

# Eigenimages

- Unitary transforms
- Karhunen-Loève transform and eigenimages
- Sirovich and Kirby method
- Eigenfaces for gender recognition
- Fisher linear discriminant analysis
- Fisherimages and varying illumination
- Fisherfaces vs. eigenfaces

# Unitary transforms

- Sort pixels  $f[x,y]$  of an image into column vector  $\vec{f}$  of length  $N$
- Calculate  $N$  transform coefficients

$$\vec{c} = A\vec{f}$$

where  $A$  is a matrix of size  $N \times N$

- The transform  $A$  is unitary, iff

$$A^{-1} = \underbrace{A^{*T}}_{\text{Hermitian conjugate}} \equiv A^H$$

- If  $A$  is real-valued, i.e.,  $A=A^*$ , transform is „orthonormal“

# Energy conservation with unitary transforms

- For any unitary transform  $\vec{c} = A\vec{f}$  we obtain

$$\|\vec{c}\|^2 = \vec{c}^H \vec{c} = \vec{f}^H A^H A \vec{f} = \|\vec{f}\|^2$$

- Interpretation: every unitary transform is simply a rotation of the coordinate system (and, possibly, sign flips)
- Vector length is conserved.
- Energy (mean squared vector length) is conserved.

# Energy distribution for unitary transforms

- Energy is conserved, but, in general, unevenly distributed among coefficients.
- Autocorrelation matrix

$$R_{cc} = E[\vec{c}\vec{c}^H] = E[A\vec{f} \cdot \vec{f}^H A^H] = AR_{ff}A^H$$

- Diagonal of  $R_{cc}$  comprises mean squared values („energies“) of the coefficients  $c_i$

$$E[c_i^2] = [R_{cc}]_{i,i} = [AR_{ff}A^H]_{i,i}$$

# Eigenmatrix of the autocorrelation matrix

Definition: eigenmatrix  $\Phi$  of autocorrelation matrix  $R_{ff}$

- $\Phi$  is unitary
- The columns of  $\Phi$  form a set of eigenvectors of  $R_{ff}$ , i.e.,

$$R_{ff} \Phi = \Phi \Lambda \quad \leftarrow \quad \Lambda \text{ is a diagonal matrix of eigenvalues } \lambda_i$$

$$\Lambda = \begin{pmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_{N-1} \end{pmatrix}$$

- $R_{ff}$  is normal matrix, i.e.,  $R_{ff}^H R_{ff} = R_{ff} R_{ff}^H$ , hence unitary eigenmatrix exists („spectral theorem“)
- $R_{ff}$  is symmetric nonnegative definite, hence  $\lambda_i \geq 0$  for all  $i$

# Karhunen-Loève transform

- Unitary transform with matrix

$$A = \Phi^H$$

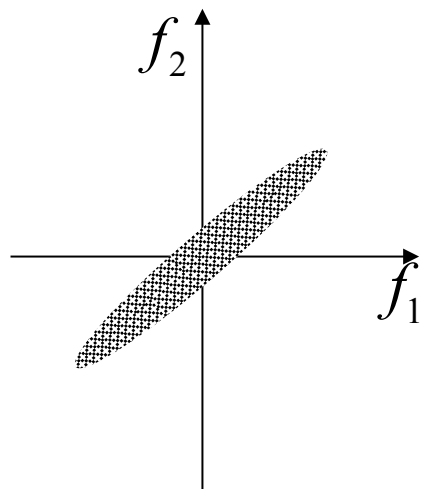
- Transform coefficients are pairwise uncorrelated

$$R_{cc} = AR_{ff}A^H = \Phi^H R_{ff} \Phi = \Phi^H \Phi \Lambda = \Lambda$$

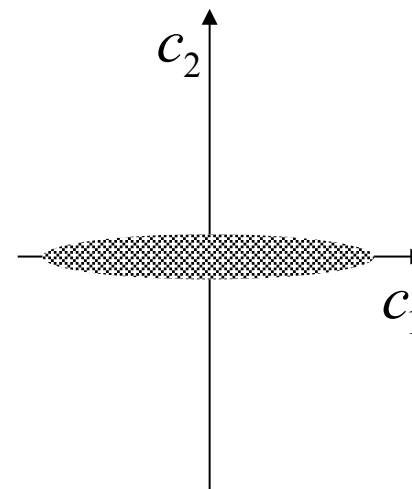
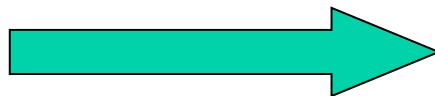
- Columns of  $\Phi$  are ordered according to decreasing eigenvalues.
- Energy concentration property:
  - No other unitary transform packs as much energy into the first  $J$  coefficients.
  - Mean squared approximation error by keeping only first  $J$  coefficients is minimized.
  - Holds for any  $J$ .

