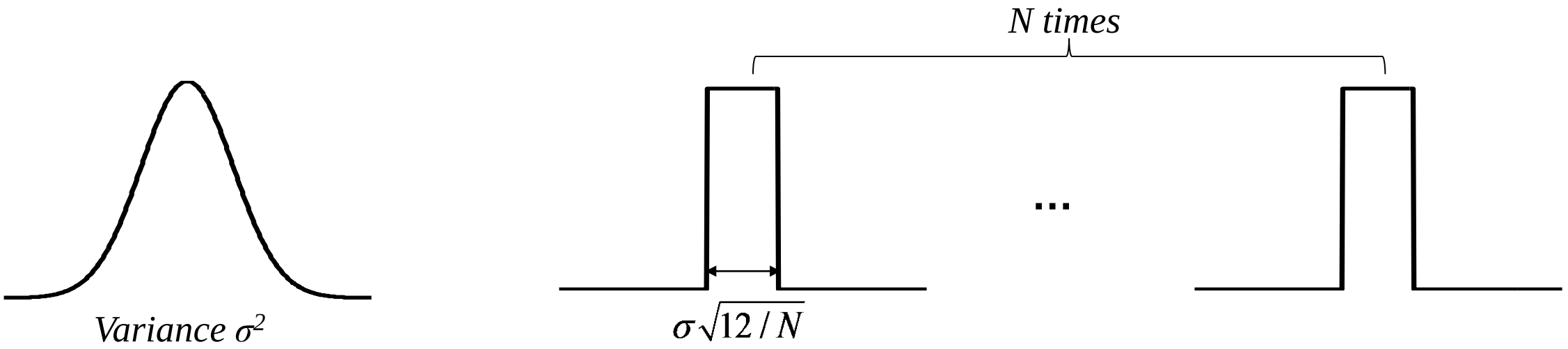
Box filtering with integral image





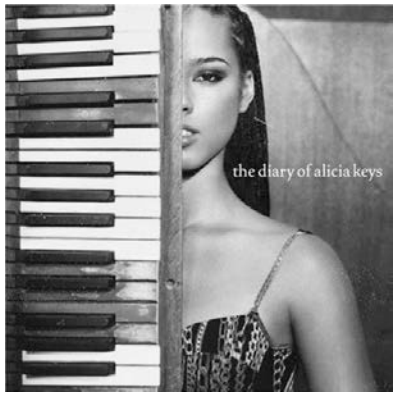
Sum = A + B – C – D

Gaussian filtering by repeated box filtering

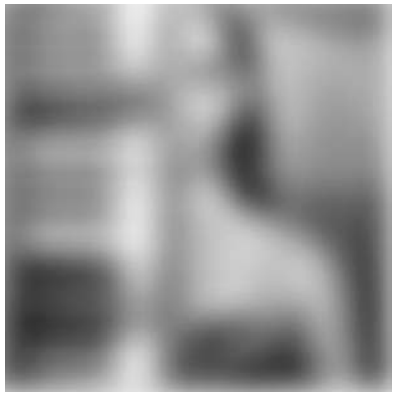


- Approximate convolution with Gaussian kernel by convolution with box kernel, repeated N times
- Convolution with box kernel can be computed efficiently using integral image

Gaussian filtering by repeated box filtering



Original image
500x500



Gaussian filtered
σ = 20, 81x81
kernel

Direct implementation:
6561 multiplications and
6560 additions per pixel

x-y separable filtering:
162 multiplications and
160 additions per pixel

20 additions or subtractions per pixel

Box filtered
after N = 1

Box filtered
after N = 2

Box filtered
after N = 3

Box filtered
after N = 4

1-d discrete-time Fourier transform

- Given a 1-d sequence $s[k], \quad k \in \mathbf{Z} = \{\ldots, -1, 0, 1, 2, 3, \ldots\}$
- Fourier transform

$$S(e^{j\omega}) = \sum_{k=-\infty}^{\infty} s[k] e^{-j\omega k} \qquad \omega \in \Re$$

- Fourier transform is periodic with 2π
- Inverse Fourier transform

$$s[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(e^{j\omega}) e^{j\omega k} d\omega$$

2-d discrete-space Fourier transform

- Given a 2-d array of image samples

$$s[m,n], \quad m,n \in \mathbf{Z}^2$$

- Fourier transform

$$S\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n] e^{-j\omega_x m - j\omega_y n} \qquad \omega_x, \omega_y \in \Re^2$$

- Fourier transform is 2π-periodic both in ω_x and ω_y
- Inverse Fourier transform

$$s[m,n] = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S\left(e^{j\omega_x}, e^{j\omega_y}\right) e^{j\omega_x m + j\omega_y n} \, d\omega_x d\omega_y$$

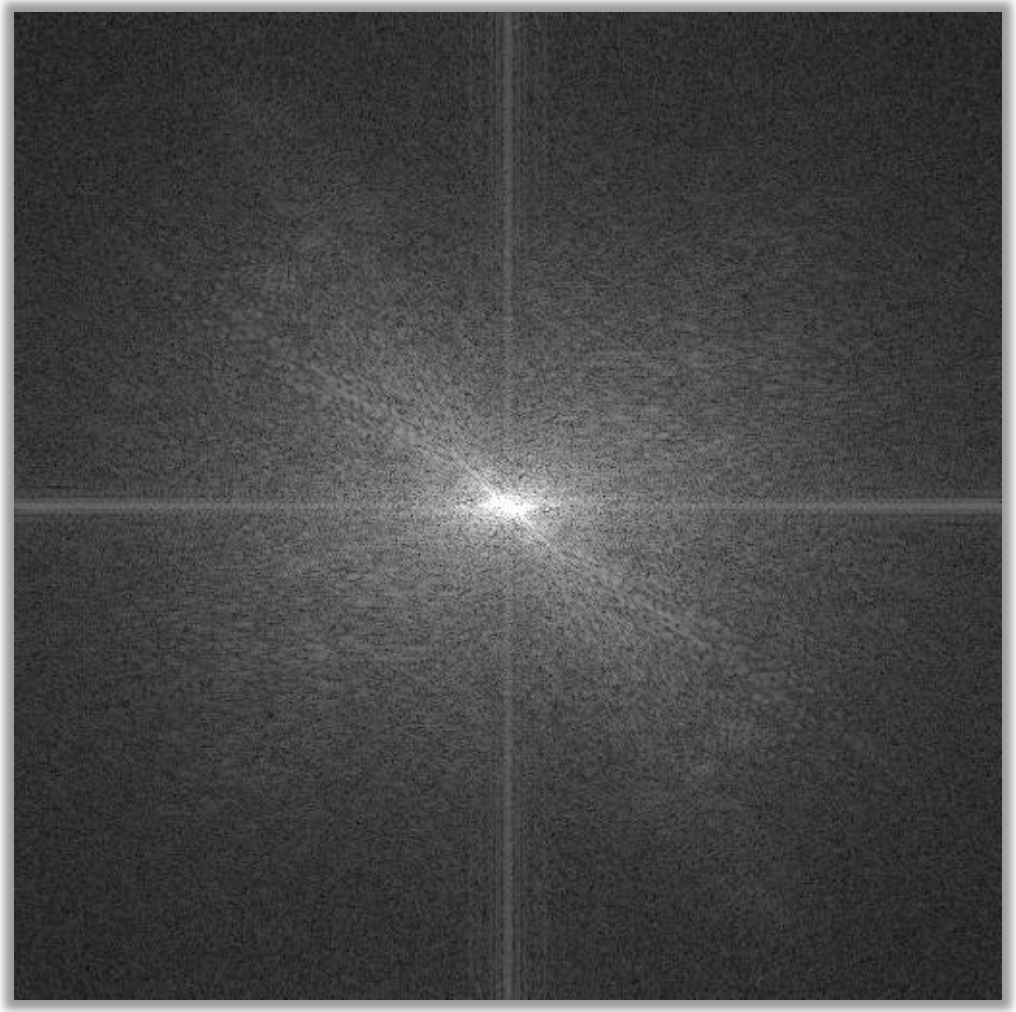
Fourier transform – interesting properties

- DFT matrix is orthonormal
- conceptually write as DFT-matrix vector multiplication O(N^2) but in practice use FFT O(NlogN)
- FT of symmetric function is always real; FT of real function is always symmetric
- convolution theorem is super useful:

$$x * g = F^{-1}\left\{F\{x\} \cdot F\{g\}\right\}$$

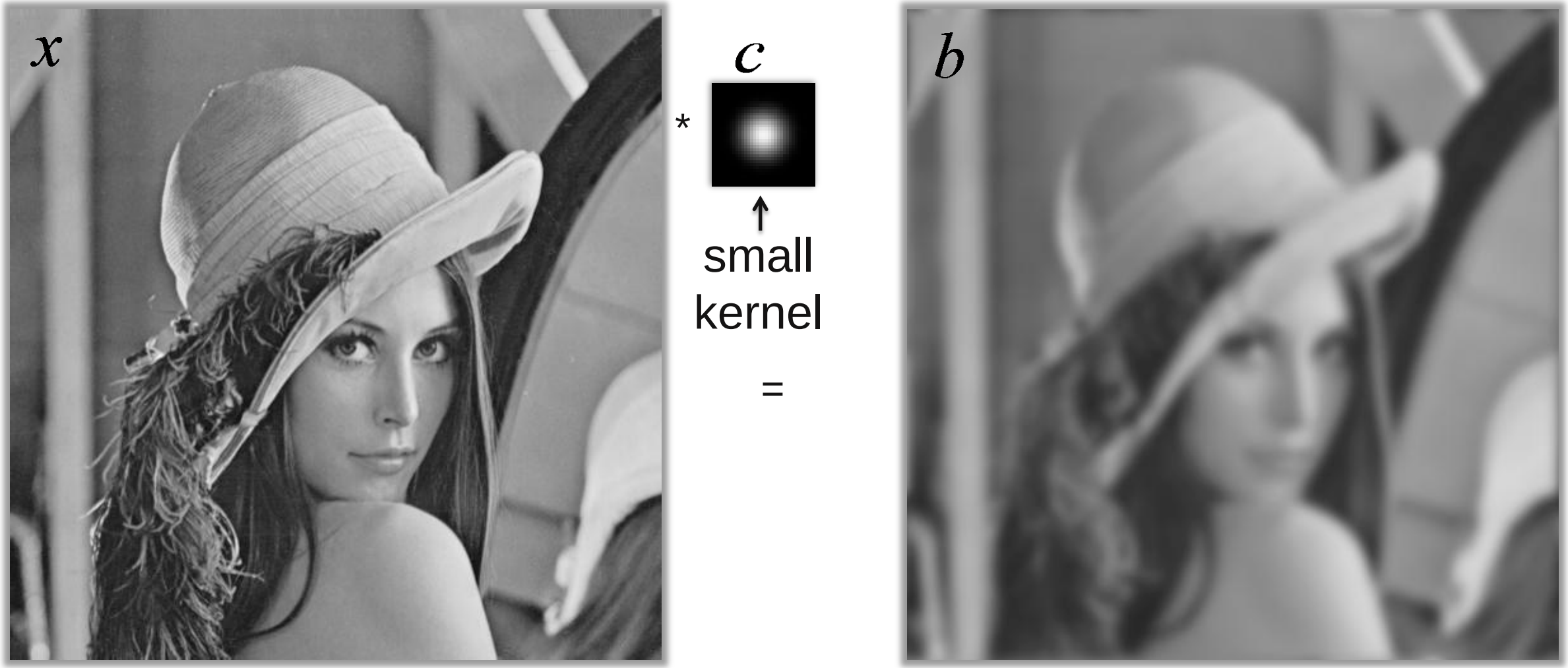
Discrete Fourier Transform

- What is this?



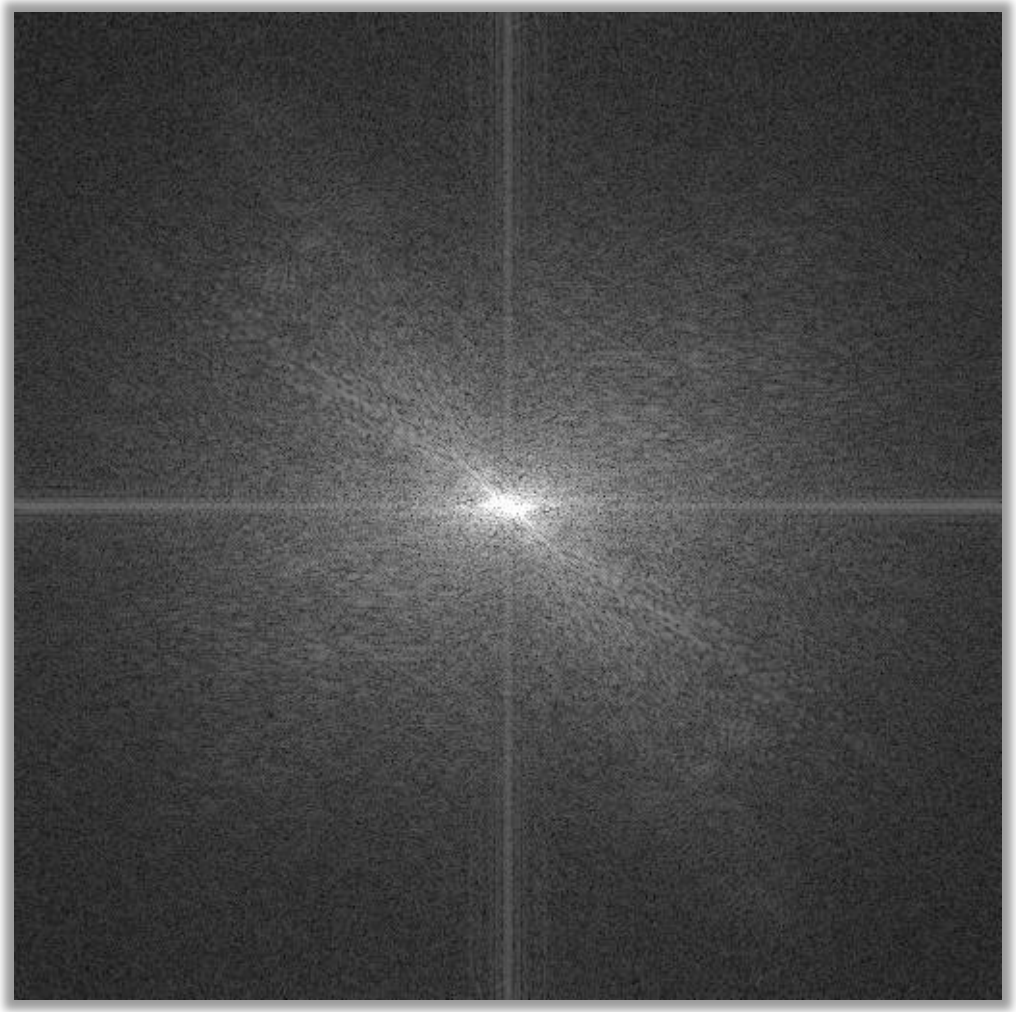
Filtering – Low-pass Filter

- low-pass filter: convolution in primal domain $b = x * c$



Filtering – Low-pass Filter

- low-pass filter: multiplication in frequency domain $F\{b\} = F\{x\} \cdot F\{c\}$



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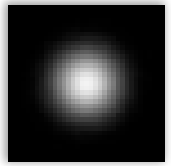

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big

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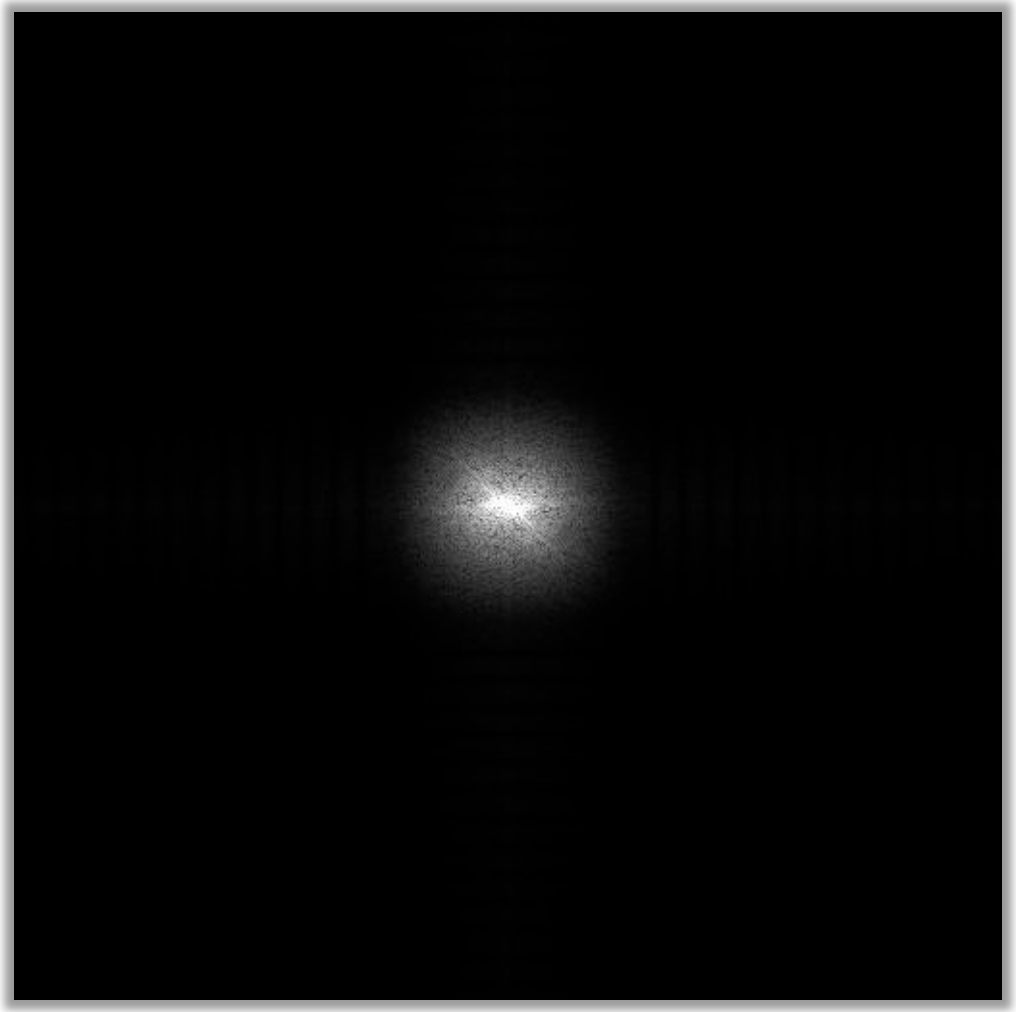
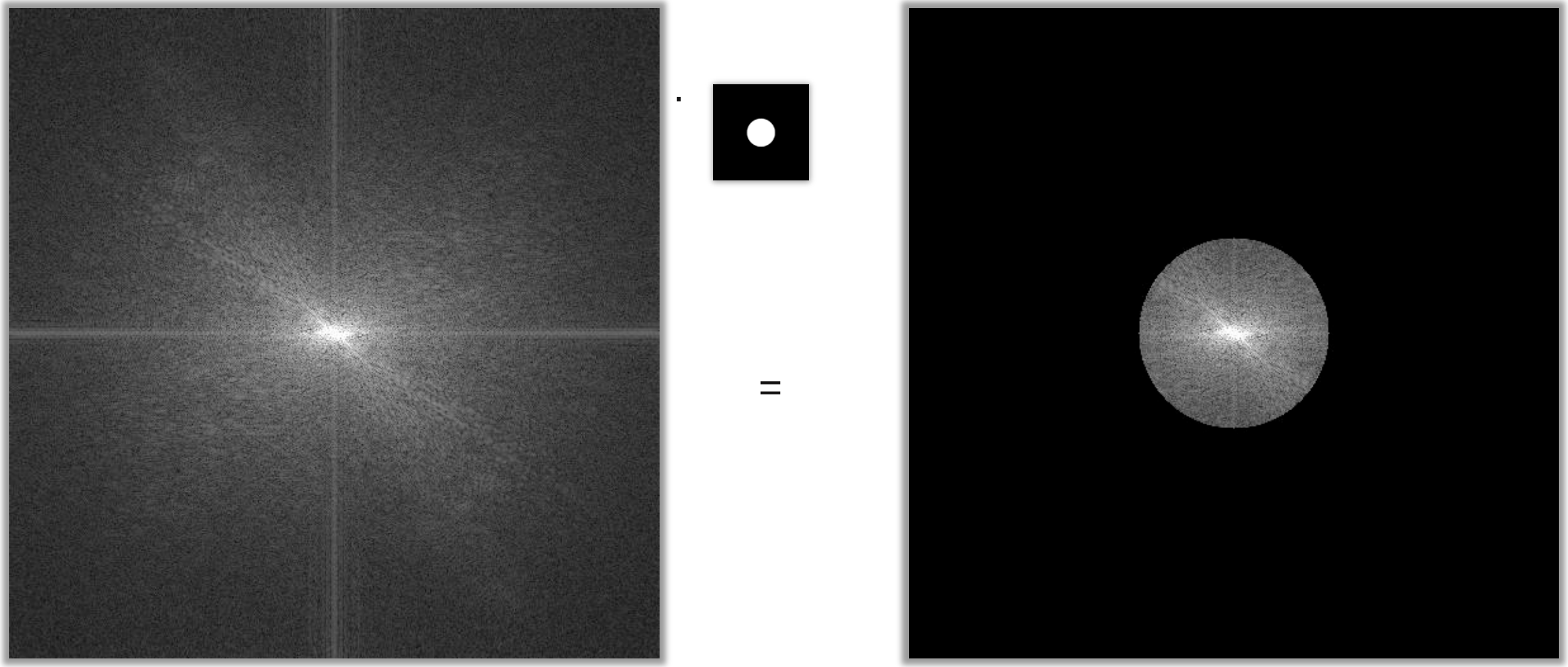


