

Eigenimages

- Unitary transforms
- Karhunen-Loève transform and eigenimages
- Sirovich and Kirby method
- Eigenfaces for gender recognition
- Fisher linear discriminant analysis
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Unitary transforms

- Sort pixels $f[x,y]$ of an image into column vector \vec{f} of length N
- Calculate N transform coefficients

$$\vec{c} = A\vec{f}$$

where A is a matrix of size $N \times N$

- The transform A is unitary, iff

$$A^{-1} = \underbrace{A^{*T}}_{\text{Hermitian conjugate}} \equiv A^H$$

- If A is real-valued, i.e., $A=A^*$, transform is „orthonormal“

Energy conservation with unitary transforms

- For any unitary transform $\vec{c} = A\vec{f}$ we obtain

$$\|\vec{c}\|^2 = \vec{c}^H \vec{c} = \vec{f}^H A^H A \vec{f} = \|\vec{f}\|^2$$

- Interpretation: every unitary transform is simply a rotation of the coordinate system (and, possibly, sign flips)
- Vector length is conserved.
- Energy (mean squared vector length) is conserved.

Energy distribution for unitary transforms

- Energy is conserved, but, in general, unevenly distributed among coefficients.
- Autocorrelation matrix

$$R_{cc} = E[\vec{c}\vec{c}^H] = E[A\vec{f} \cdot \vec{f}^H A^H] = AR_{ff}A^H$$

- Diagonal of R_{cc} comprises mean squared values („energies“) of the coefficients c_i

$$E[c_i^2] = [R_{cc}]_{i,i} = [AR_{ff}A^H]_{i,i}$$

Eigenmatrix of the autocorrelation matrix

Definition: eigenmatrix Φ of autocorrelation matrix R_{ff}

- Φ is unitary
- The columns of Φ form a set of eigenvectors of R_{ff} , i.e.,

$$R_{ff} \Phi = \Phi \Lambda \quad \leftarrow \quad \Lambda \text{ is a diagonal matrix of eigenvalues } \lambda_i$$

$$\Lambda = \begin{pmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_{N-1} \end{pmatrix}$$

- R_{ff} is normal matrix, i.e., $R_{ff}^H R_{ff} = R_{ff} R_{ff}^H$,
hence unitary eigenmatrix exists („spectral theorem“)
- R_{ff} is symmetric nonnegative definite, hence $\lambda_i \geq 0$ for all i

Karhunen-Loève transform

- Unitary transform with matrix

$$A = \Phi^H$$

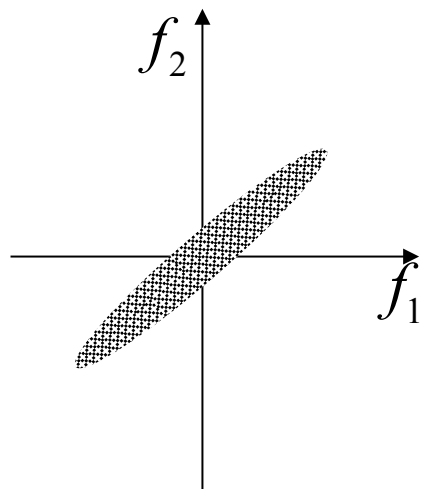
- Transform coefficients are pairwise uncorrelated

$$R_{cc} = AR_{ff}A^H = \Phi^H R_{ff} \Phi = \Phi^H \Phi \Lambda = \Lambda$$

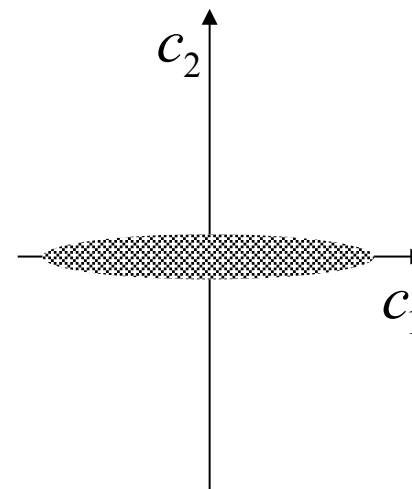
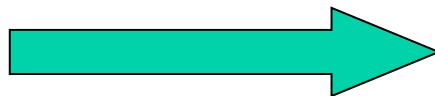
- Columns of Φ are ordered according to decreasing eigenvalues.
- Energy concentration property:
 - No other unitary transform packs as much energy into the first J coefficients.
 - Mean squared approximation error by keeping only first J coefficients is minimized.
 - Holds for any J .