# Illustration of energy concentration

Strongly correlated samples,  
equal energies



$$\mathbf{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$



After KLT:  
uncorrelated samples,  
most of the energy in  
first coefficient

# Basis images and eigenimages

- For any transform, the inverse transform

$$\vec{f} = A^{-1}\vec{c}$$

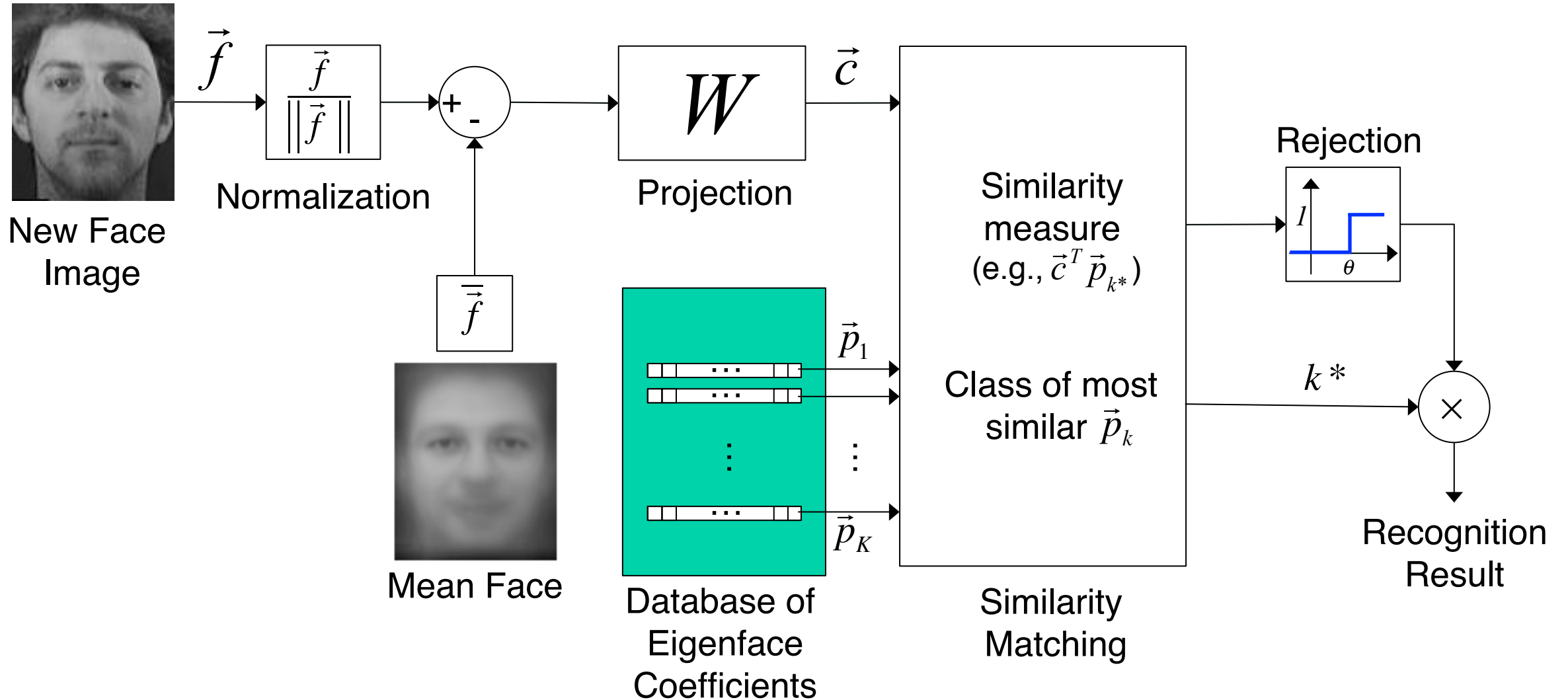
can be interpreted in terms of the superposition of columns of  $A^{-1}$  („basis images“)

- For the KL transform, the basis images are the eigenvectors of the autocorrelation matrix  $R_{ff}$  and are called „eigenimages.“
- If energy concentration works well, only a limited number of eigenimages is needed to approximate a set of images with small error. These eigenimages span an optimal linear subspace of dimensionality  $J$ .