Filtering – Low-pass Filter

- low-pass filter: hard cutoff

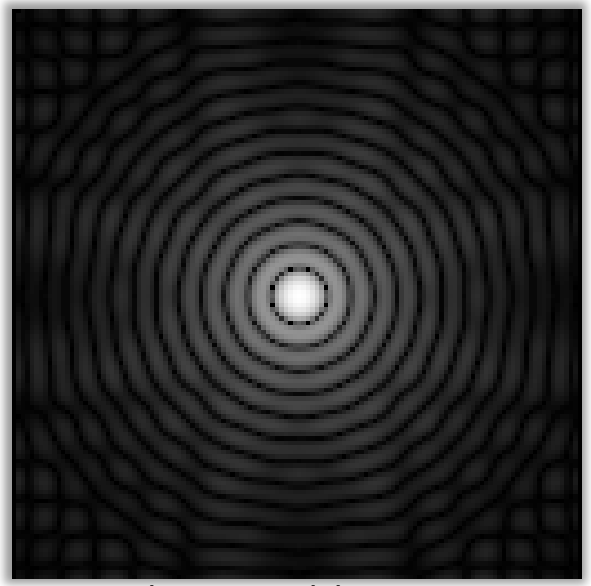
$$F\{b\} = F\{x\} \cdot F\{c\}$$



Filtering – Low-pass Filter

- Bessel function of the first kind or “jinc”

$$F^{-1}\left\{ \bullet \right\}$$



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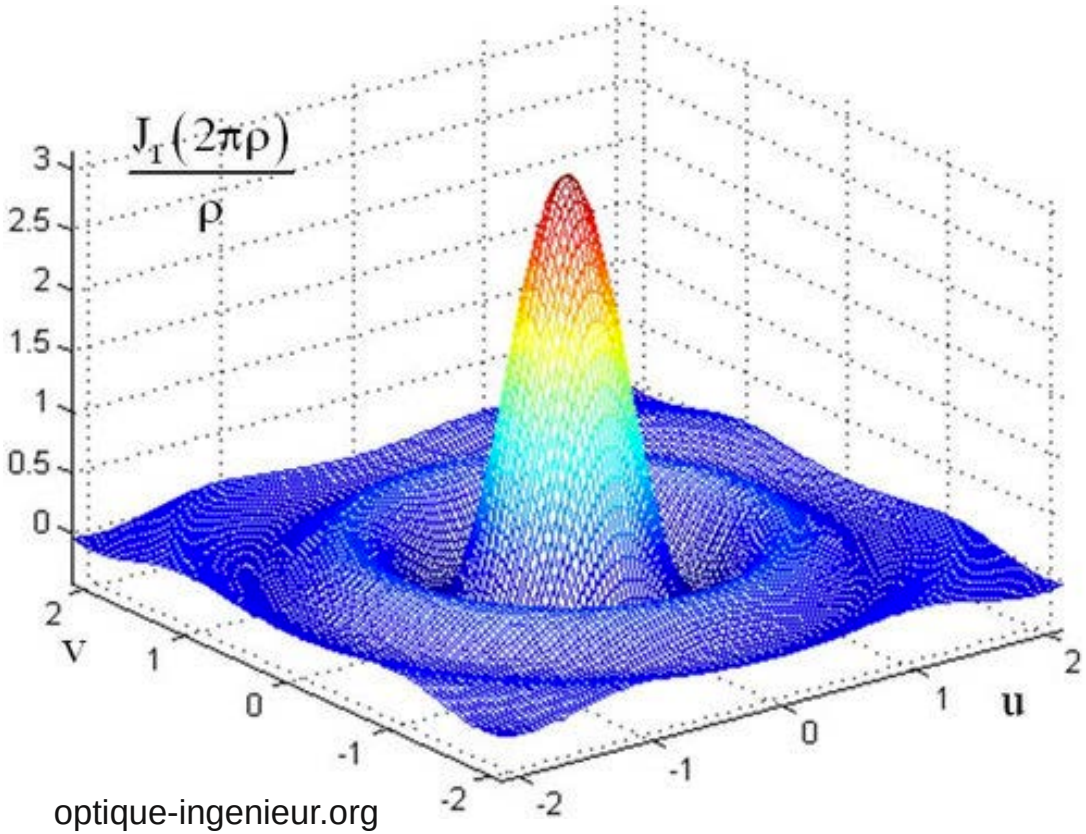

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Filtering – Low-pass Filter

- hard frequency filters often introduce ringing

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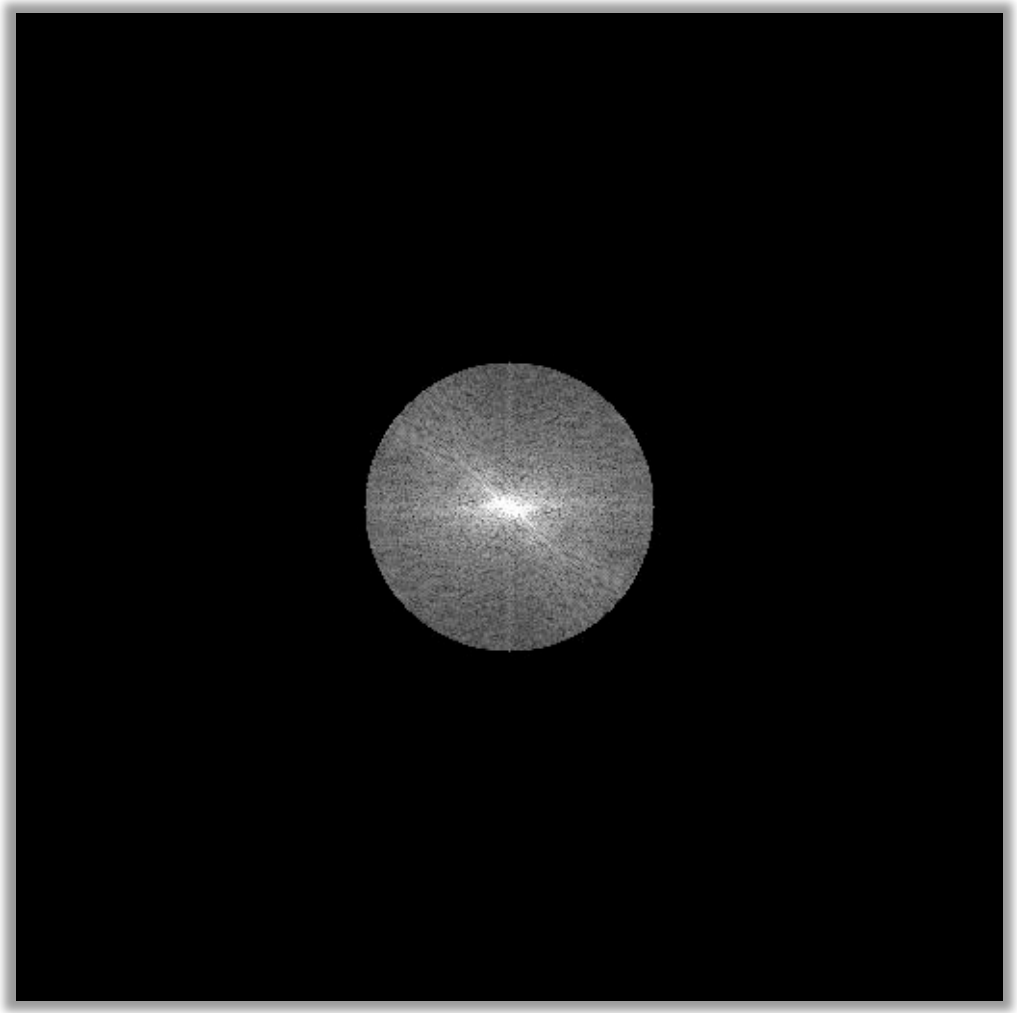

Filtering – High-pass Filter

- sharpening (possibly with ringing, but don’t see any here)

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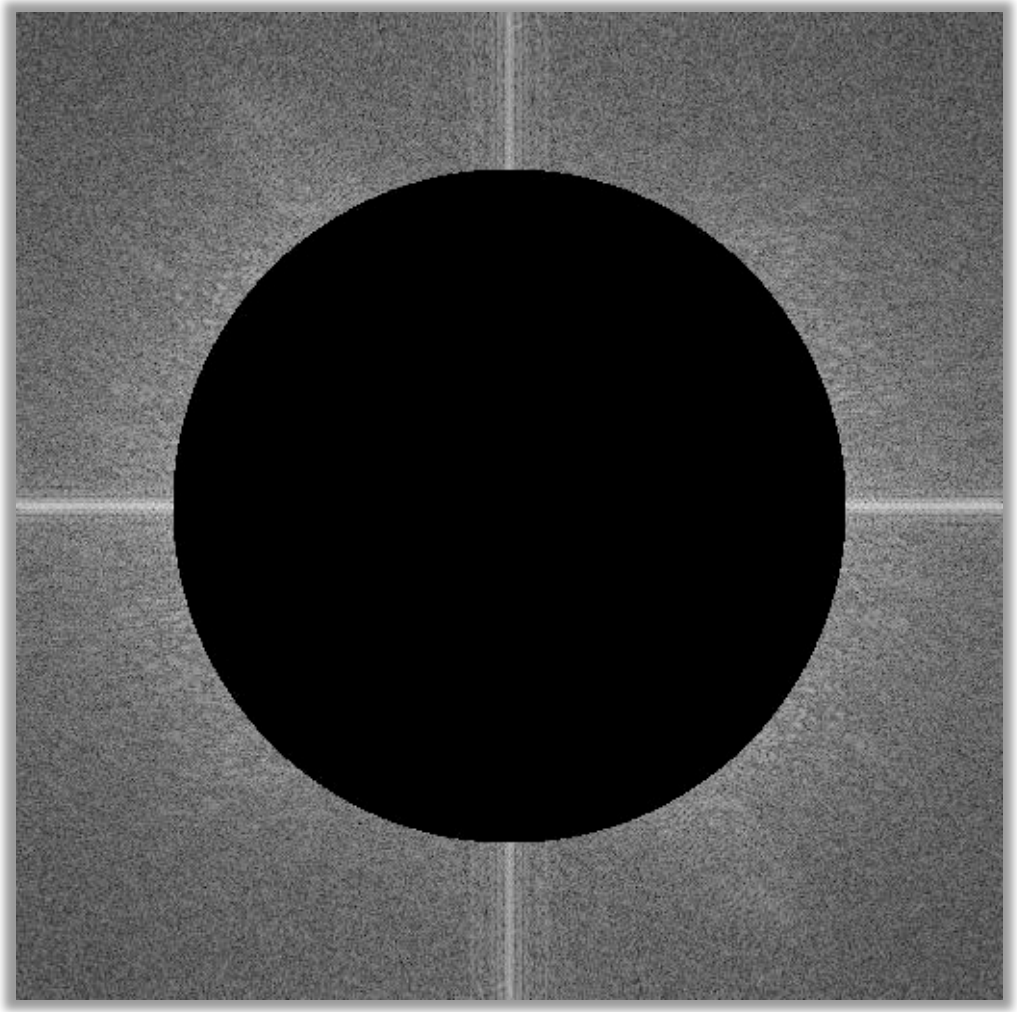

Filtering – Unsharp Masking

- sharpening (without ringing): unsharp masking, e.g. in Photoshop

$$b = x * (\delta - c_{lowpass\_gauss}) = x - x * c_{lowpass\_gauss}$$

or

$$b = x * (\delta + c_{highpass}) = x + x * c_{highpass}$$

Filtering – Band-pass Filter

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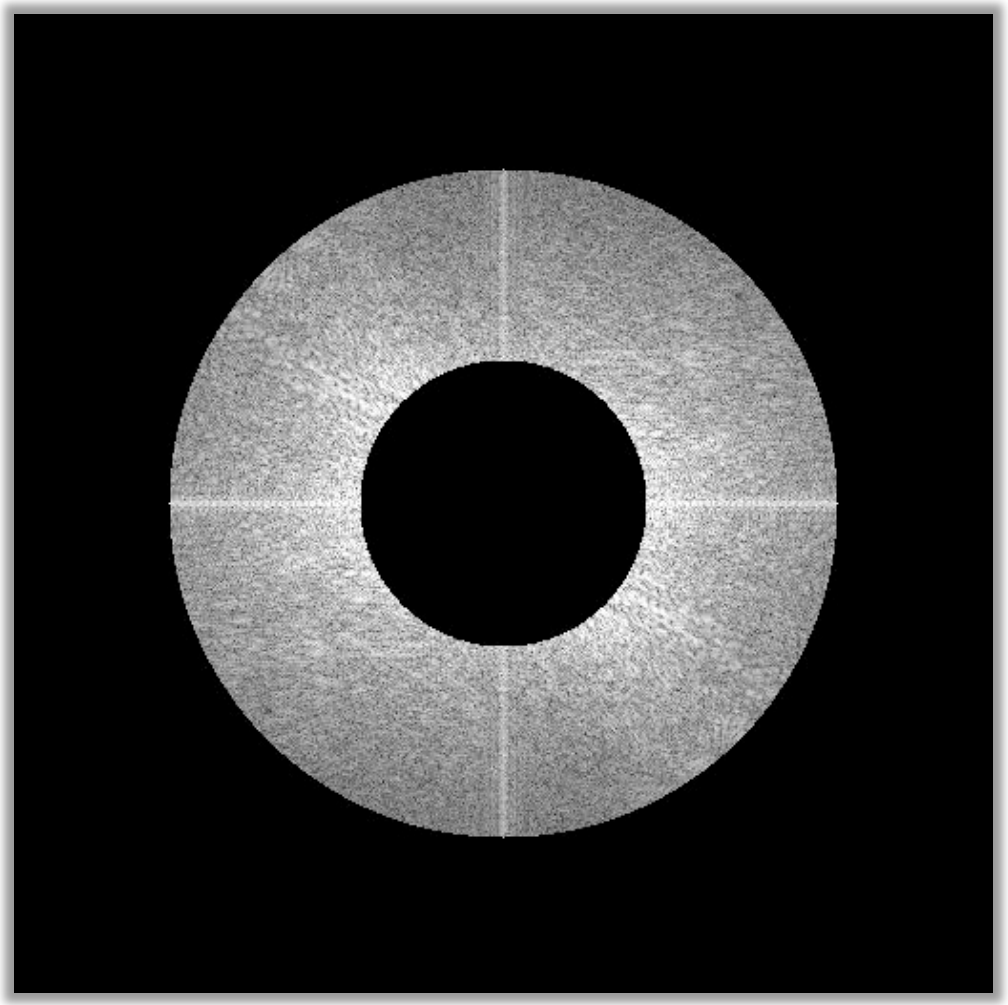

Filtering – Oriented Band-pass Filter

- edges with specific orientation (e.g., hat) are gone!

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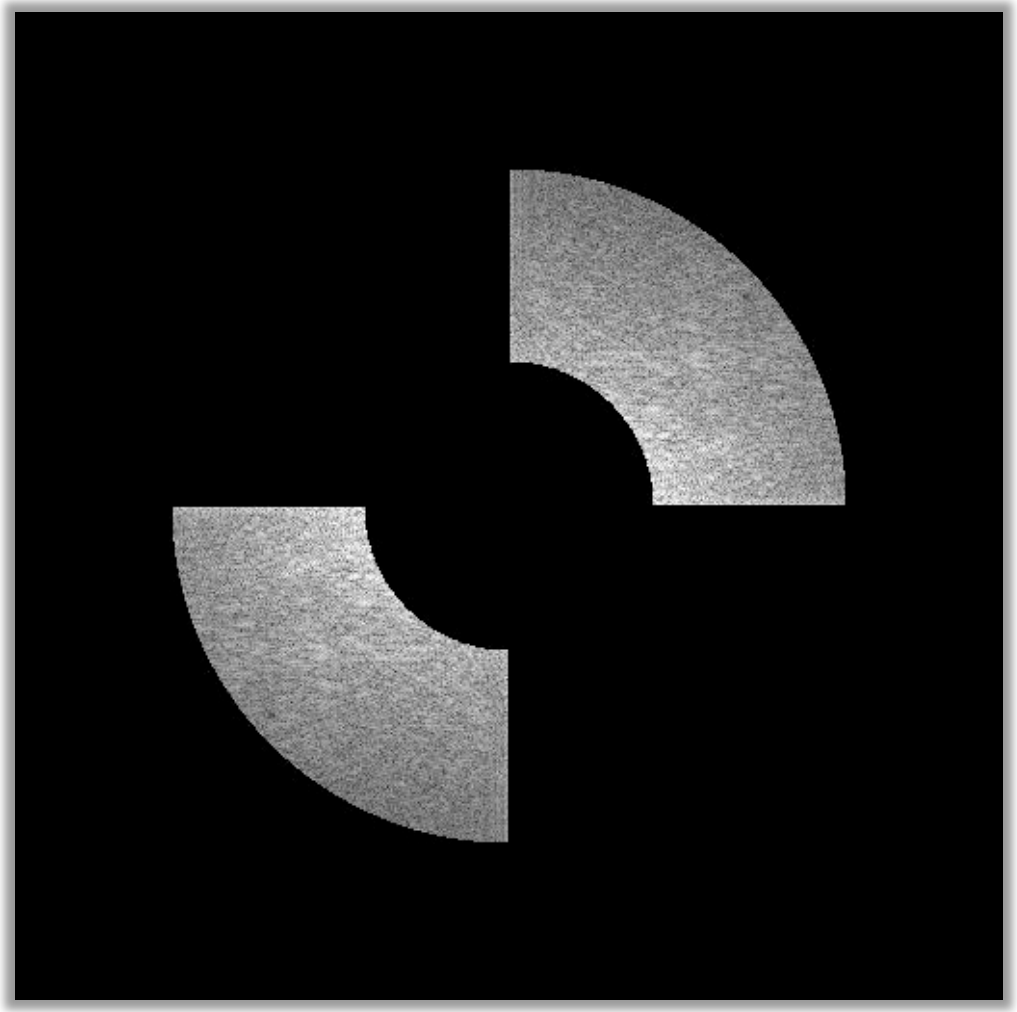
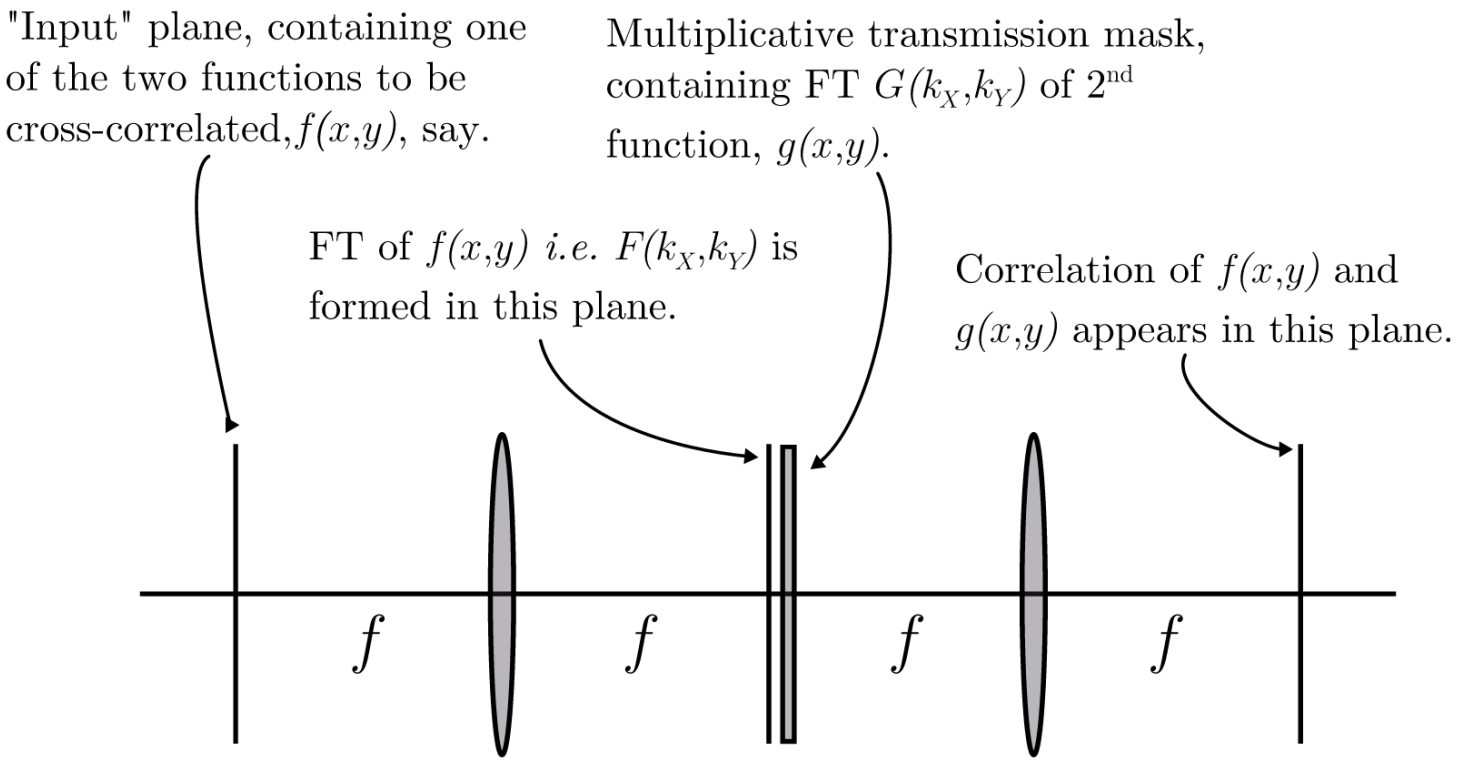


Optical Filtering with Fourier Optics

- can do all of this optically (with coherent light)!
- Fourier optics – not part of this course



http://en.wikipedia.org/wiki/Fourier_optics

Image Downsampling (& Upsampling)

- “anti-aliasing” → before re-sampling, apply appropriate filter!

- how much filtering? Nyquist-Shannon sampling theorem:

$$f_s \geq 2 f_{\max}$$

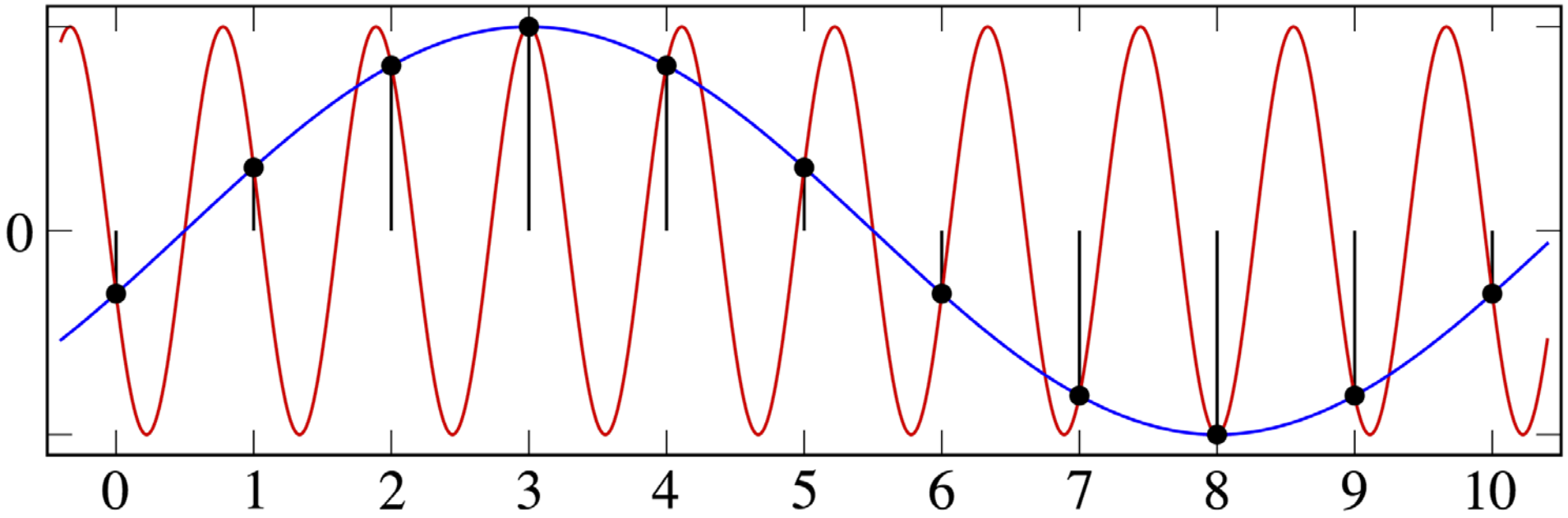
Other Forms of Aliasing

- wagon wheel effect (temporal aliasing):

youtube.com/watch?v=jHS9JGkEOmA

Other Forms of Aliasing

- wagon wheel effect (temporal aliasing)
- sampling frequency was lower than $2 f_{\max}$



wikipedia

Other Forms of Aliasing

- photography – optical AA filter removed (“hot rodding” camera)

John Shafer

mosaicengineering.com

Other Forms of Aliasing

- photography – optical AA filter removed (“hot rodding” camera)

without AA filter

with AA filter (standard)

maxmax.com

Other Forms of Aliasing

- photography – optical AA filter removed (“hot rodding” camera)

without AA filter

with AA filter (standard)

maxmax.com

5x5 box filter revisited

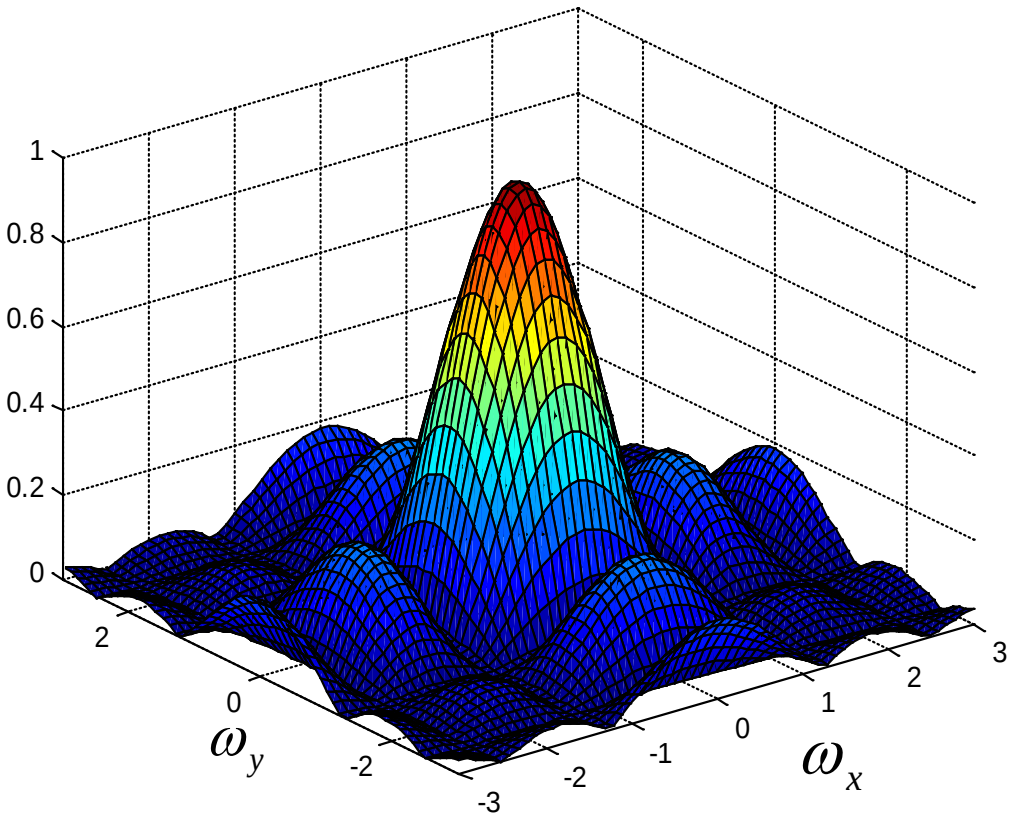
Original
Bike

Bike blurred by convolution

Impulse response „box filter“

$$\frac{1}{25}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & [1] & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Frequency response of 5x5 lowpass filter



$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h[m,n]\, e^{-j\omega_x m - j\omega_y n}$$

$$= \frac{1}{25}\sum_{m=-2}^{2}\sum_{n=-2}^{2} e^{-j\omega_x m - j\omega_y n} = \frac{1}{25}\sum_{m=-2}^{2} e^{-j\omega_x m}\sum_{n=-2}^{2} e^{-j\omega_y n}$$

$$= \frac{1}{25}\left(1+2\cos\omega_x+2\cos(2\omega_x)\right)\left(1+2\cos\omega_y+2\cos(2\omega_y)\right)$$

Separable filter:
1-d frequency responses
are multiplied

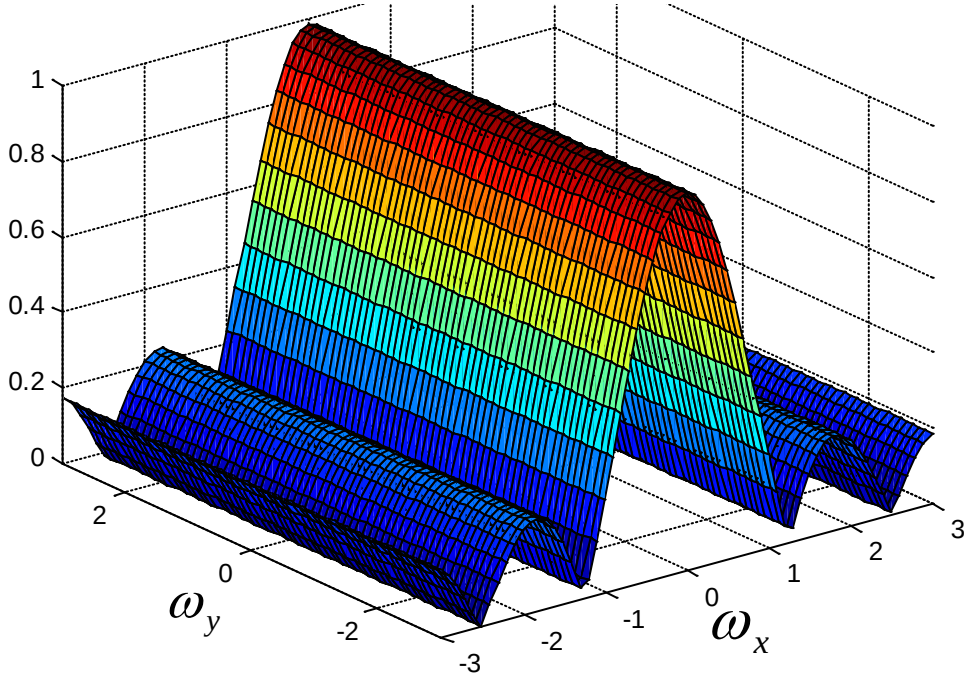
Horizontal lowpass filter

Bike blurred horizontally

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 & 1 & \boxed{1} & 1 & 1 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{1}{5}\left(1 + 2\cos\omega_x + 2\cos(2\omega_x)\right)$$



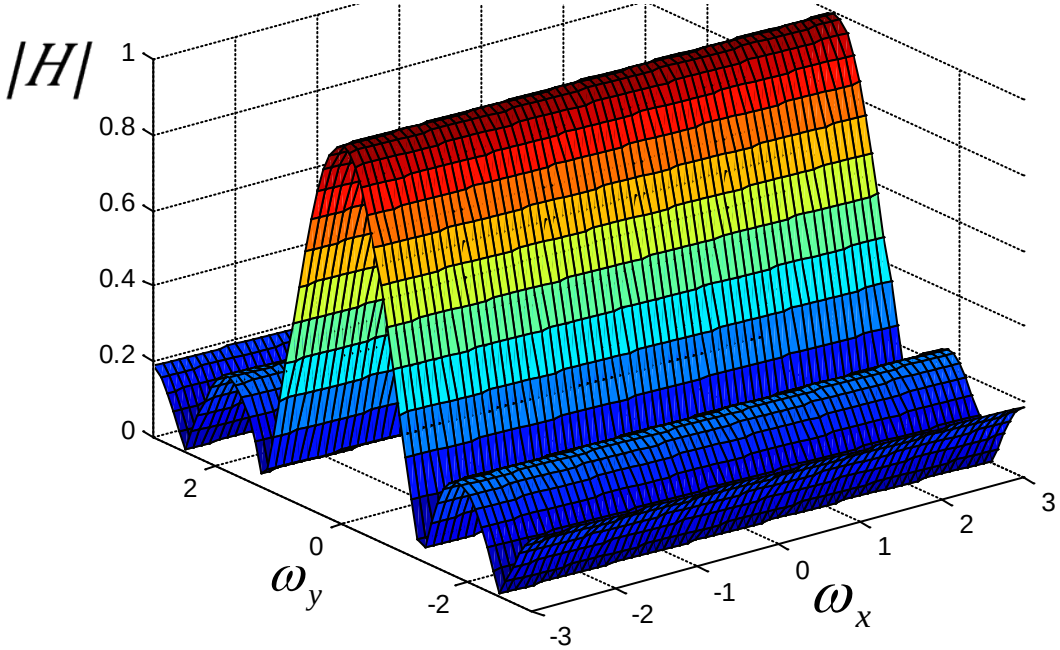
Vertical lowpass filter

Bike blurred vertically

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ [1] \\ 1 \\ 1 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{1}{5}\left(1 + 2\cos\omega_y + 2\cos(2\omega_y)\right)$$



Sharpening filter

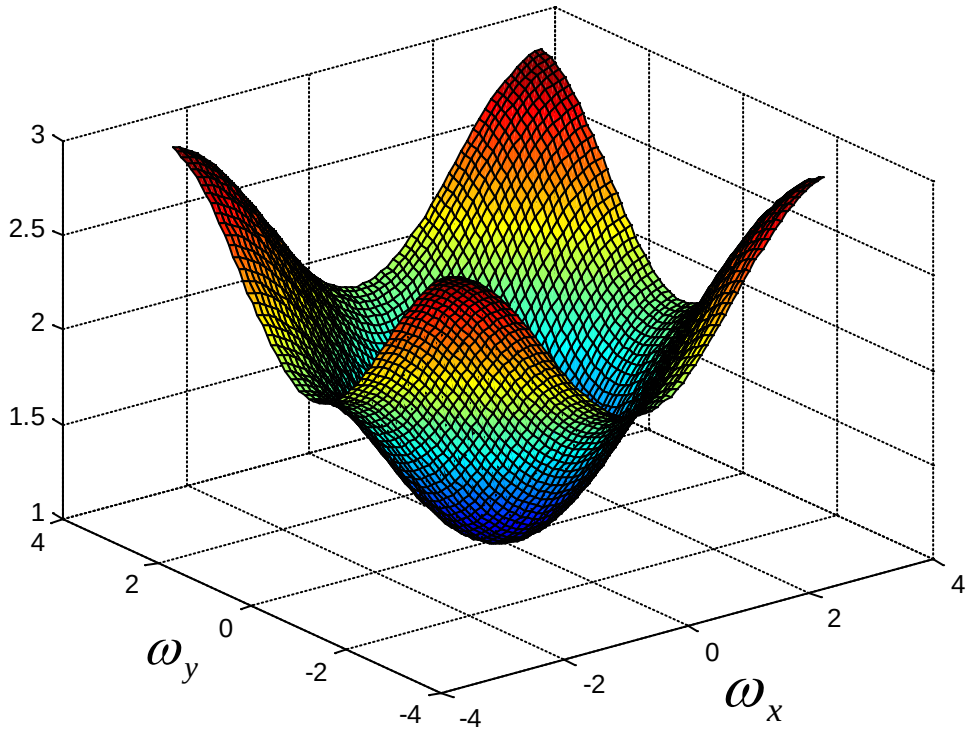
Original
Bike

Bike sharpened

Filter impulse response

$$\frac{1}{4}\begin{pmatrix} 0 & -1 & 0 \\ -1 & [8] & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Frequency response of sharpening filter