# Eigenimages for recognition

- To recognize complex patterns (e.g., faces), large portions of an image have to be considered
- High dimensionality of “image space” means high computational burden for many recognition techniques
  - Example: nearest-neighbor search requires pairwise comparison with every image in a database
- Transform  $\vec{c} = W\vec{f}$  can reduce dimensionality from  $N$  to  $J$  by representing the image by  $J$  coefficients
- Idea: tailor a KLT to a specific set of training images representative of the recognition task to preserve the salient features

# Eigenimages for recognition



# Computing eigenimages from a training set

## ■ How to obtain $N \times N$ covariance matrix?

- Use training set  $\vec{\Gamma}_1, \vec{\Gamma}_2, \dots, \vec{\Gamma}_{L+1}$  (each column vector represents one image)
- Let  $\vec{\mu}$  be the mean image of all  $L+1$  training images
- Define training set matrix  $S = (\vec{\Gamma}_1 - \vec{\mu}, \vec{\Gamma}_2 - \vec{\mu}, \vec{\Gamma}_3 - \vec{\mu}, \dots, \vec{\Gamma}_L - \vec{\mu})$ ,

and calculate scatter matrix  $R = \sum_{l=1}^L (\vec{\Gamma}_l - \vec{\mu})(\vec{\Gamma}_l - \vec{\mu})^H = SS^H$

Problem 1: Training set size should be  $L + 1 \gg N$

If  $L < N$ , scatter matrix  $R$  is rank-deficient

Problem 2: Finding eigenvectors of an  $N \times N$  matrix.

- ## ■ Can we find a small set of the most important eigenimages from a small training set $L \ll N$ ?

# Sirovich and Kirby algorithm

- Instead of eigenvectors of  $SS^H$ , consider the eigenvectors of  $S^H S$ , i.e.,

$$S^H S \vec{v}_i = \lambda_i \vec{v}_i$$

- Premultiply both sides by  $S$

$$SS^H S \vec{v}_i = \lambda_i S \vec{v}_i$$

- By inspection, we find that  $S \vec{v}_i$  are eigenvectors of  $SS^H$

## Sirovich and Kirby Algorithm (for $L \ll N$ )

- Compute the  $L \times L$  matrix  $S^H S$
- Compute  $L$  eigenvectors  $\vec{v}_i$  of  $S^H S$
- Compute eigenimages corresponding to the  $L_0 \leq L$  largest eigenvalues as a linear combination of training images  $S \vec{v}_i$

L. Sirovich and M. Kirby, "Low-dimensional procedure for the characterization of human faces," *Journal of the Optical Society of America A*, 4(3), pp. 519-524, 1987.

# Example: eigenfaces

- The first 8 eigenfaces obtained from a training set of 100 male and 100 female training images