$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m,n] e^{-j\omega_x m - j\omega_y n}$$

$$= \frac{1}{4}\left(8 - e^{-j\omega_x} - e^{j\omega_x} - e^{-j\omega_y} - e^{j\omega_y}\right)$$

$$= 2 - \frac{1}{2}\cos\omega_x - \frac{1}{2}\cos\omega_y$$

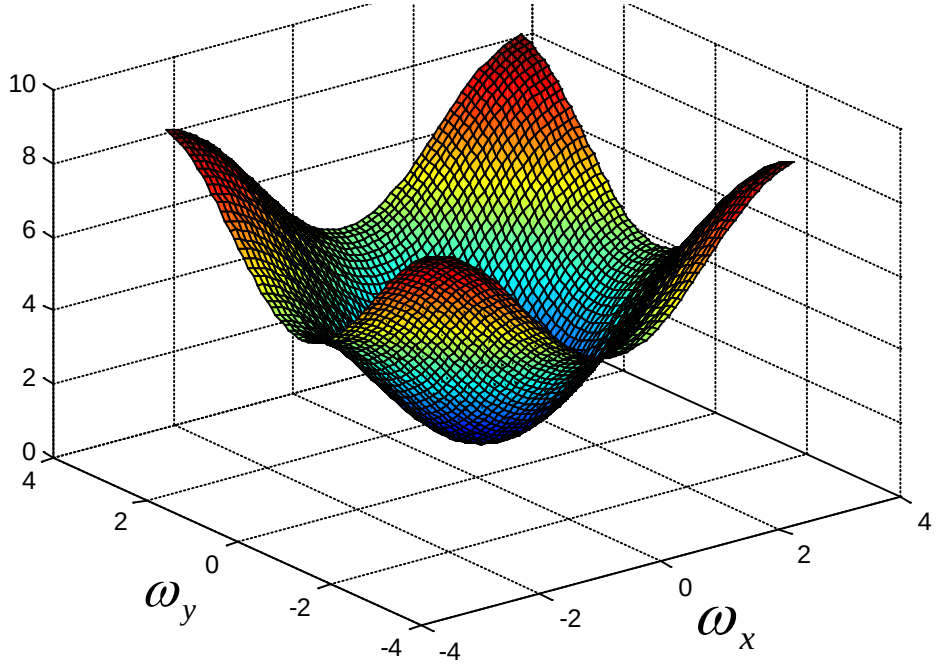
More aggressive sharpening

Bike sharpened

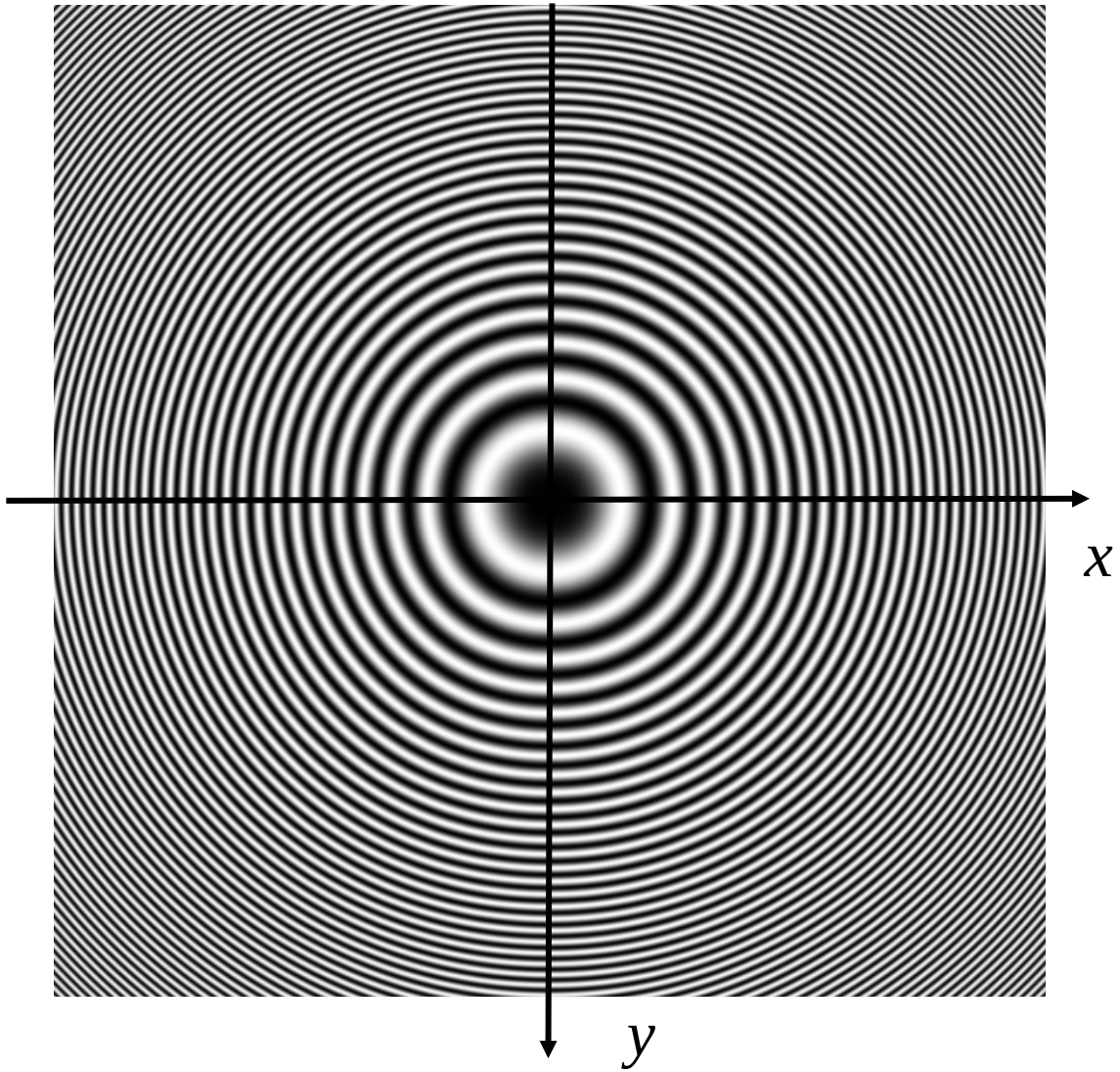
Filter impulse response

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & [5] & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = 5 - 2\cos\omega_x - 2\cos\omega_y$$



Zoneplate pattern to visualize frequency plane



Equation to generate pattern:

$$s(x,y) = \hat{s}\cos(a_x x^2 + a_y y^2) + s_0$$

Local frequency at (x,y)

$$\frac{\partial}{\partial x}\left(a_x x^2 + a_y y^2\right) = 2a_x x$$

$$\frac{\partial}{\partial y}\left(a_x x^2 + a_y y^2\right) = 2a_y y$$

$$\mathrm{s}[x,y] \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} s(x,y)$$

Sampling interpretation of 2-d discrete-space Fourier transform

- How is the Fourier transform of $s[m,n]$ related to the Fourier transform of the continuous signal

$$s(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n]\, \delta_2(x-m, y-n)$$

continuous

discrete

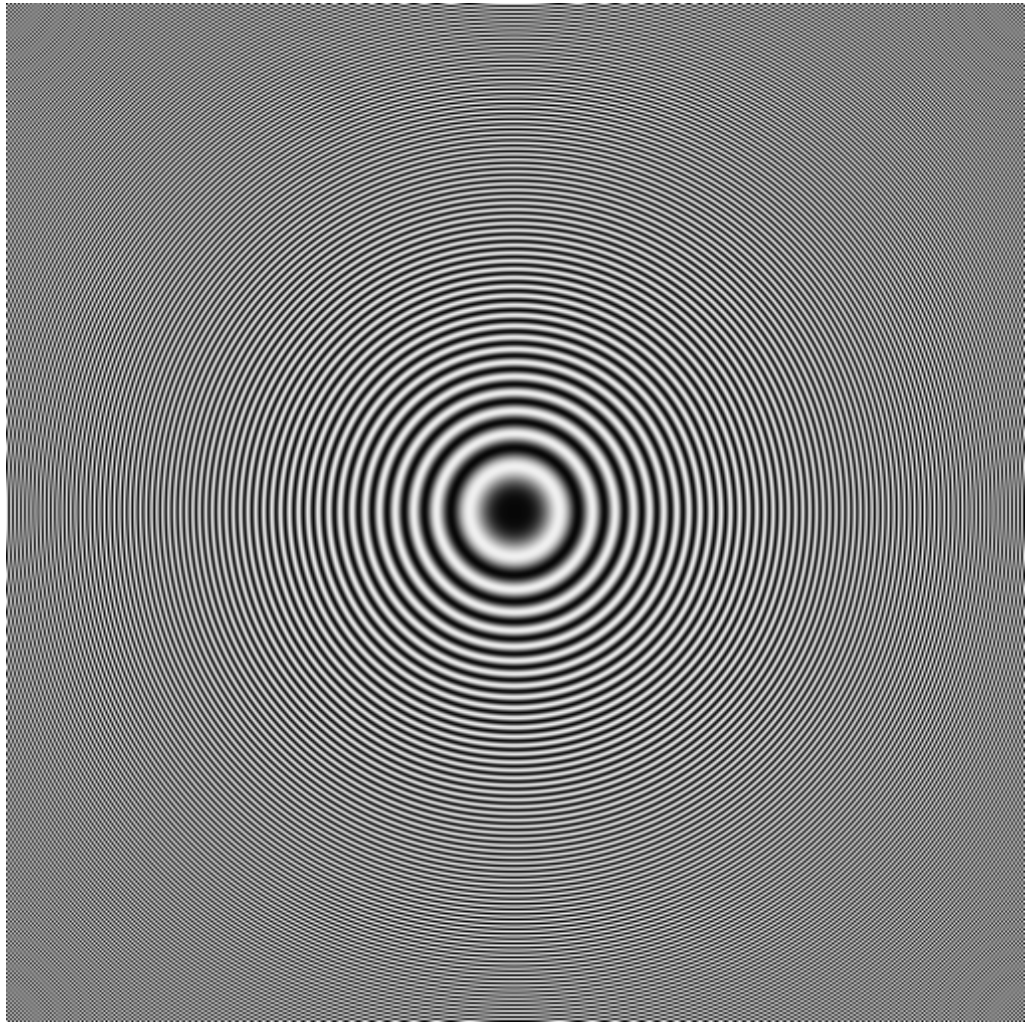
„continuous“ delta function

- Continuous-space 2-d Fourier transform

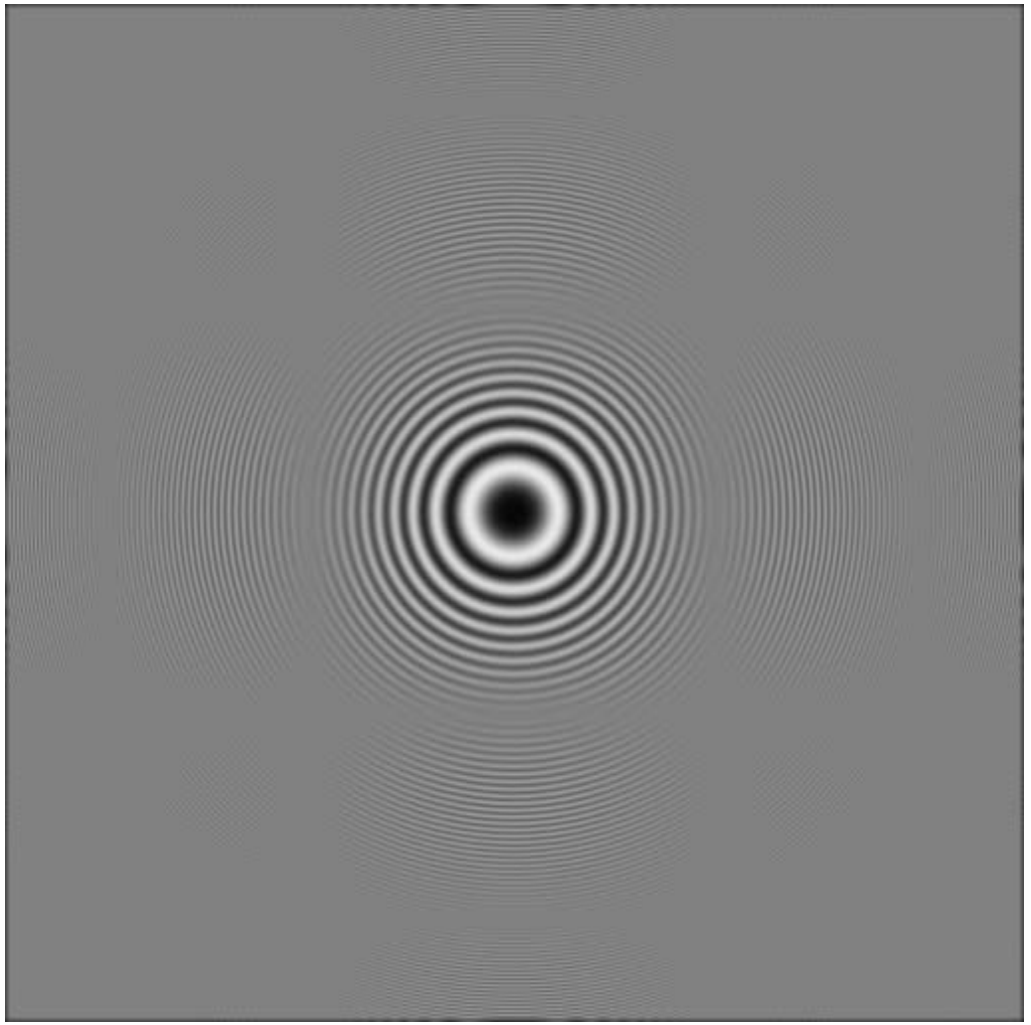
$$\hat{S}(\omega_x,\omega_y) = \int_x \int_y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n]\, \delta_2(x-m, y-n)\, e^{-j\omega_x x - j\omega_y y}\, dx\, dy$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n] e^{-j\omega_x m - j\omega_y n} = S\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

Lowpass filtering

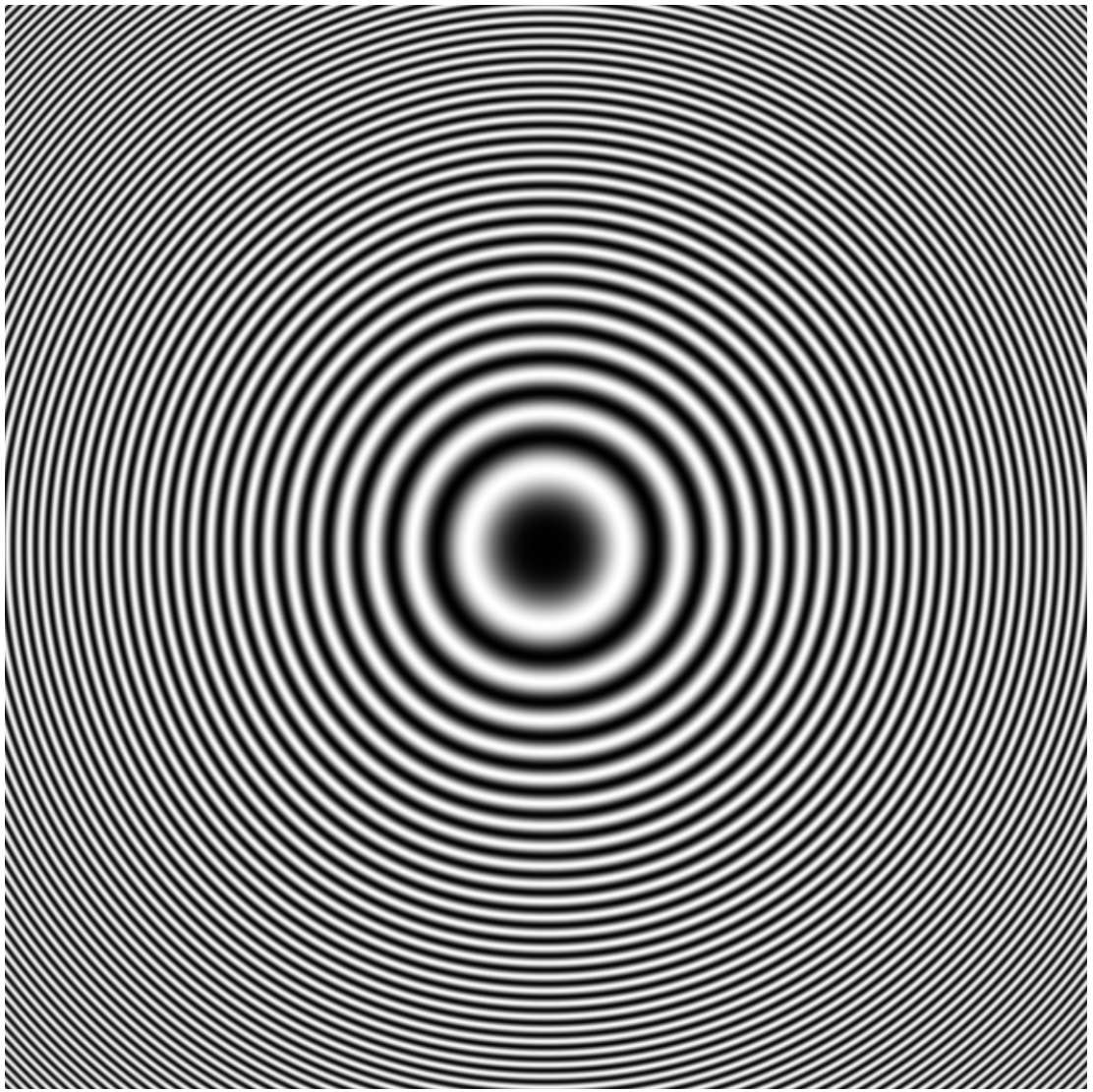


Original *Zoneplate* (512x512)

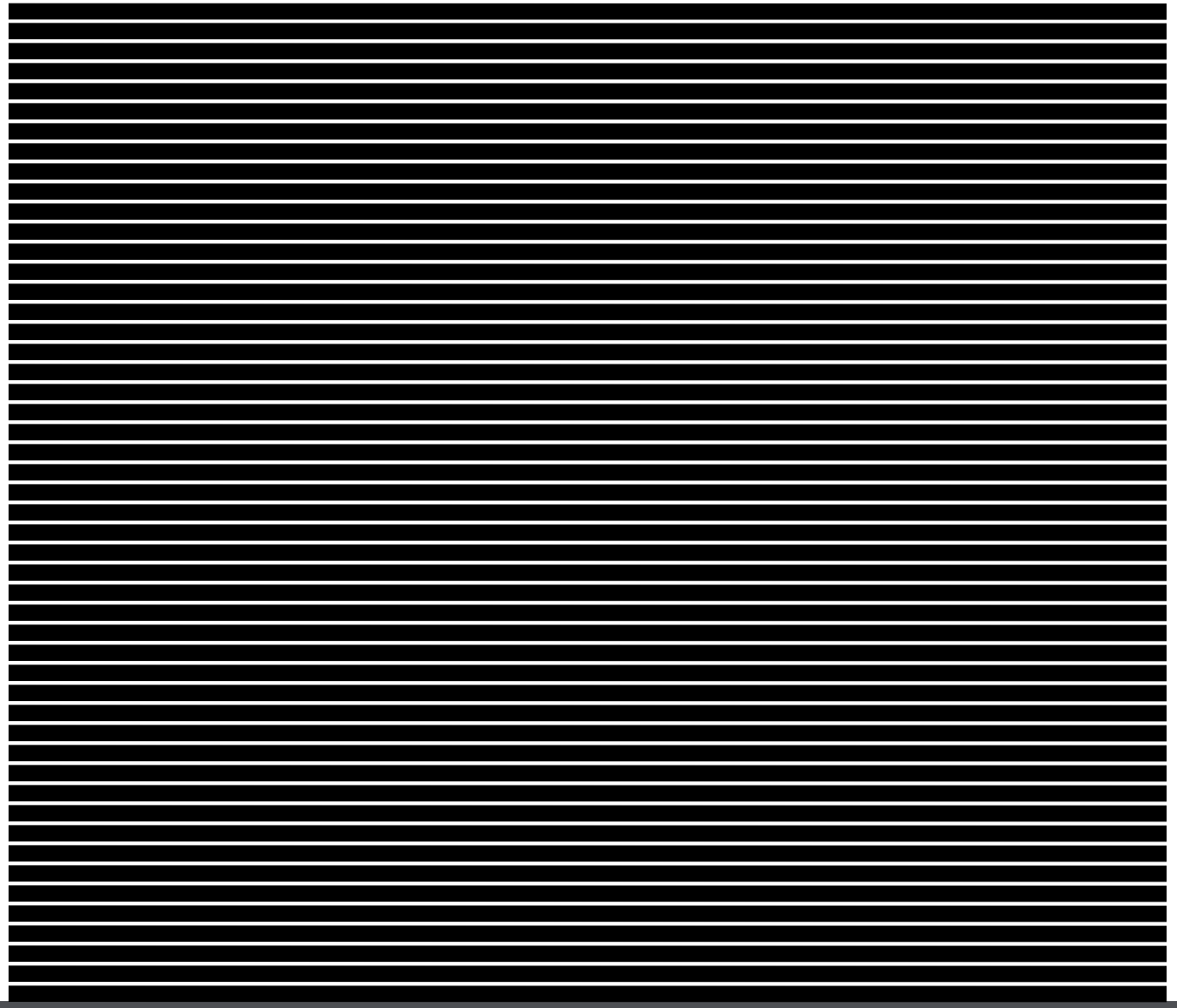


Lowpass filtered with 5x5 box filter

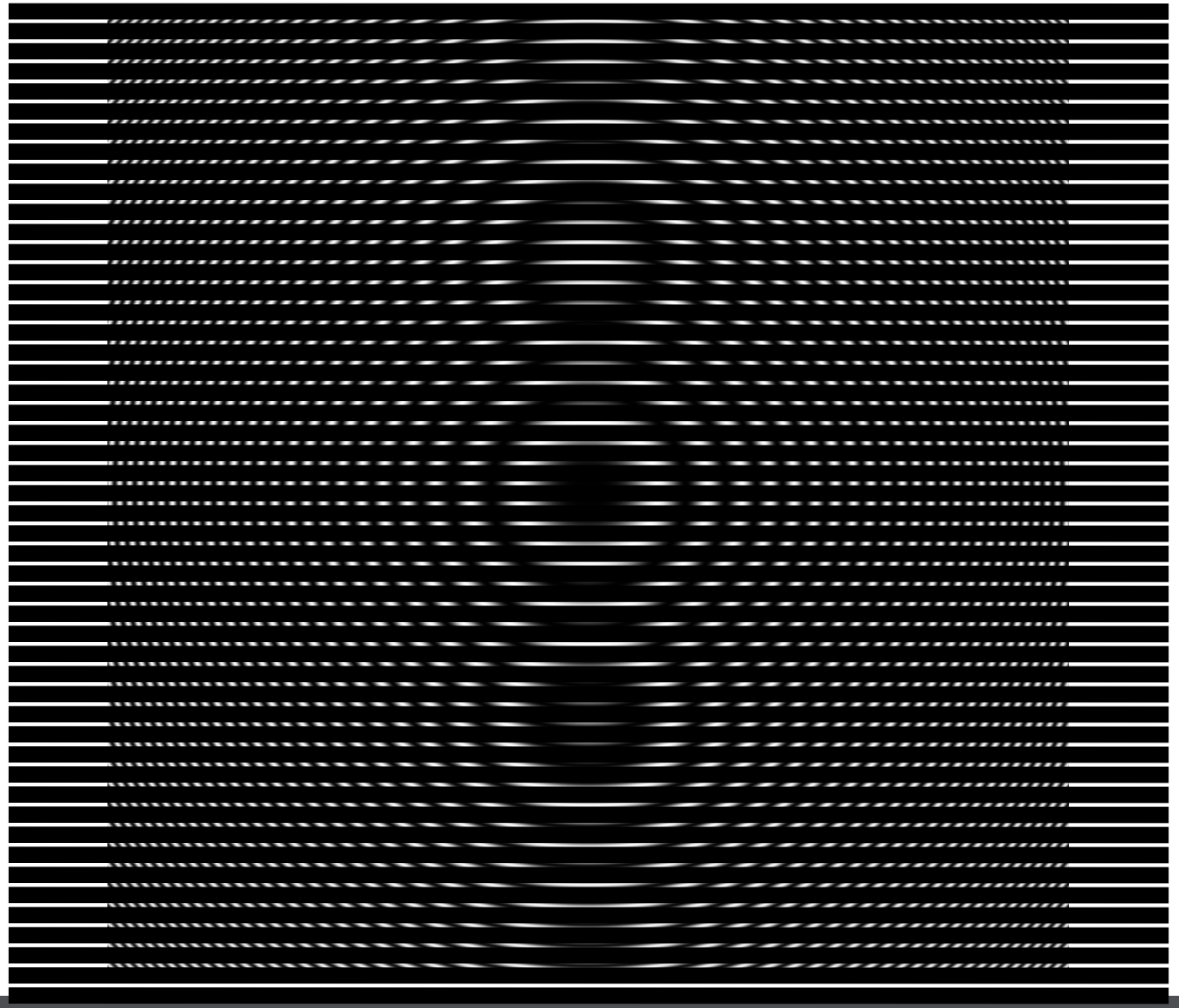
Zoneplate demonstration: 1-d sampling of 2-d signals



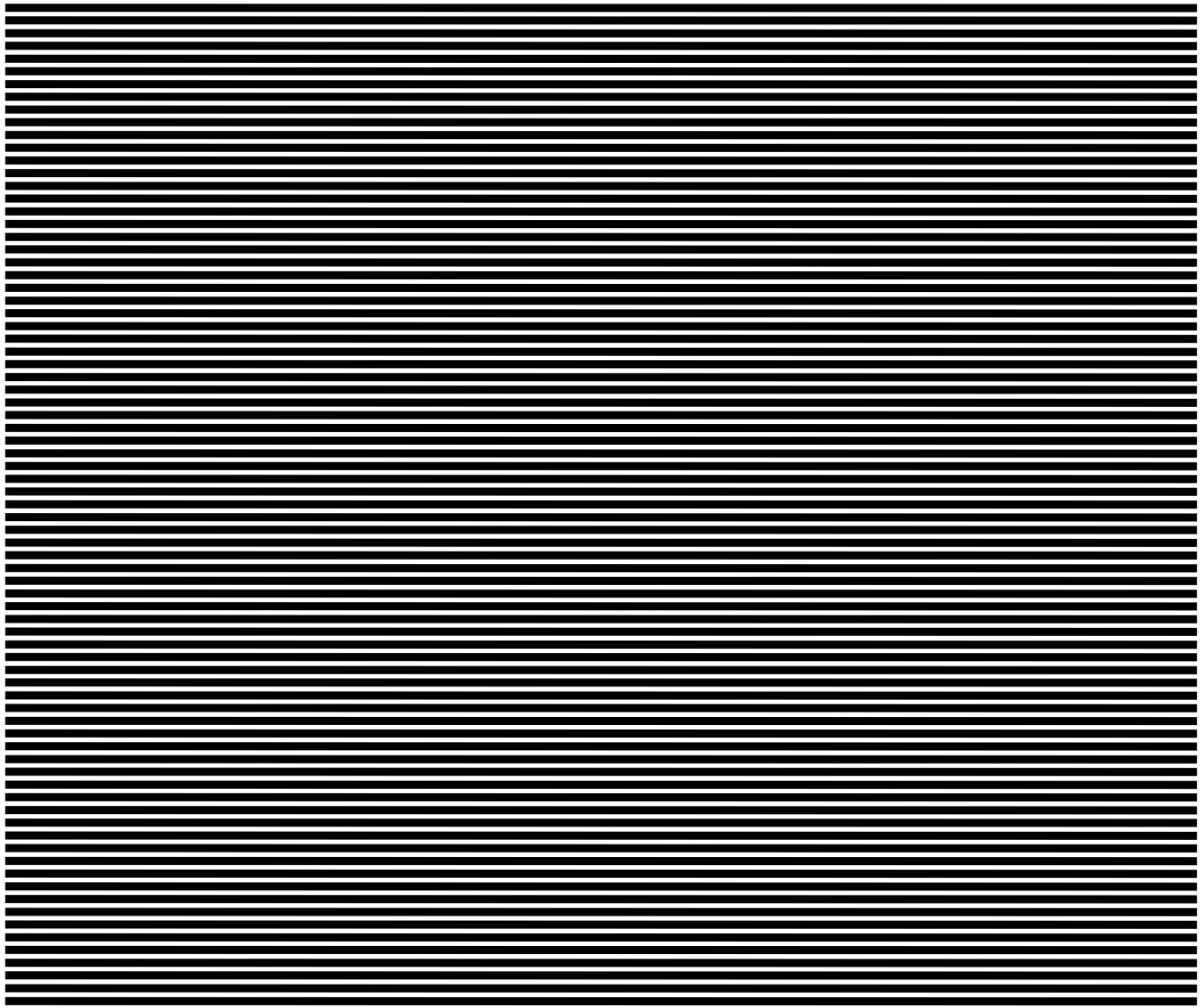
Zoneplate demonstration: 1-d sampling of 2-d signals



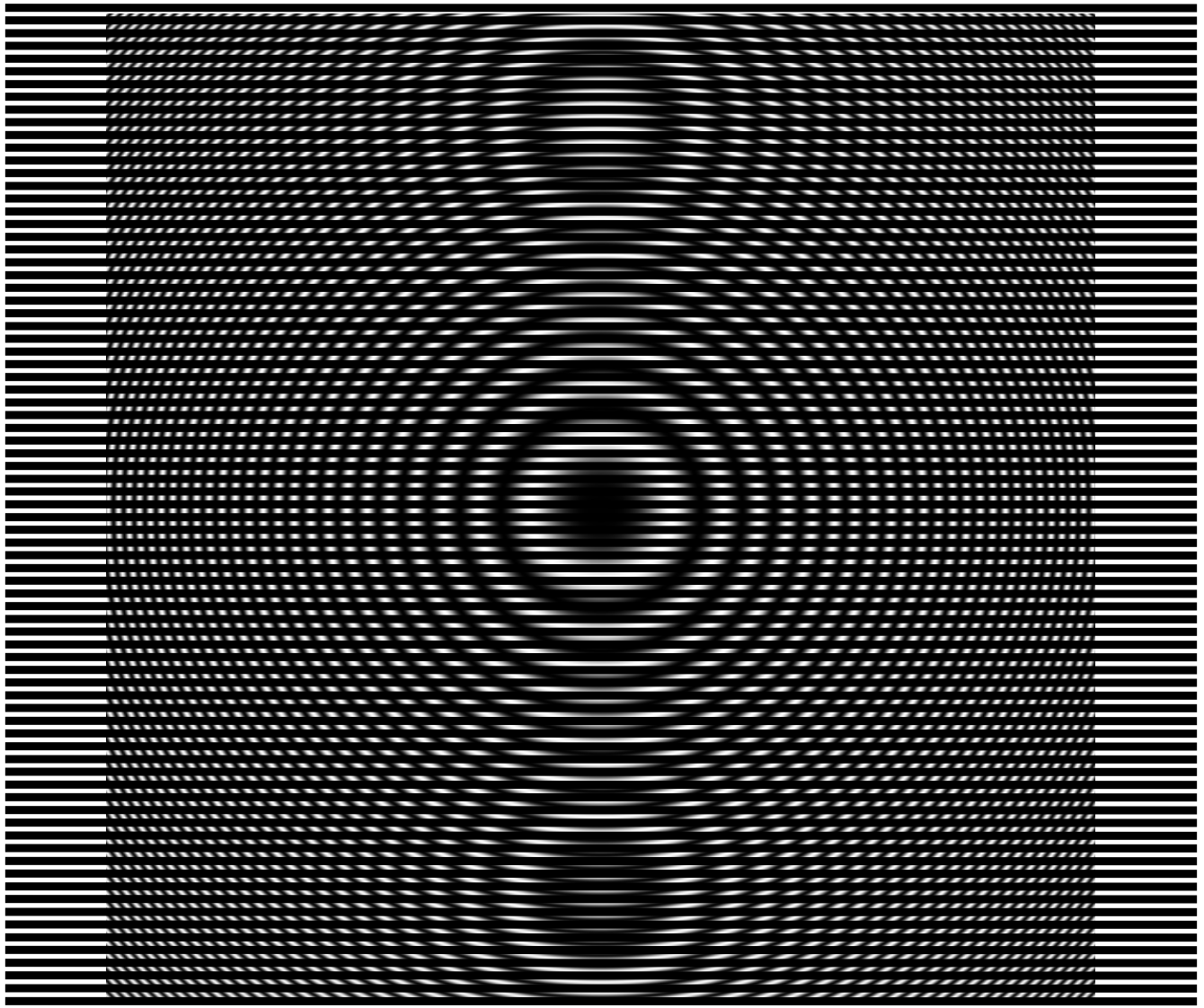
Zoneplate demonstration: 1-d sampling of 2-d signals



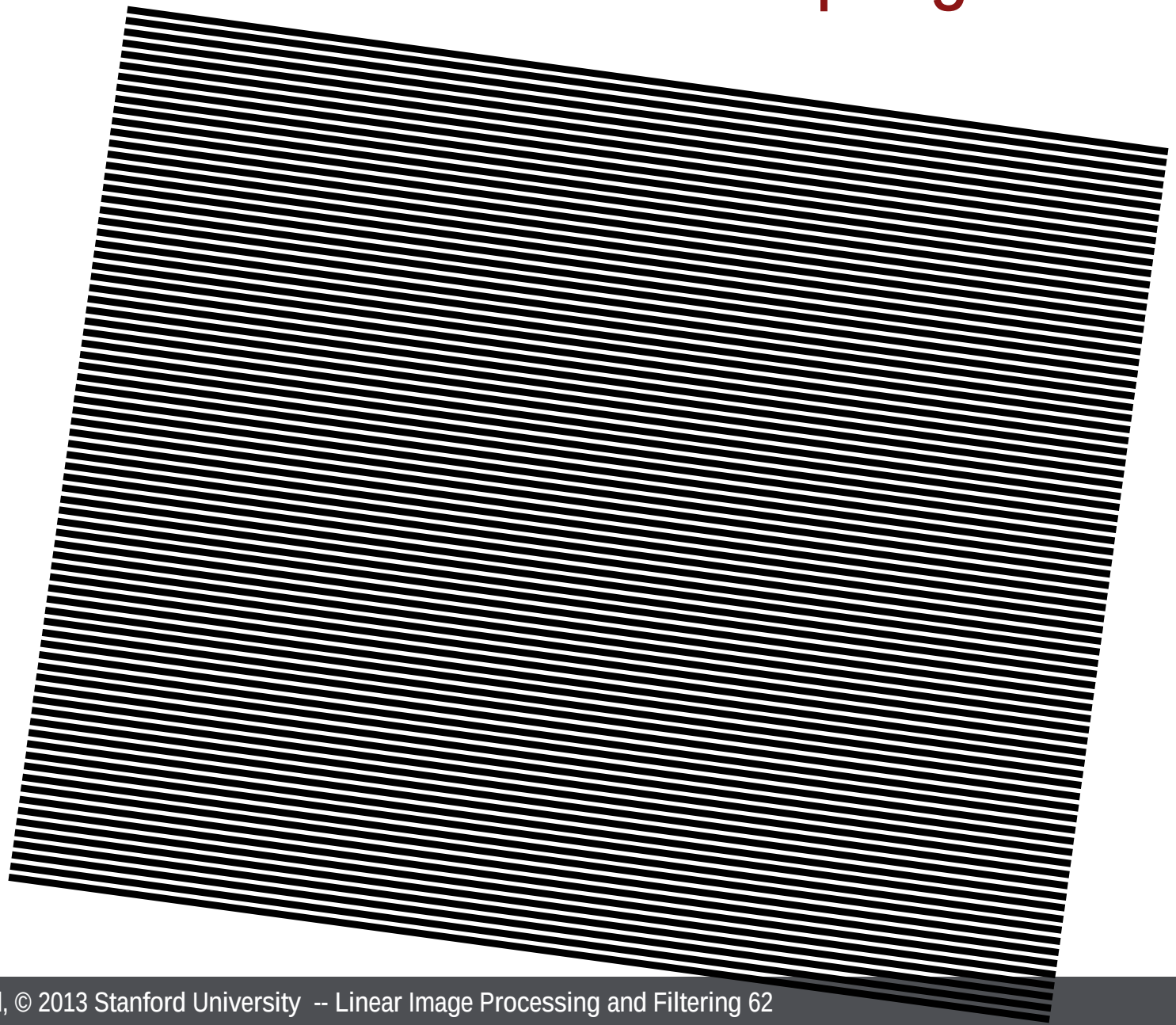
Zoneplate demonstration: 1-d sampling of 2-d signals



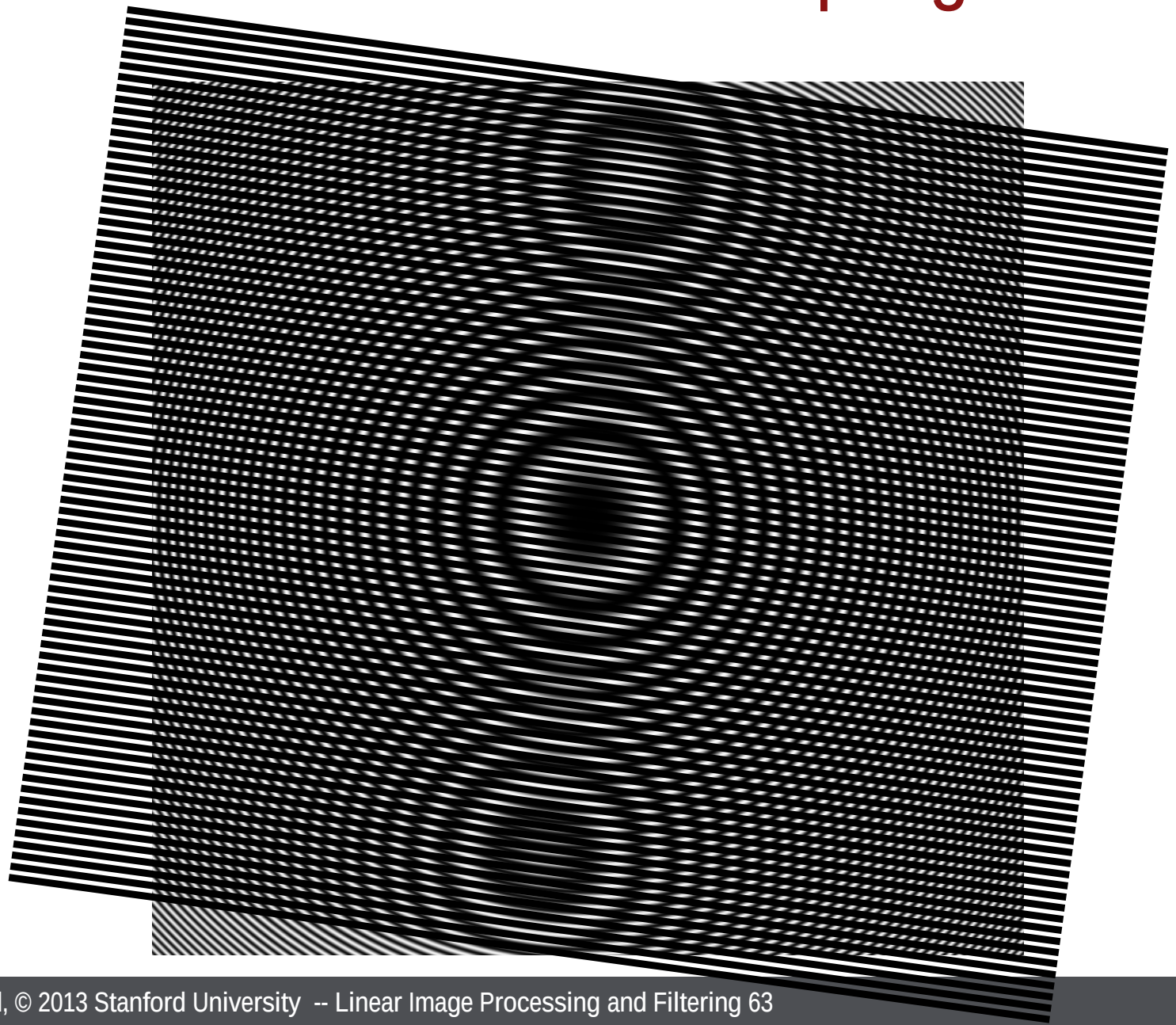
Zoneplate demonstration: 1-d sampling of 2-d signals