Mean Face



Eigenface 1



Eigenface 2



Eigenface 3



Eigenface 4



Eigenface 5



Eigenface 6



Eigenface 7



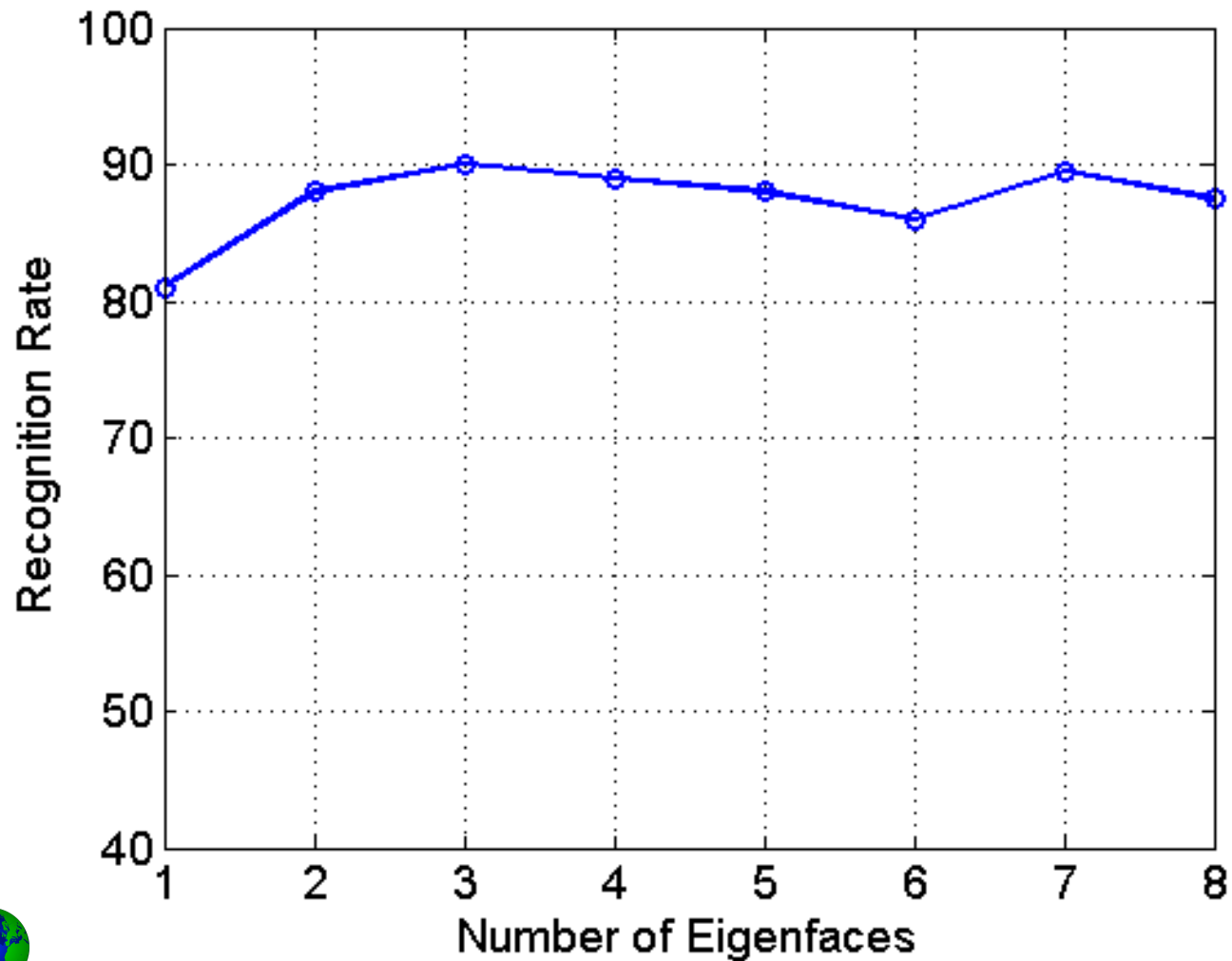
Eigenface 8

- Can be used to generate faces by adjusting 8 coefficients.
- Can be used for face recognition by nearest-neighbor search in 8-d „face space.“



# Gender recognition using eigenfaces

Nearest neighbor search "face space"



Female face samples



Male face samples



# Fisher linear discriminant analysis

- Eigenimage method maximizes “scatter” within the linear subspace over the entire image set – regardless of classification task

$$W_{opt} = \arg \max_W \left( \det(WR_W W^H) \right)$$

- Fisher linear discriminant analysis (1936): maximize between-class scatter, while minimizing within-class scatter

$$W_{opt} = \arg \max_W \left( \frac{\det(WR_B W^H)}{\det(WR_W W^H)} \right)$$

$$R_B = \sum_{i=1}^c N_i (\bar{\mu}_i - \bar{\mu})(\bar{\mu}_i - \bar{\mu})^H$$

Samples  
in class  $i$

Mean in class  $i$

$$R_W = \sum_{i=1}^c \sum_{\Gamma_l \in \text{Class}(i)} (\bar{\Gamma}_l - \bar{\mu}_i)(\bar{\Gamma}_l - \bar{\mu}_i)^H$$

# Fisher linear discriminant analysis (cont.)

- Solution: Generalized eigenvectors  $\vec{w}_i$  corresponding to the  $J$  largest eigenvalues  $\{\lambda_i \mid i = 1, 2, \dots, J\}$ , i.e.

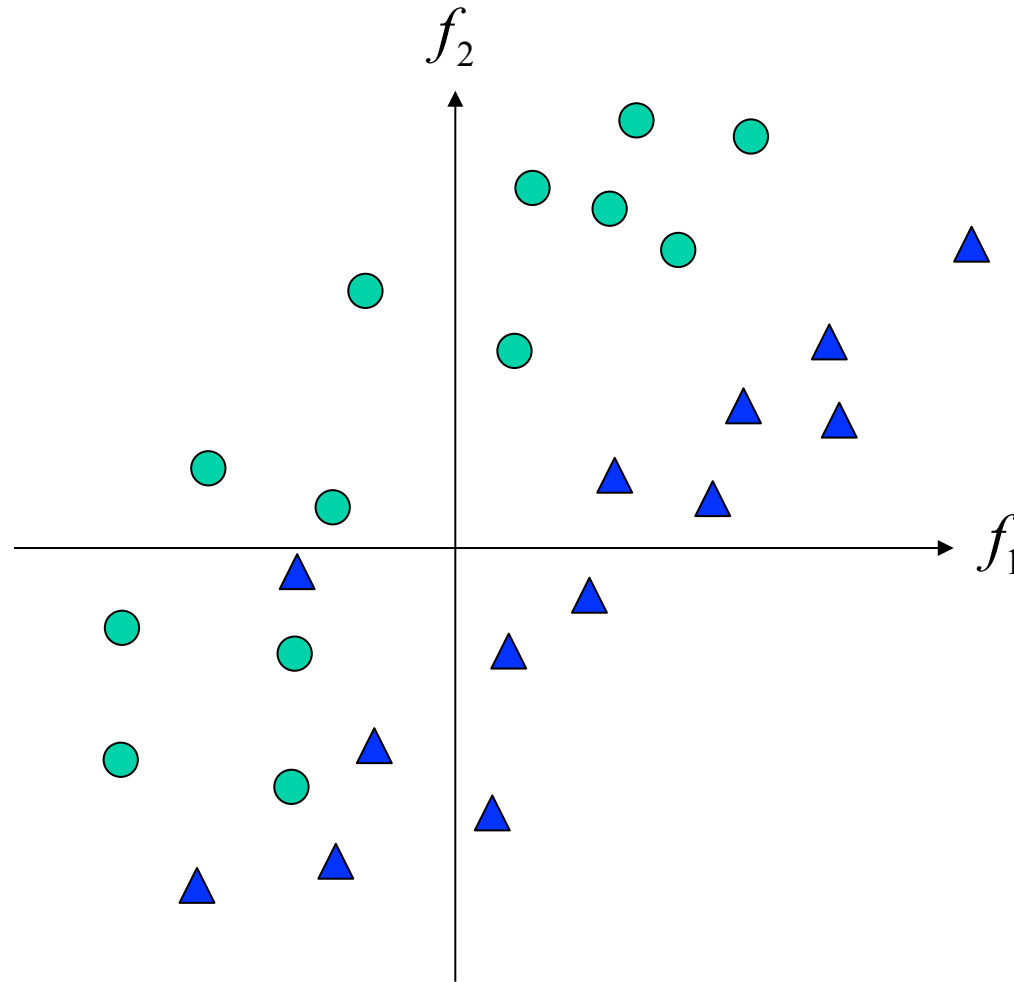
$$R_B \vec{w}_i = \lambda_i R_W \vec{w}_i, \quad i = 1, 2, \dots, J$$