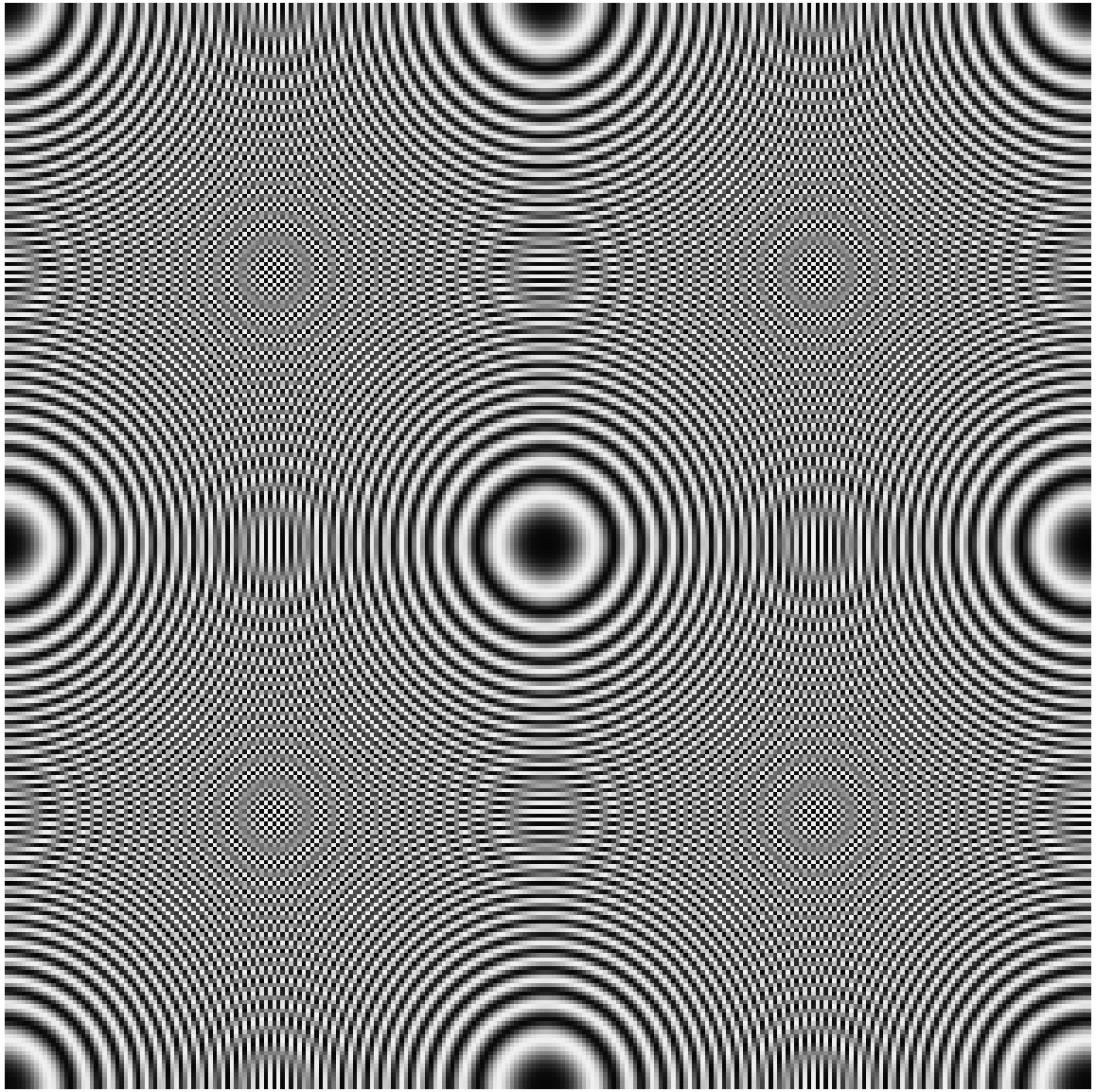
Zoneplate demonstration: 1-d sampling of 2-d signals



Zoneplate demonstration: 1-d sampling of 2-d signals



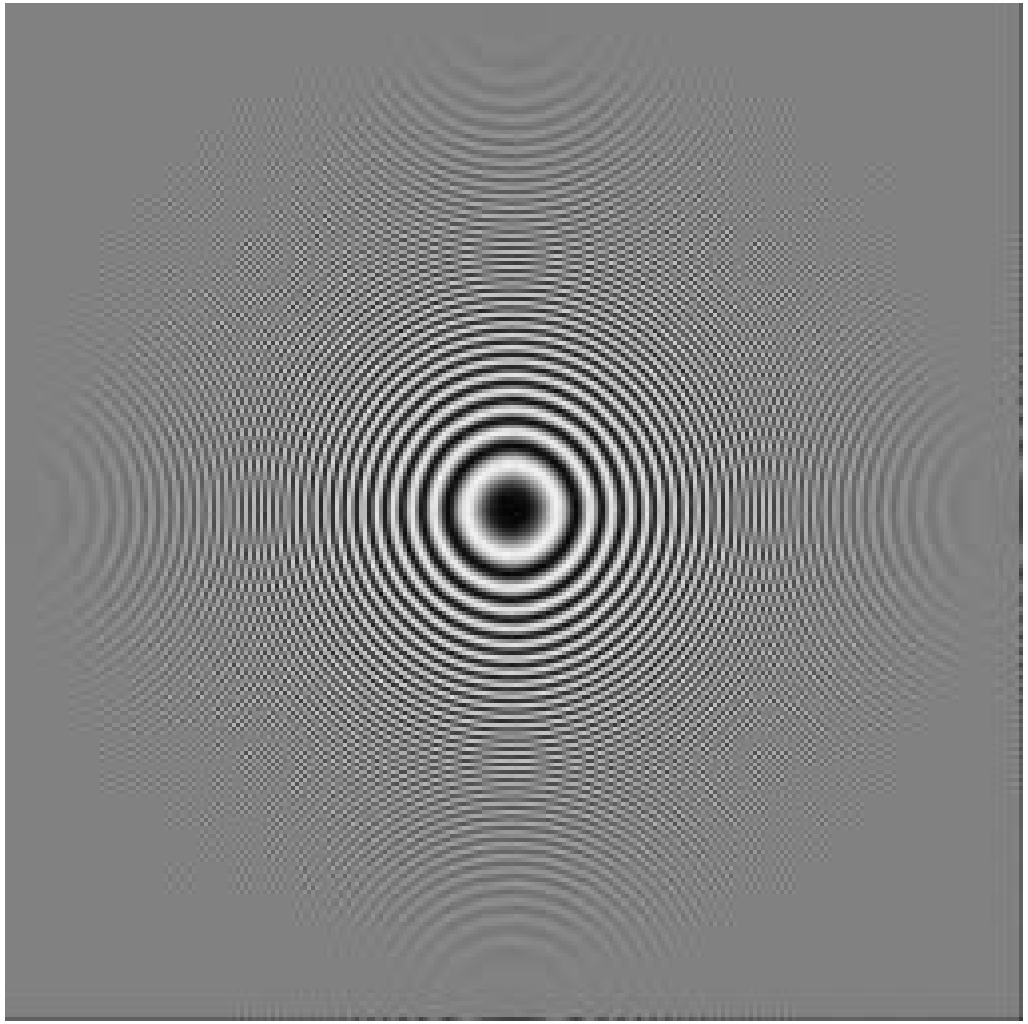
Horizontal/vertical 2:1 subsampling without prefiltering



Horizontal/vertical 2:1 subsampling with prefiltering

2d impulse response

$$\begin{pmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{pmatrix}$$



Frequency response

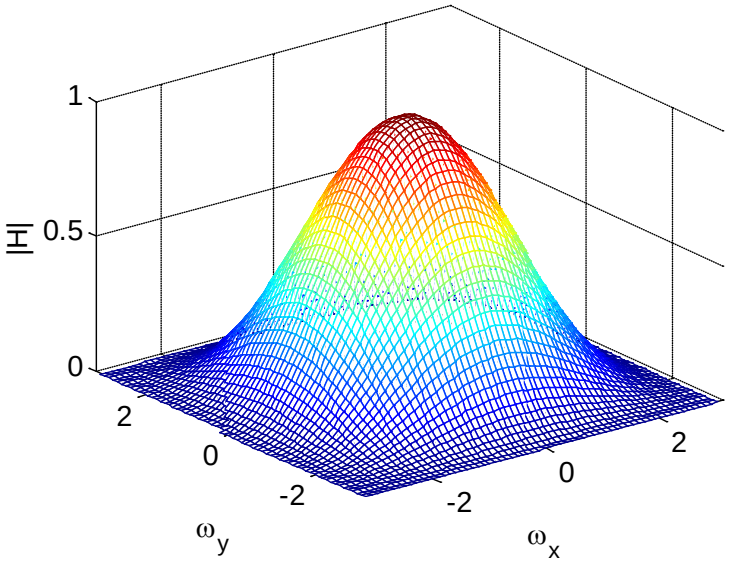
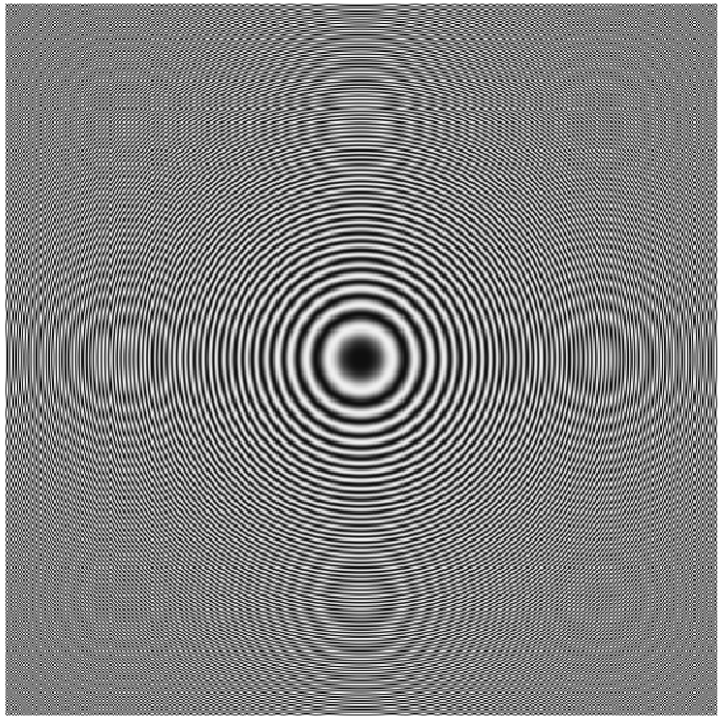
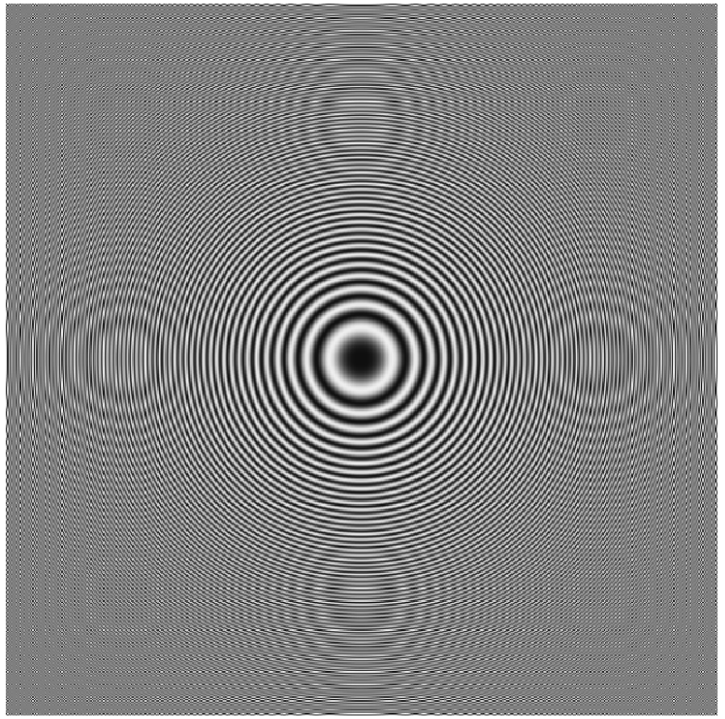


Image magnification (10%)



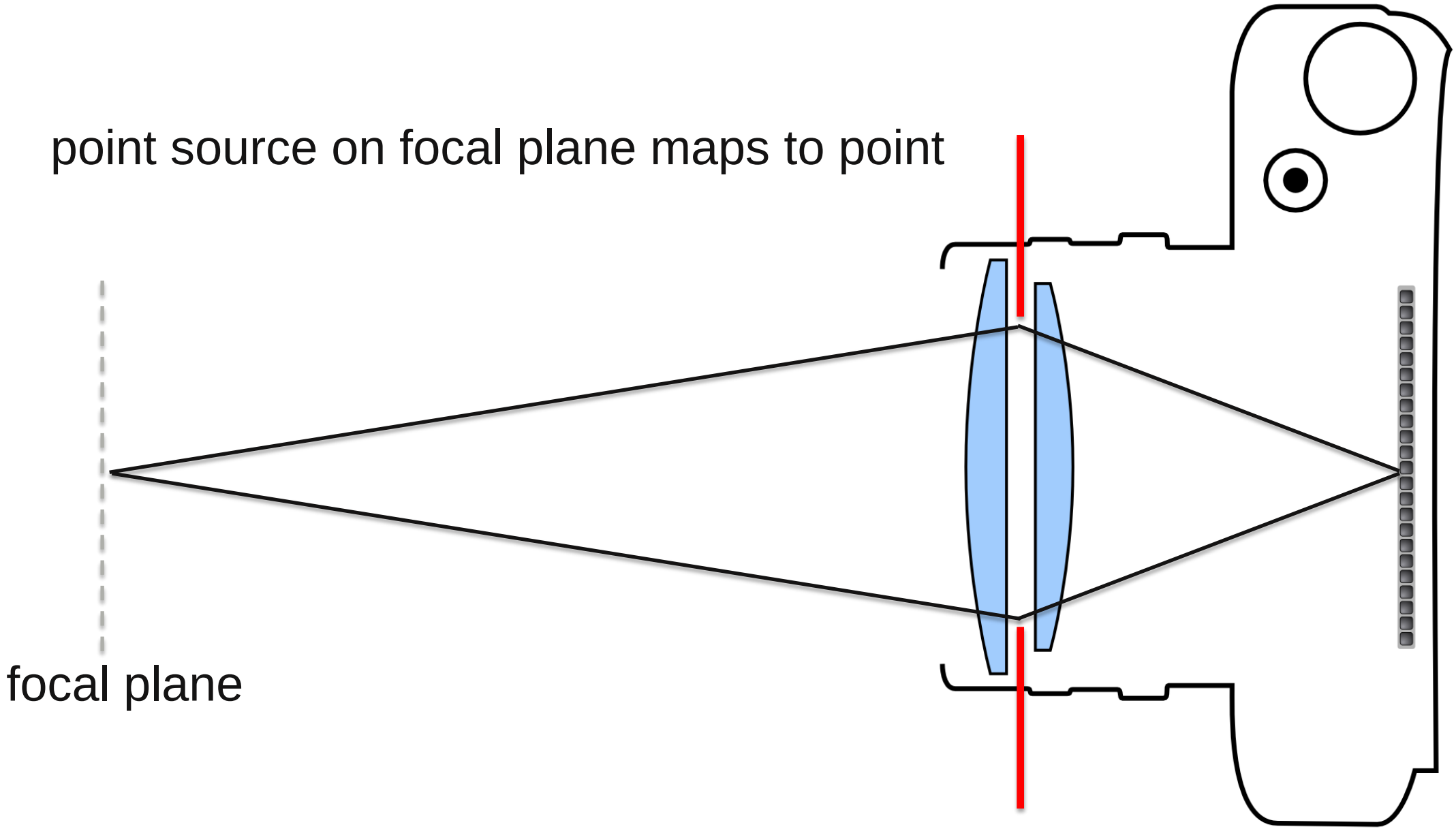
Nearest neighbor interpolation



Bilinear interpolation

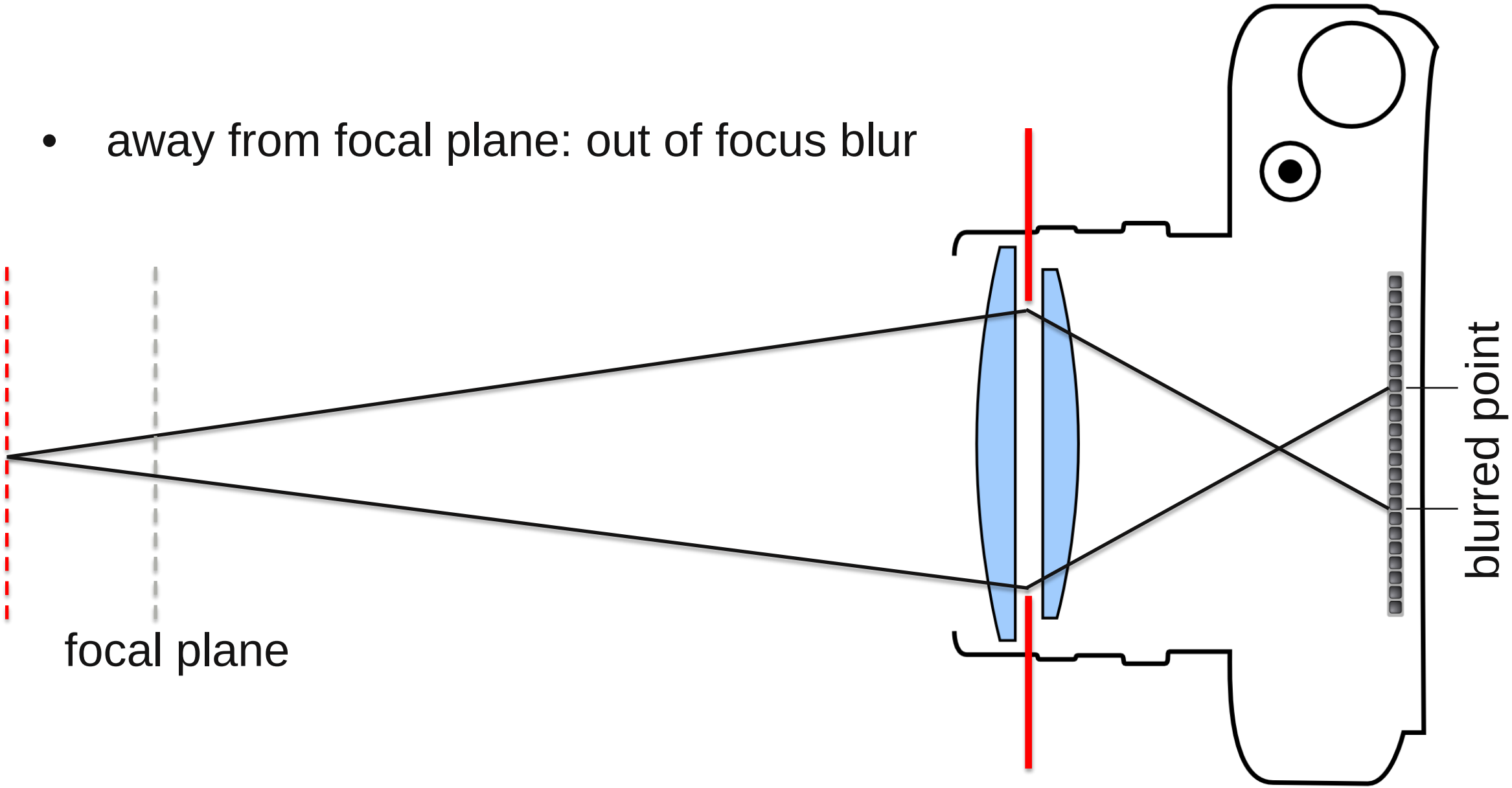
Lens as Optical Low-pass Filter

- point source on focal plane maps to point



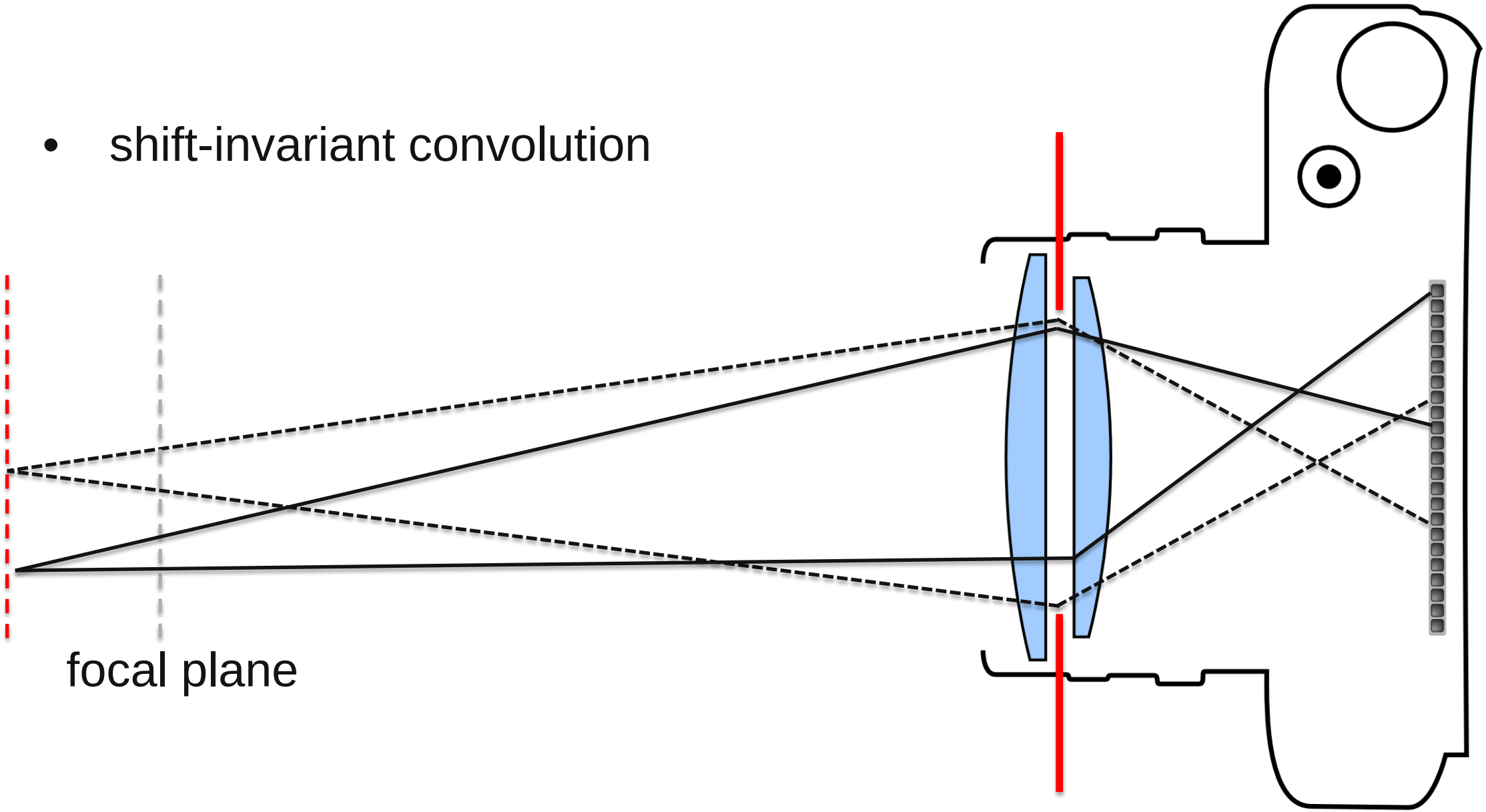
Lens as Optical Low-pass Filter

- away from focal plane: out of focus blur



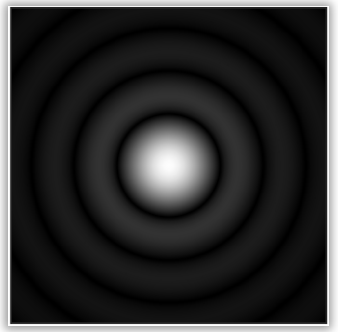
Lens as Optical Low-pass Filter

- shift-invariant convolution



Lens as Optical Low-pass Filter

diffraction-limited PSF of circular aperture (aka “Airy” pattern):



$$b = c * x$$

Deconvolution

- given measurements *b* and convolution kernel *c*, what is *x*?

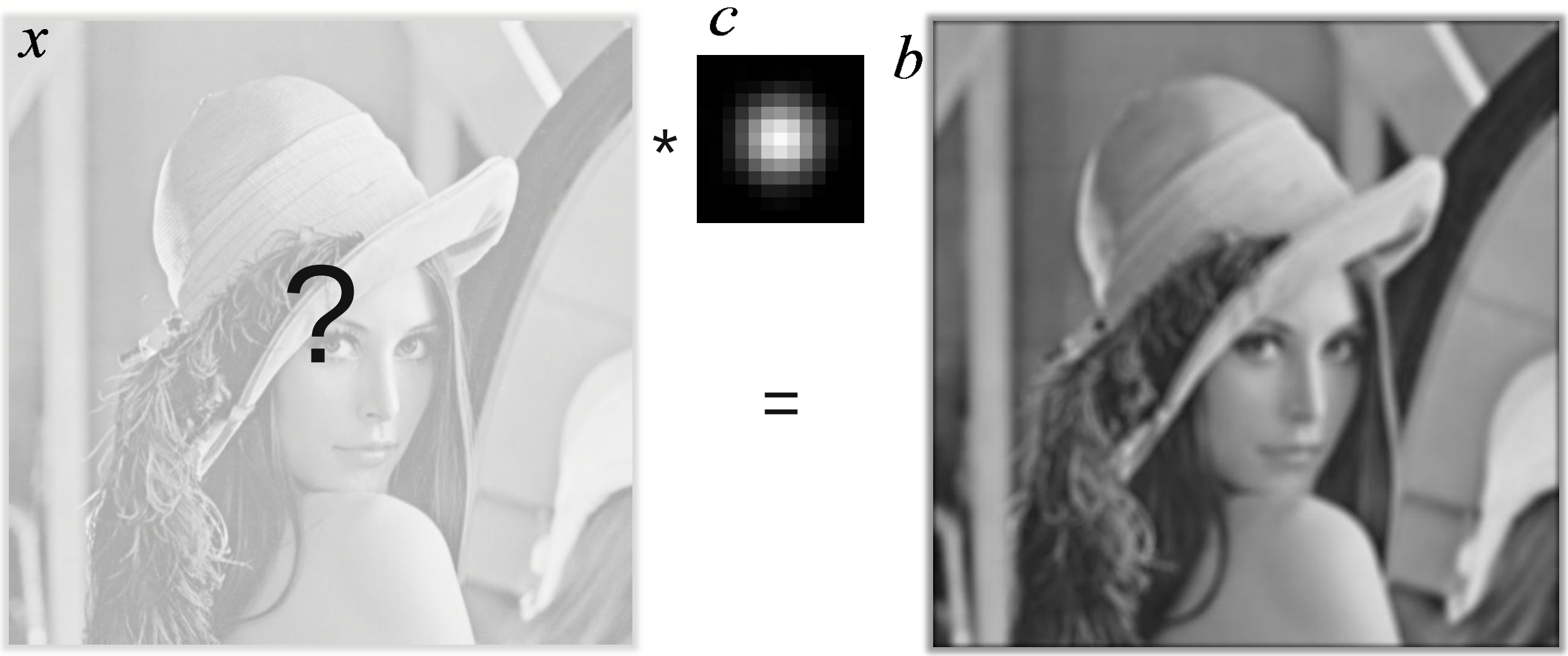
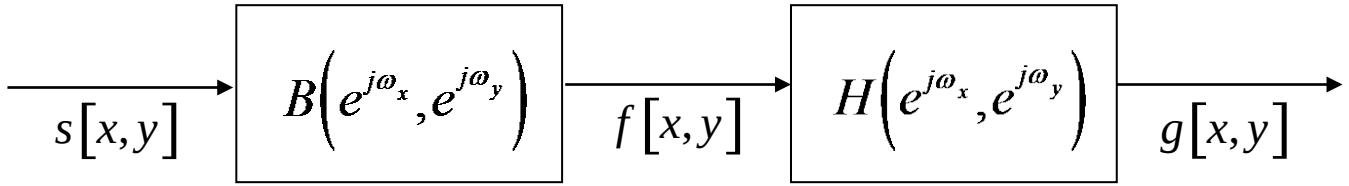


Image deconvolution

- Given an image $f[x,y]$ that is a blurred version of original image $s[x,y]$, recover the original image.
- Assume linear shift-invariant blur, transfer function $B\left(e^{j\omega_x}, e^{j\omega_y}\right)$



- Naive solution: inverse filter

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{1}{B\left(e^{j\omega_x}, e^{j\omega_y}\right)}$$

- Problem: $B\left(e^{j\omega_x}, e^{j\omega_y}\right)$ might be zero, noise amplification

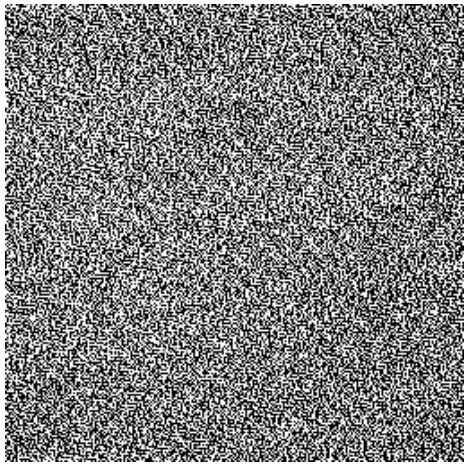
Naïve deconvolution

Original

Blurred w/ $B\left(e^{j\omega_x}, e^{j\omega_y}\right)$

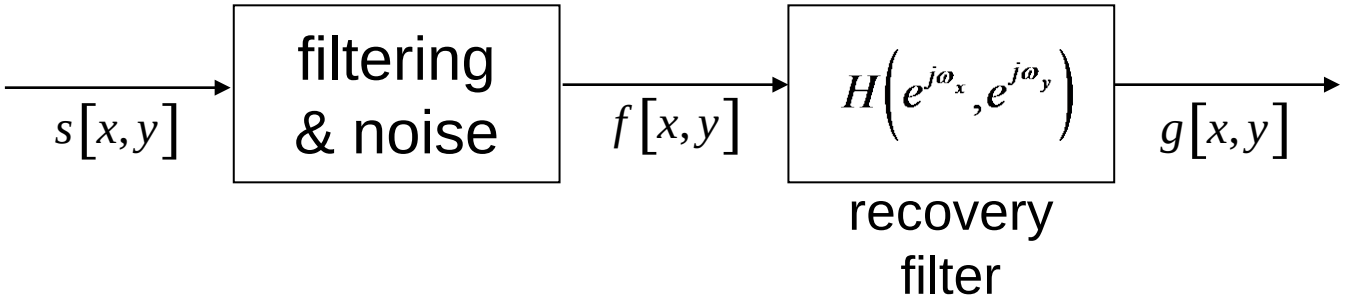
Deblurred w/ $B^{-1}\left(e^{j\omega_x}, e^{j\omega_y}\right)$

+ white noise $rms = 2.5$



Wiener filtering

- Model



- Minimize mean squared estimation error

$$E\left\{e^2[x,y]\right\} = E\left\{\left(g[x,y]-s[x,y]\right)^2\right\} \xrightarrow{H\left(e^{j\omega_x}, e^{j\omega_y}\right)} \min.$$

- Power spectral density of estimation error

$$\begin{aligned}\Phi_{ee}\left(e^{j\omega_x}, e^{j\omega_y}\right) &= \Phi_{gg}\left(e^{j\omega_x}, e^{j\omega_y}\right) - \Phi_{gs}\left(e^{j\omega_x}, e^{j\omega_y}\right) - \Phi_{sg}\left(e^{j\omega_x}, e^{j\omega_y}\right) + \Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) \\ &= \Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right) H\left(e^{j\omega_x}, e^{j\omega_y}\right) H^*\left(e^{j\omega_x}, e^{j\omega_y}\right) - \Phi_{fs}\left(e^{j\omega_x}, e^{j\omega_y}\right) H\left(e^{j\omega_x}, e^{j\omega_y}\right) \\ &\quad - \Phi_{sf}\left(e^{j\omega_x}, e^{j\omega_y}\right) H^*\left(e^{j\omega_x}, e^{j\omega_y}\right) + \Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right)\end{aligned}$$

Review: Power spectrum and cross spectrum

- 2-d discrete-space cross correlation function for ergodic, stationary signals

$$\varphi_{fg}\left[m,n\right] = E\left\{f\left[x+m,y+n\right]g^{*}\left[x,y\right]\right\}$$

- Special case: autocorrelation function

$$\varphi_{ff}\left[m,n\right] = E\left\{f\left[x+m,y+n\right]f^{*}\left[x,y\right]\right\}$$

- Cross spectral density

$$\Phi_{fg}\left(e^{j\omega_x},e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\varphi_{fg}\left[m,n\right]e^{-j\omega_x m - j\omega_y n}$$

- Power spectral density

$$\Phi_{ff}\left(e^{j\omega_x},e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\varphi_{ff}\left[m,n\right]e^{-j\omega_x m - j\omega_y n}$$

Wiener filtering (cont.)

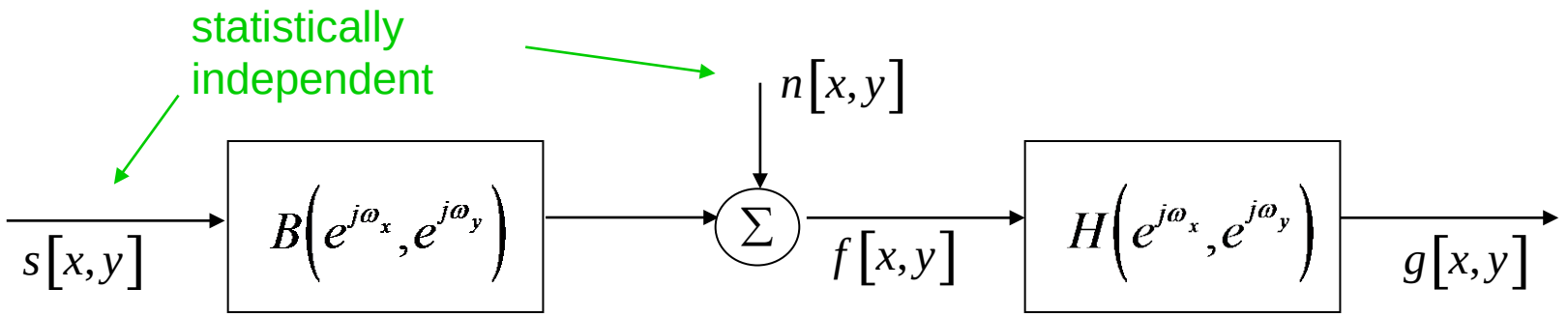
- Power spectrum Φ_{ee} is minimized separately at each frequency ω_x, ω_y if

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{\Phi^*_{fs}\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right)} = \frac{\Phi_{sf}\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right)}$$

- Can be shown to be global minimum by considering filter

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{\Phi^*_{fs}\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right)} + \Delta H\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

Wiener filter for linear distortion and additive noise



$$\Phi_{sf}\left(e^{j\omega_x}, e^{j\omega_y}\right) = \Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) B^*\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

$$\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right) = \Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right)\left|B\left(e^{j\omega_x}, e^{j\omega_y}\right)\right|^2 + \Phi_{nn}\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) B^*\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right)\left|B\left(e^{j\omega_x}, e^{j\omega_y}\right)\right|^2 + \Phi_{nn}\left(e^{j\omega_x}, e^{j\omega_y}\right)}$$

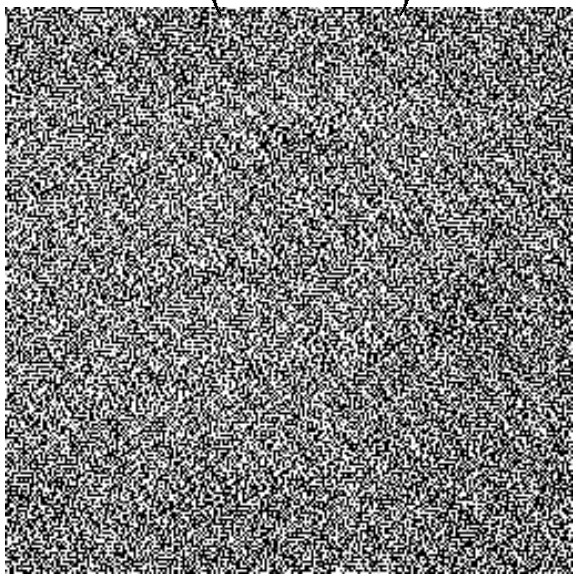
Wiener filter for linear distortion and additive noise

Blurred w/ $B\left(e^{j\omega_x}, e^{j\omega_y}\right)$

Add. white noise, *rms* = 2.5

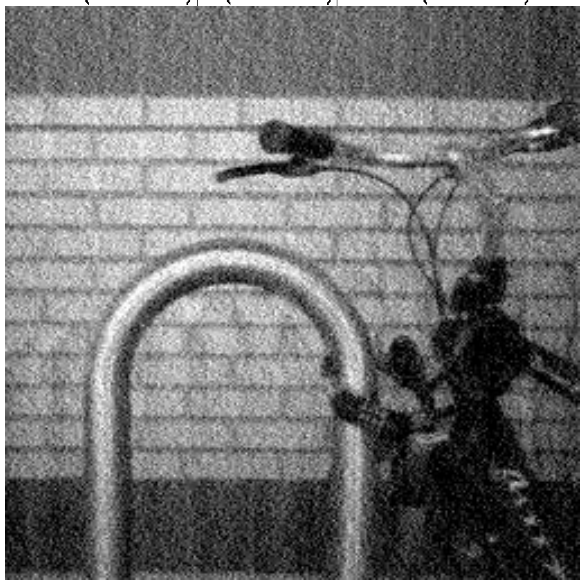
Deblurred

$$B^{-1}\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

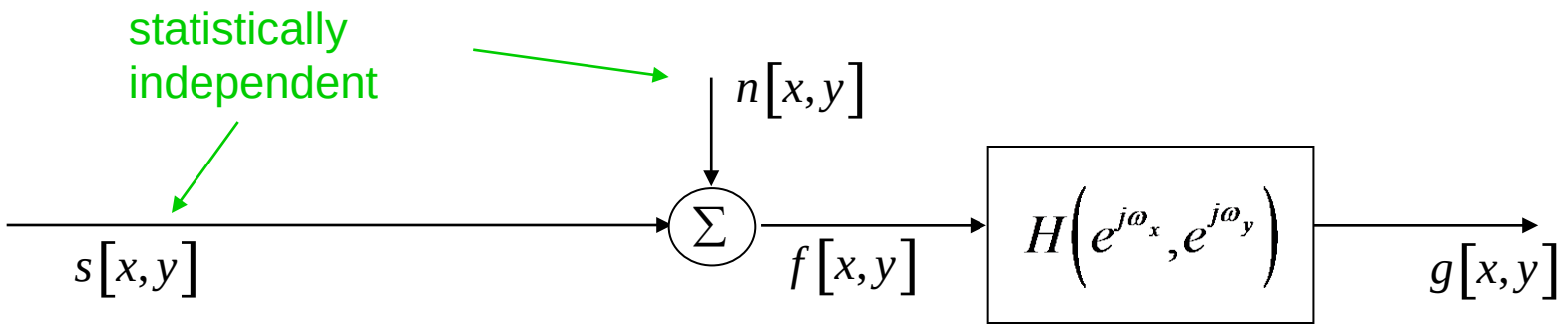


Wiener Filter

$$\frac{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) B^*\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) \left|B\left(e^{j\omega_x}, e^{j\omega_y}\right)\right|^2 + \Phi_{nn}\left(e^{j\omega_x}, e^{j\omega_y}\right)}$$



Wiener filter for additive noise



$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) + \Phi_{nn}\left(e^{j\omega_x}, e^{j\omega_y}\right)}$$