- Problem: within-class scatter matrix  $R_W$  at most of rank  $L-c$ , hence usually singular.
- Apply KLT first to reduce dimensionality of feature space to  $L-c$  (or less), proceed with Fisher LDA in lower-dimensional space

# Eigenimages vs. Fisherimages

## 2-d example:

Goal: project samples on a 1-d subspace, then perform classification.

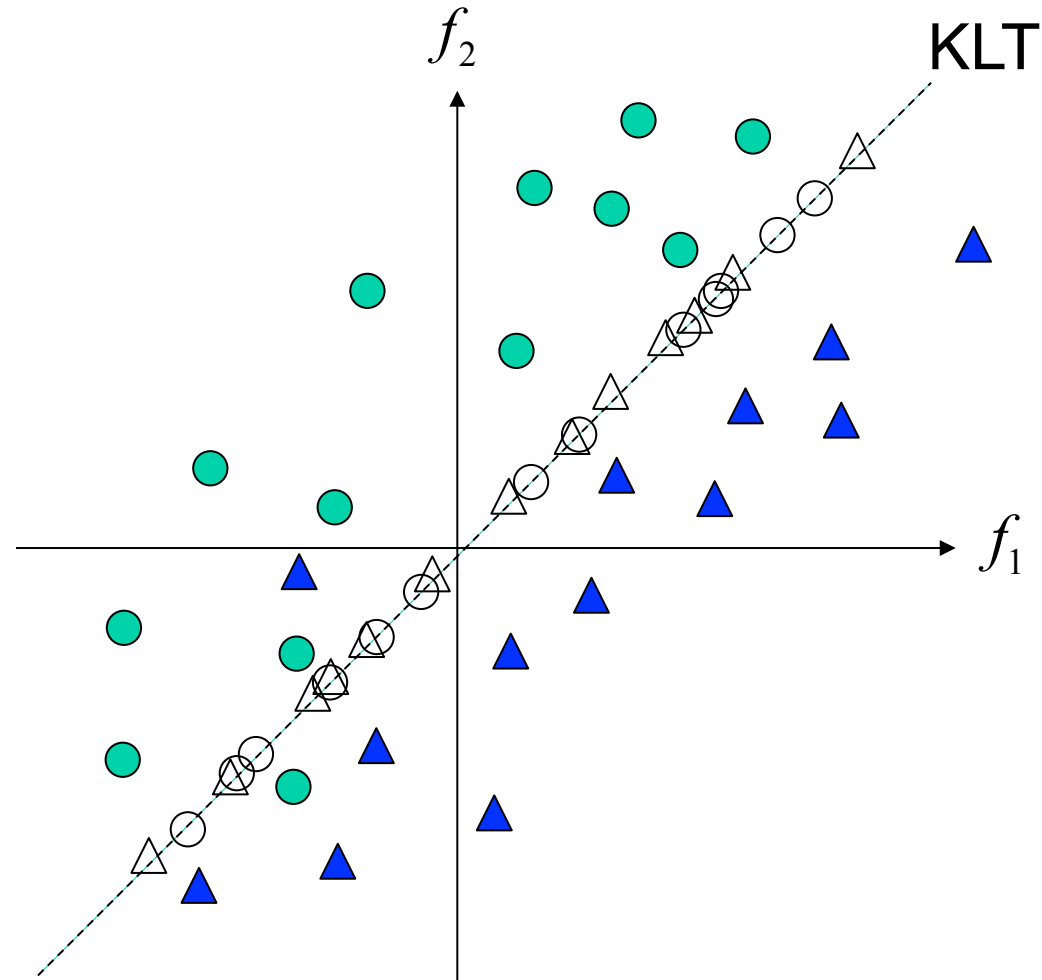


# Eigenimages vs. Fisherimages

## 2-d example:

Goal: project samples on a 1-d subspace, then perform classification.