Wiener filter for additive noise

Original

Additive white noise

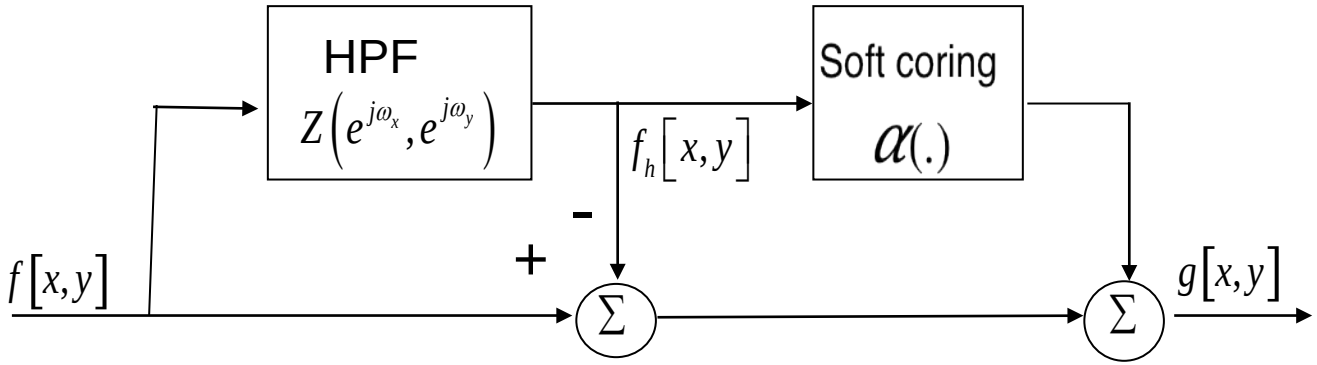
$rms = 17.7$

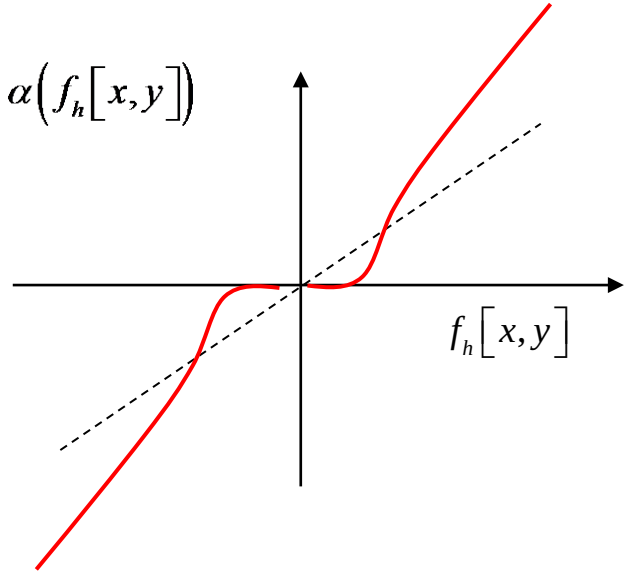
Noise reduction by Wiener filtering

$rms = 8.8$

Nonlinear noise reduction/sharpening

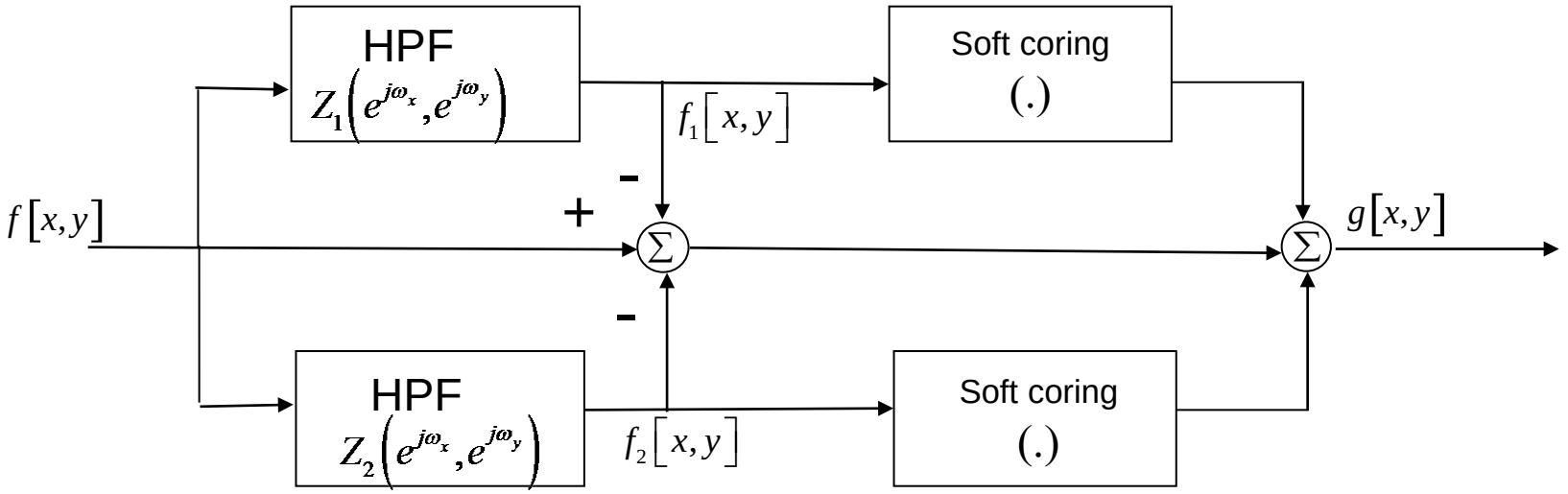
- Noise reduction: smooth the image, lowpass filtering
- Deblurring: sharpen edges, highpass filtering
- How can both be achieved simultaneously?
- Key insight: large amplitude of highpass filtered image indicates presence of edge

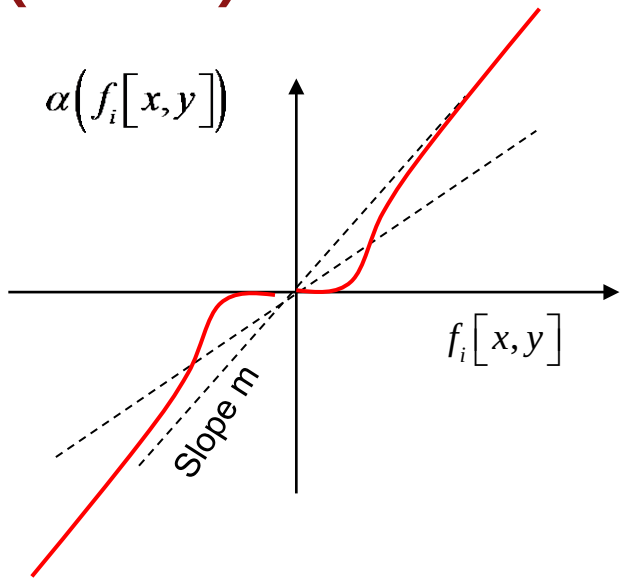




- Can be extended to multiple HPFs

Nonlinear noise reduction/sharpening (cont.)





- Flat areas:

$$\alpha \approx 0 \;\rightarrow\; H\left(e^{j\omega_x}, e^{j\omega_y}\right) \approx 1 - Z_1\left(e^{j\omega_x}, e^{j\omega_y}\right) - Z_2\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

- f_1 small; f_2 large:

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) \approx 1 - Z_1\left(e^{j\omega_x}, e^{j\omega_y}\right) + (m-1) Z_2\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

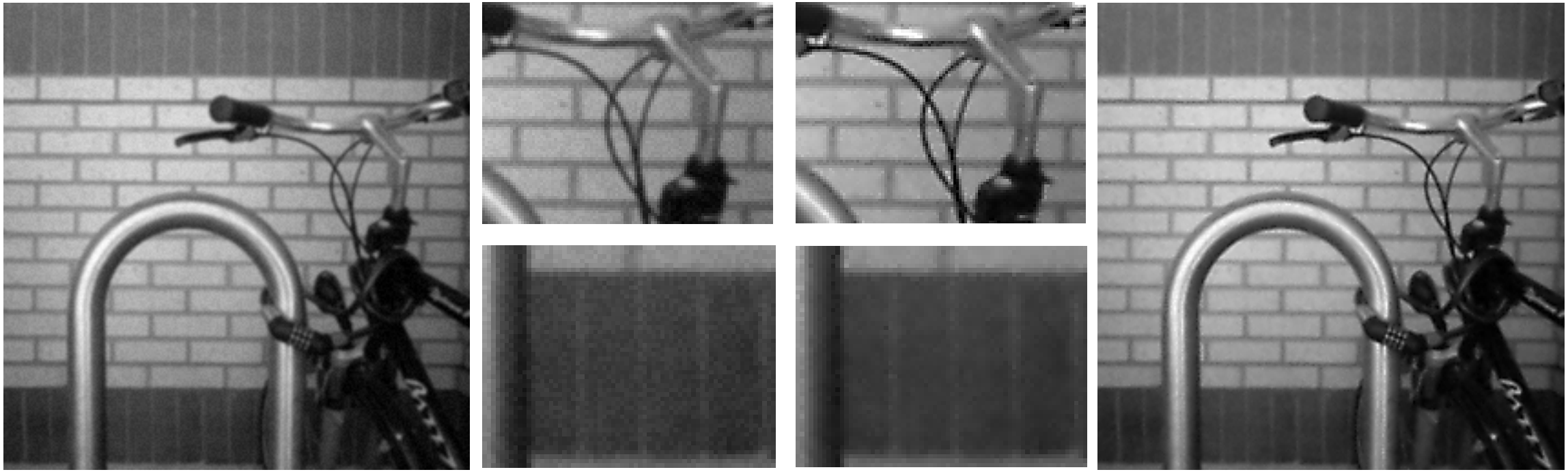
- f_1 large; f_2 small:

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) \approx 1 + (m-1) Z_1\left(e^{j\omega_x}, e^{j\omega_y}\right) - Z_2\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

- Both large:

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) \approx 1 + (m-1)\left[Z_1\left(e^{j\omega_x}, e^{j\omega_y}\right) + Z_2\left(e^{j\omega_x}, e^{j\omega_y}\right)\right]$$

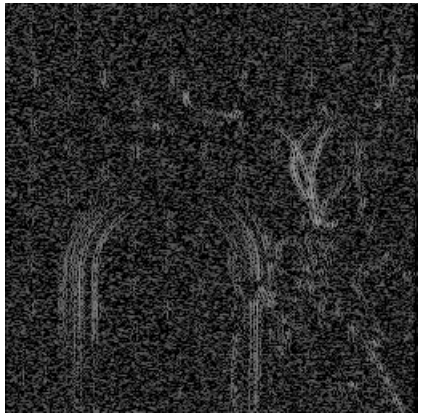
Nonlinear noise reduction/sharpening example



blurred, noisy image

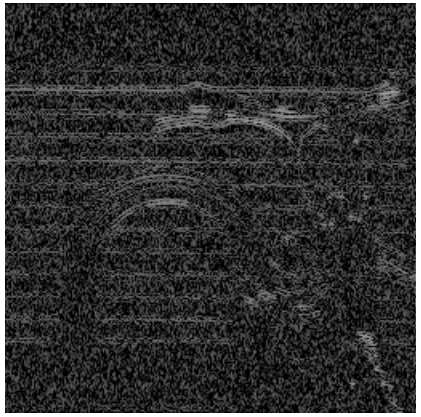
noise-reduced
and sharpened

Highpass filtered images



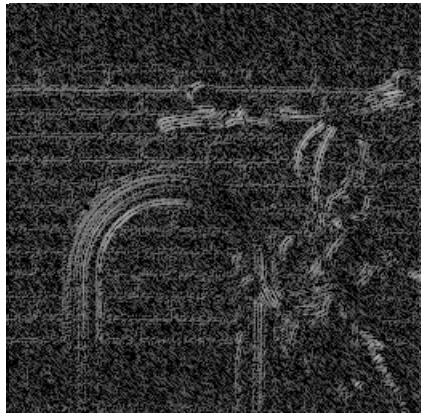
log magnitude of image filtered with

$$\begin{pmatrix} 0 & 0 & 0 \\ -0.5 & [1] & -0.5 \\ 0 & 0 & 0 \end{pmatrix}$$



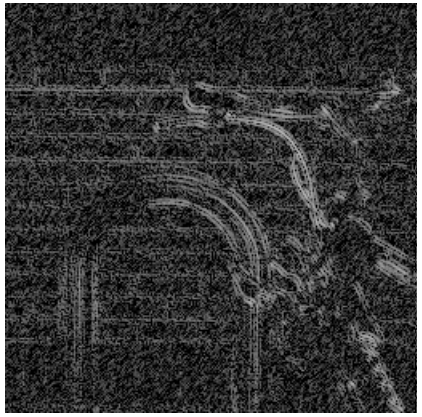
log magnitude of image filtered with

$$\begin{pmatrix} 0 & -0.5 & 0 \\ 0 & [1] & 0 \\ 0 & -0.5 & 0 \end{pmatrix}$$



log magnitude of image filtered with

$$\begin{pmatrix} -0.5 & 0 & 0 \\ 0 & [1] & 0 \\ 0 & 0 & -0.5 \end{pmatrix}$$

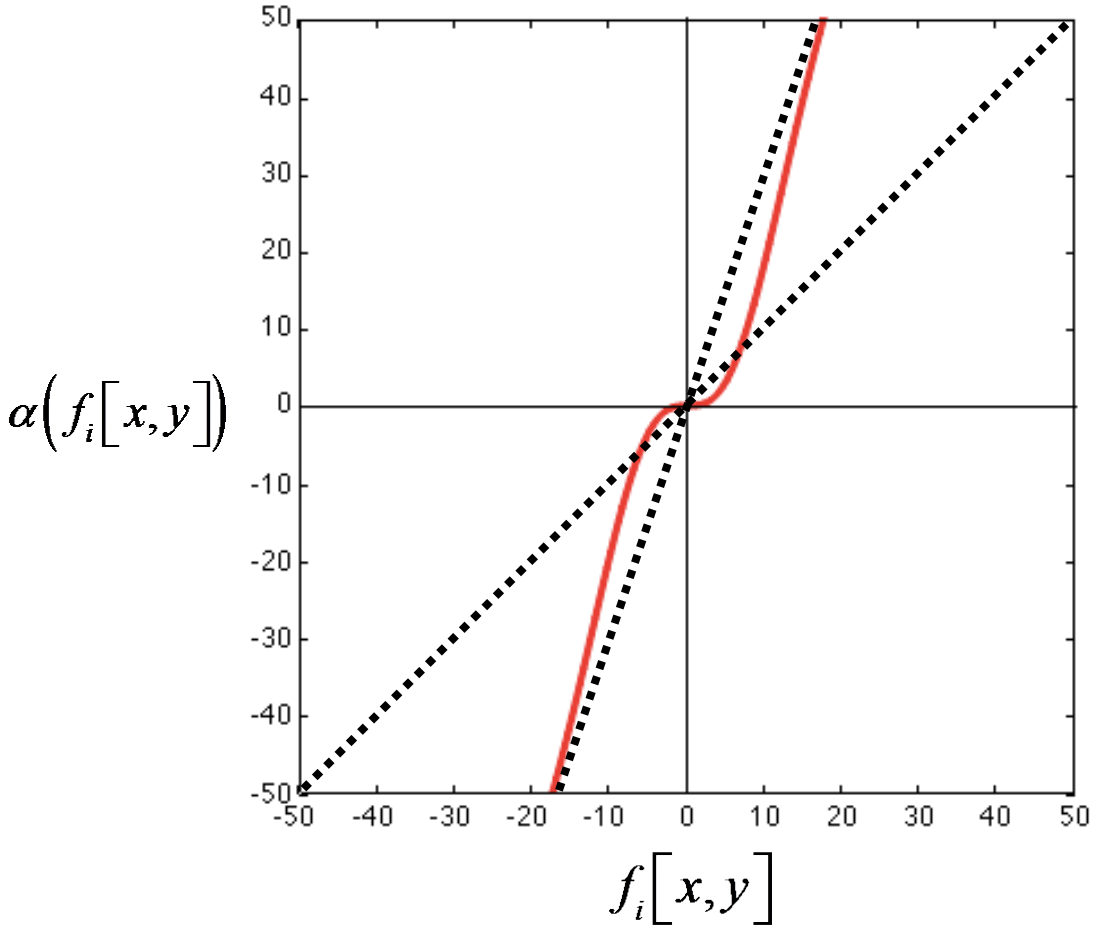


log magnitude of image filtered with

$$\begin{pmatrix} 0 & 0 & -0.5 \\ 0 & [1] & 0 \\ -0.5 & 0 & 0 \end{pmatrix}$$

Soft coring function

$$\alpha\left(f_i\left[x,y\right]\right) = m \cdot f_i\left[x,y\right] \cdot \left(1 - e^{-\left|\frac{f_i\left[x,y\right]}{\tau}\right|^{\gamma}}\right)$$



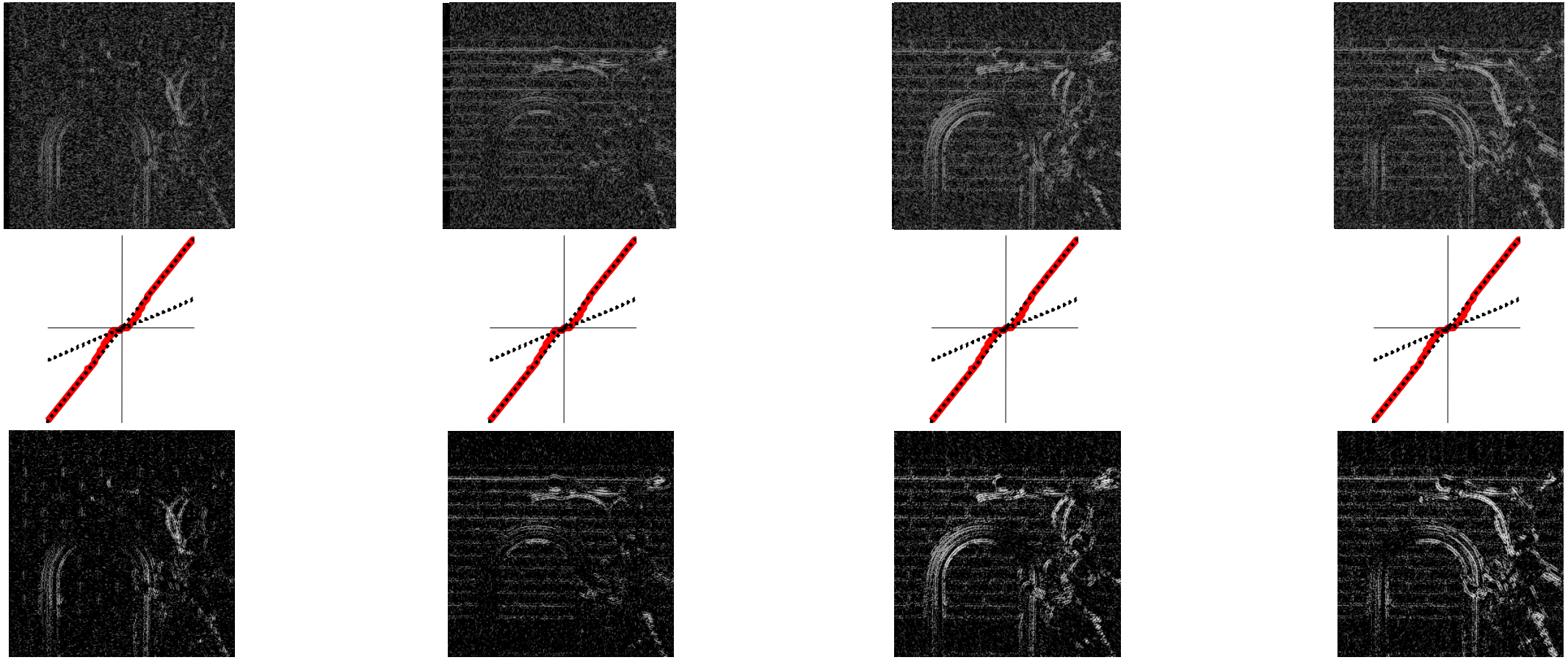
Example:

$m = 3$

$\gamma = 2$

$\tau = 10$

Soft coring of highpass filtered images



Linear vs. nonlinear noise reduction/sharpening

Noise reduction by lowpass filter (linear)

Sharpening by highpass filter (linear)

Combined noise reduction and sharpening (nonlinear)