The KLT preserves maximum energy, but the 2 classes are no longer distinguishable.



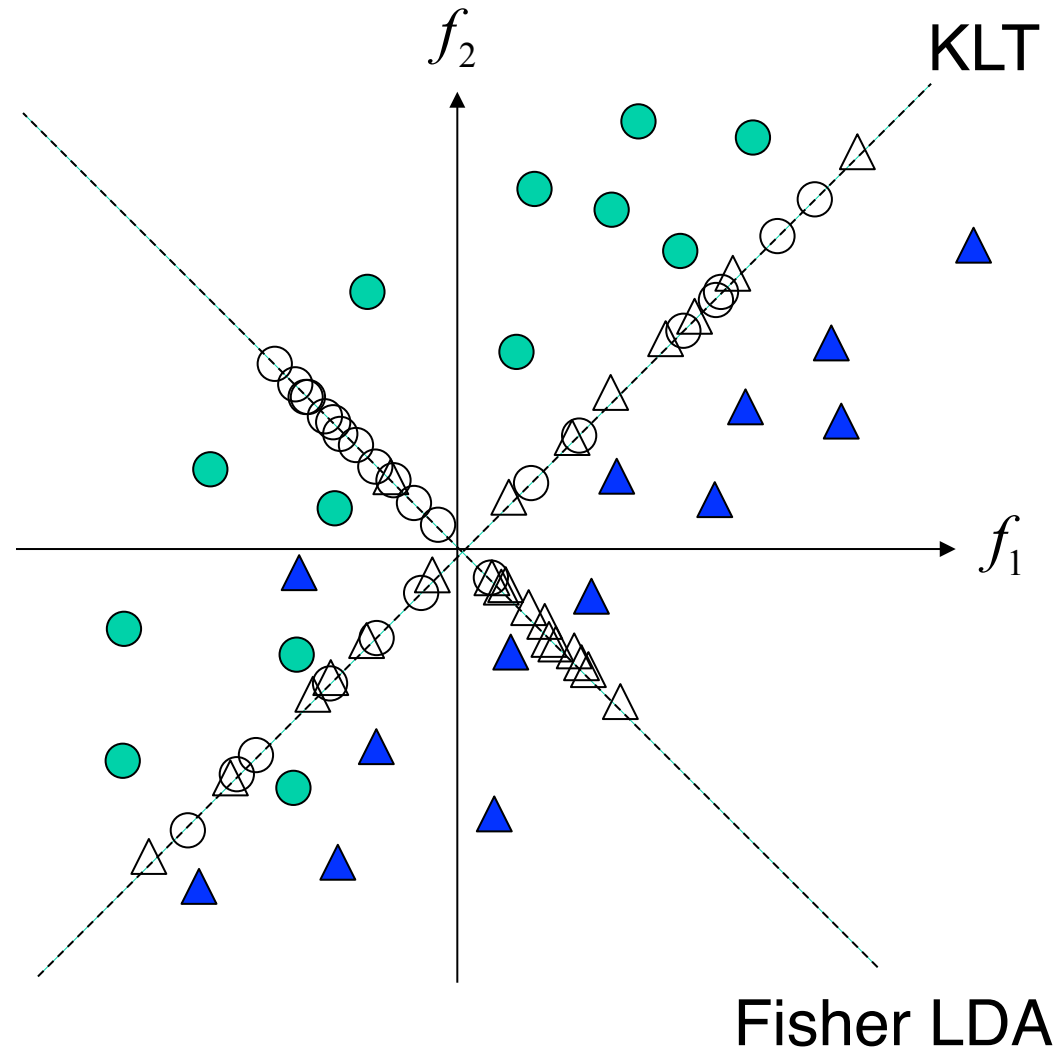
# Eigenimages vs. Fisherimages

## 2-d example:

Goal: project samples on a 1-d subspace, then perform classification.

The KLT preserves maximum energy, but the 2 classes are no longer distinguishable.

Fisher LDA separates the classes by choosing a better 1-d subspace.



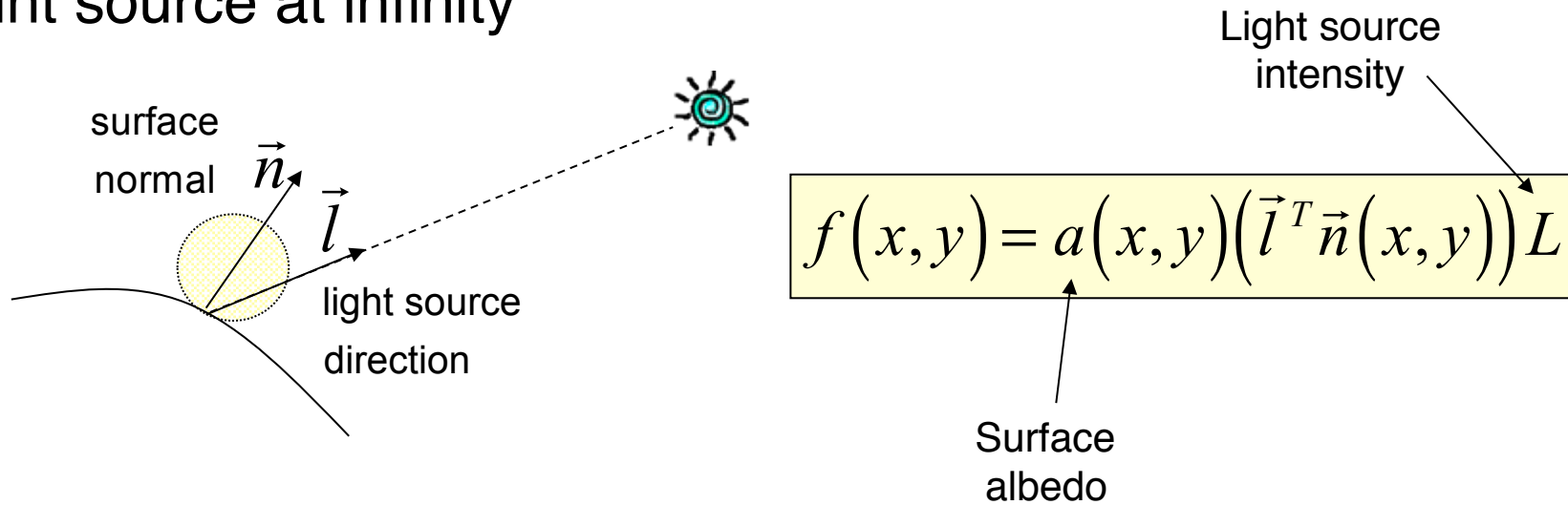
# Fisherimages and varying illumination

Differences due to varying illumination can be much larger than differences among faces!



# Fisherimages and varying illumination

- All images of same Lambertian surface with different illumination (without shadows) lie in a 3d linear subspace
- Single point source at infinity



- Superposition of arbitrary number of point sources at infinity still in same 3d linear subspace, due to linear superposition of each contribution to image
- Fisherimages can eliminate within-class scatter

# Fisherface trained to recognize gender



Female face samples



Male face samples



Mean image

$$\vec{\mu}$$



Female mean

$$\vec{\mu}_1$$



Male mean

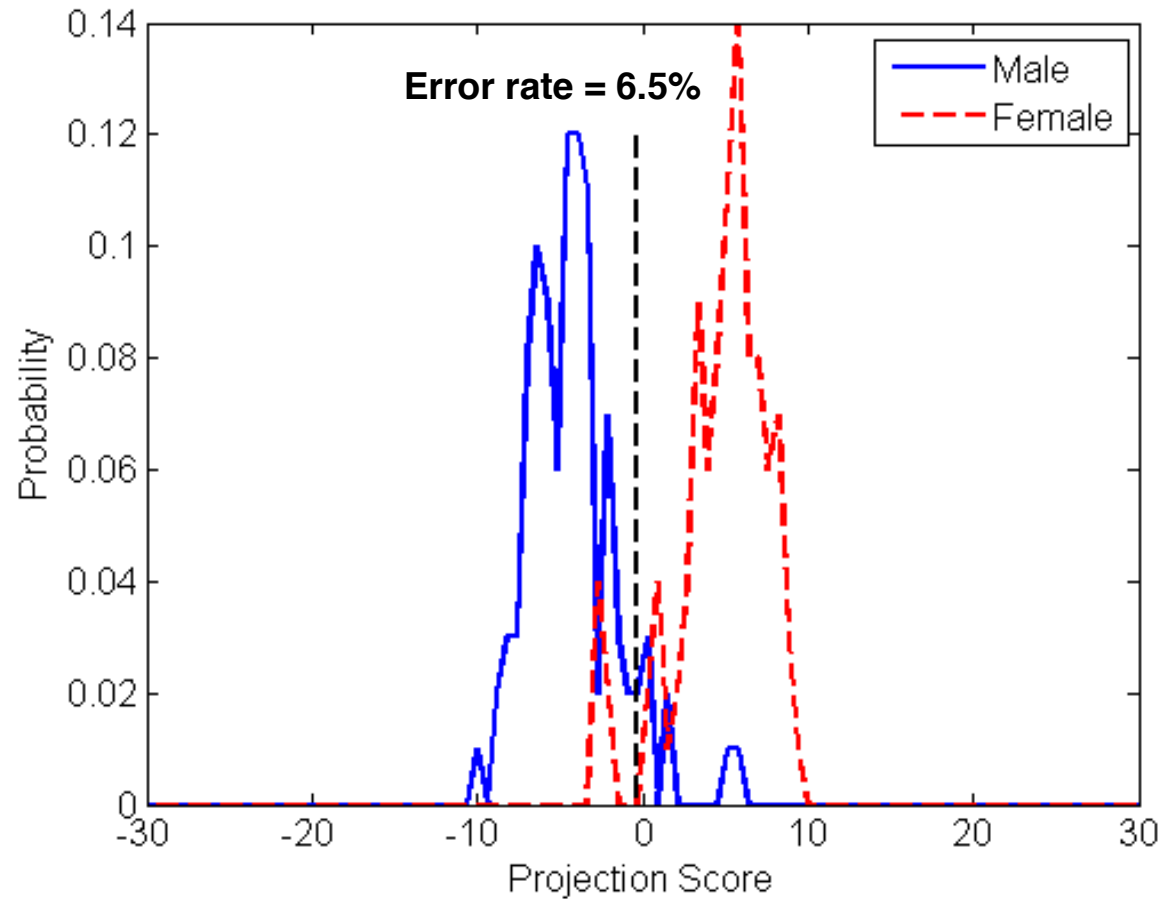
$$\vec{\mu}_2$$



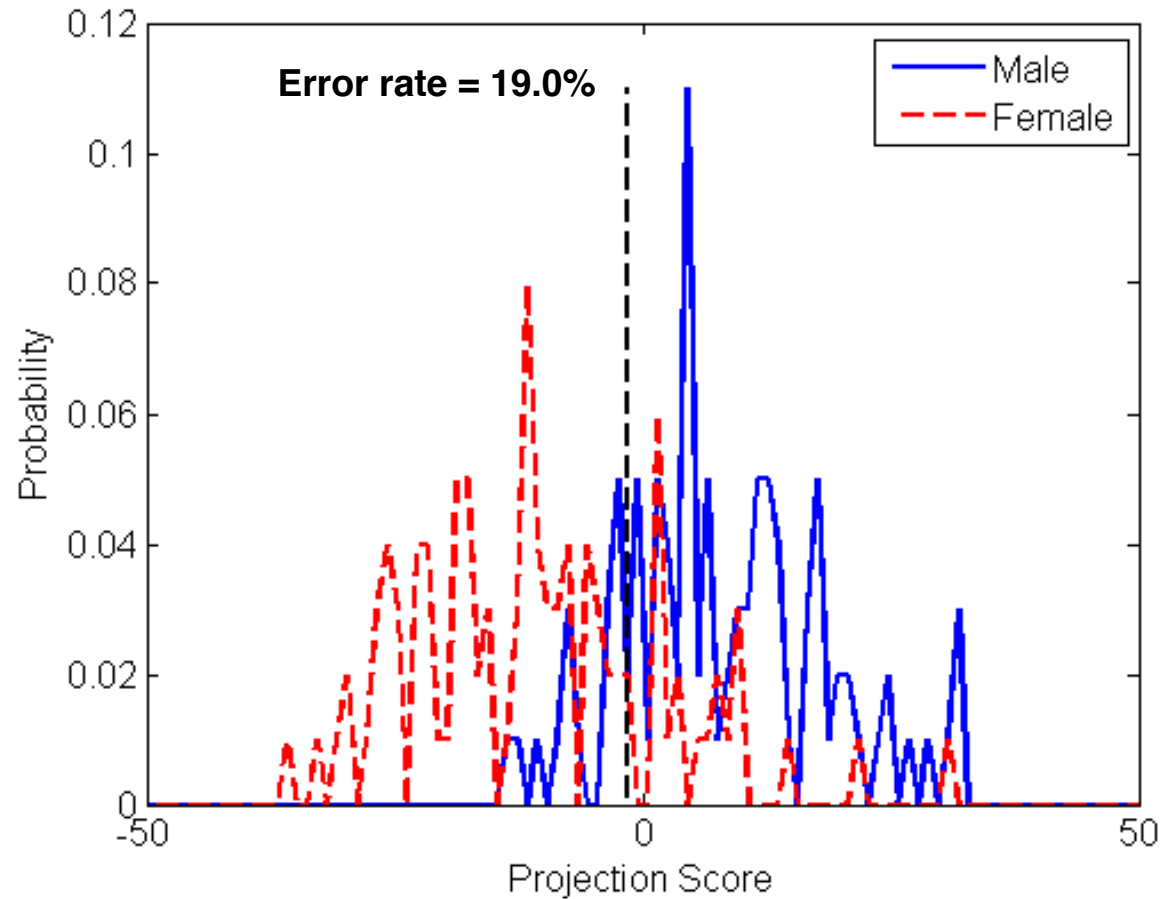
Fisherface



# Gender recognition using 1<sup>st</sup> Fisherface



# Gender recognition using 1<sup>st</sup> eigenface



# Person identification with Fisherfaces and eigenfaces