Illustration of energy concentration

Strongly correlated samples,
equal energies



$$\mathbf{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$



After KLT:
uncorrelated samples,
most of the energy in
first coefficient

Basis images and eigenimages

- For any transform, the inverse transform

$$\vec{f} = A^{-1}\vec{c}$$

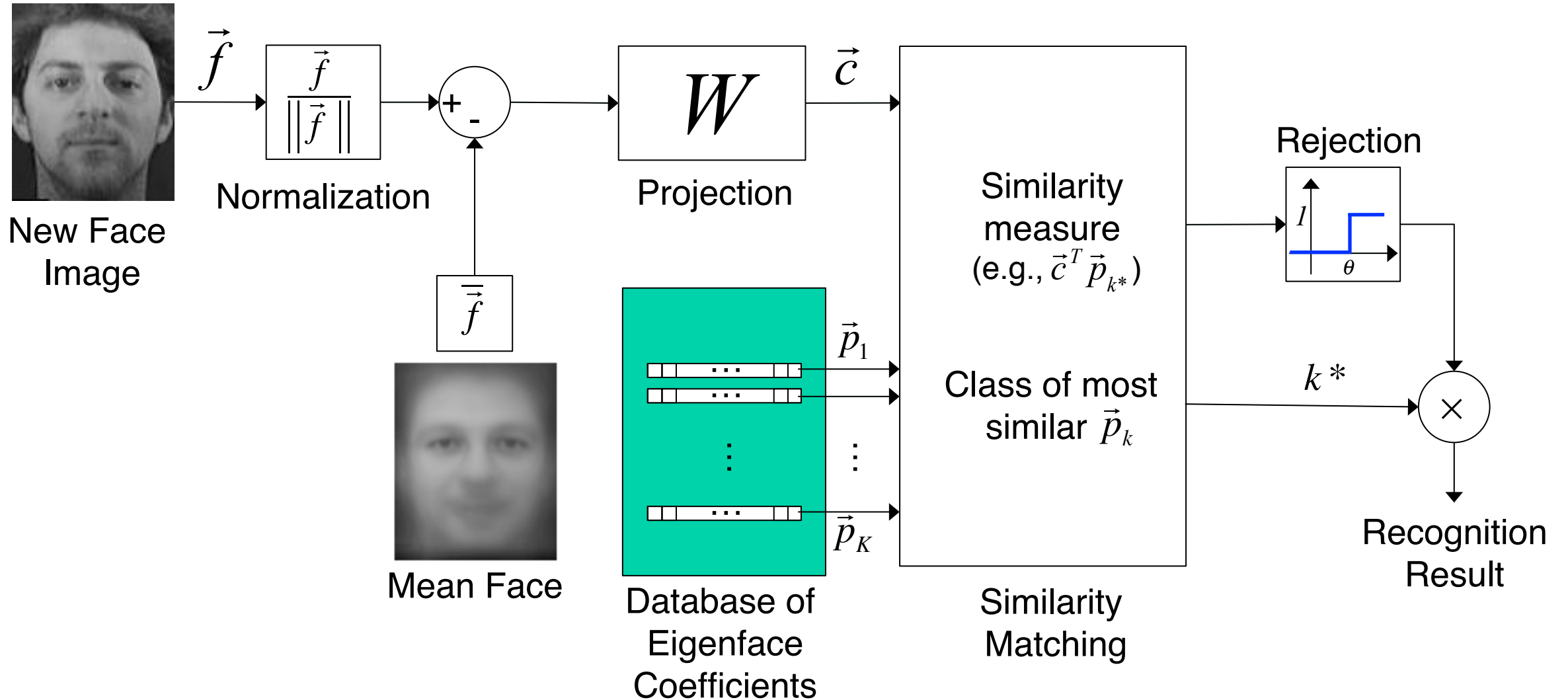
can be interpreted in terms of the superposition of columns of A^{-1} („basis images“)

- For the KL transform, the basis images are the eigenvectors of the autocorrelation matrix R_{ff} and are called „eigenimages.“
- If energy concentration works well, only a limited number of eigenimages is needed to approximate a set of images with small error. These eigenimages span an optimal linear subspace of dimensionality J .

Eigenimages for recognition

- To recognize complex patterns (e.g., faces), large portions of an image have to be considered
- High dimensionality of “image space” means high computational burden for many recognition techniques
 - Example: nearest-neighbor search requires pairwise comparison with every image in a database
- Transform $\vec{c} = W\vec{f}$ can reduce dimensionality from N to J by representing the image by J coefficients
- Idea: tailor a KLT to a specific set of training images representative of the recognition task to preserve the salient features

Eigenimages for recognition



Computing eigenimages from a training set

■ How to obtain $N \times N$ covariance matrix?

- Use training set $\vec{\Gamma}_1, \vec{\Gamma}_2, \dots, \vec{\Gamma}_L$ (each column vector represents one image)
- Let $\vec{\mu}$ be the mean image of all samples
- Define training set matrix $S = (\vec{\Gamma}_1 - \vec{\mu}, \vec{\Gamma}_2 - \vec{\mu}, \vec{\Gamma}_3 - \vec{\mu}, \dots, \vec{\Gamma}_L - \vec{\mu})$,

and calculate scatter matrix $R = \sum_{l=1}^L (\vec{\Gamma}_l - \vec{\mu})(\vec{\Gamma}_l - \vec{\mu})^H = SS^H$

Problem 1: Training set size should be $L \gg N$

If $L < N$, scatter matrix R is rank-deficient

Problem 2: Finding eigenvectors of an $N \times N$ matrix.

- ## ■ Can we find a small set of the most important eigenimages from a small training set $L \ll N$?

Sirovich and Kirby algorithm

- Instead of eigenvectors of SS^H , consider the eigenvectors of $S^H S$, i.e.,

$$S^H S \vec{v}_i = \lambda_i \vec{v}_i$$

- Premultiply both sides by S

$$SS^H S \vec{v}_i = \lambda_i S \vec{v}_i$$

- By inspection, we find that $S \vec{v}_i$ are eigenvectors of SS^H

Sirovich and Kirby Algorithm (for $L \ll N$)

- Compute the $L \times L$ matrix $S^H S$
- Compute L eigenvectors \vec{v}_i of $S^H S$
- Compute eigenimages corresponding to the $L_0 \leq L$ largest eigenvalues as a linear combination of training images $S \vec{v}_i$

L. Sirovich and M. Kirby, "Low-dimensional procedure for the characterization of human faces," *Journal of the Optical Society of America A*, 4(3), pp. 519-524, 1987.

Example: eigenfaces

- The first 8 eigenfaces obtained from a training set of 100 male and 100 female training images



Mean Face



Eigenface 1



Eigenface 2



Eigenface 3



Eigenface 4



Eigenface 5



Eigenface 6



Eigenface 7



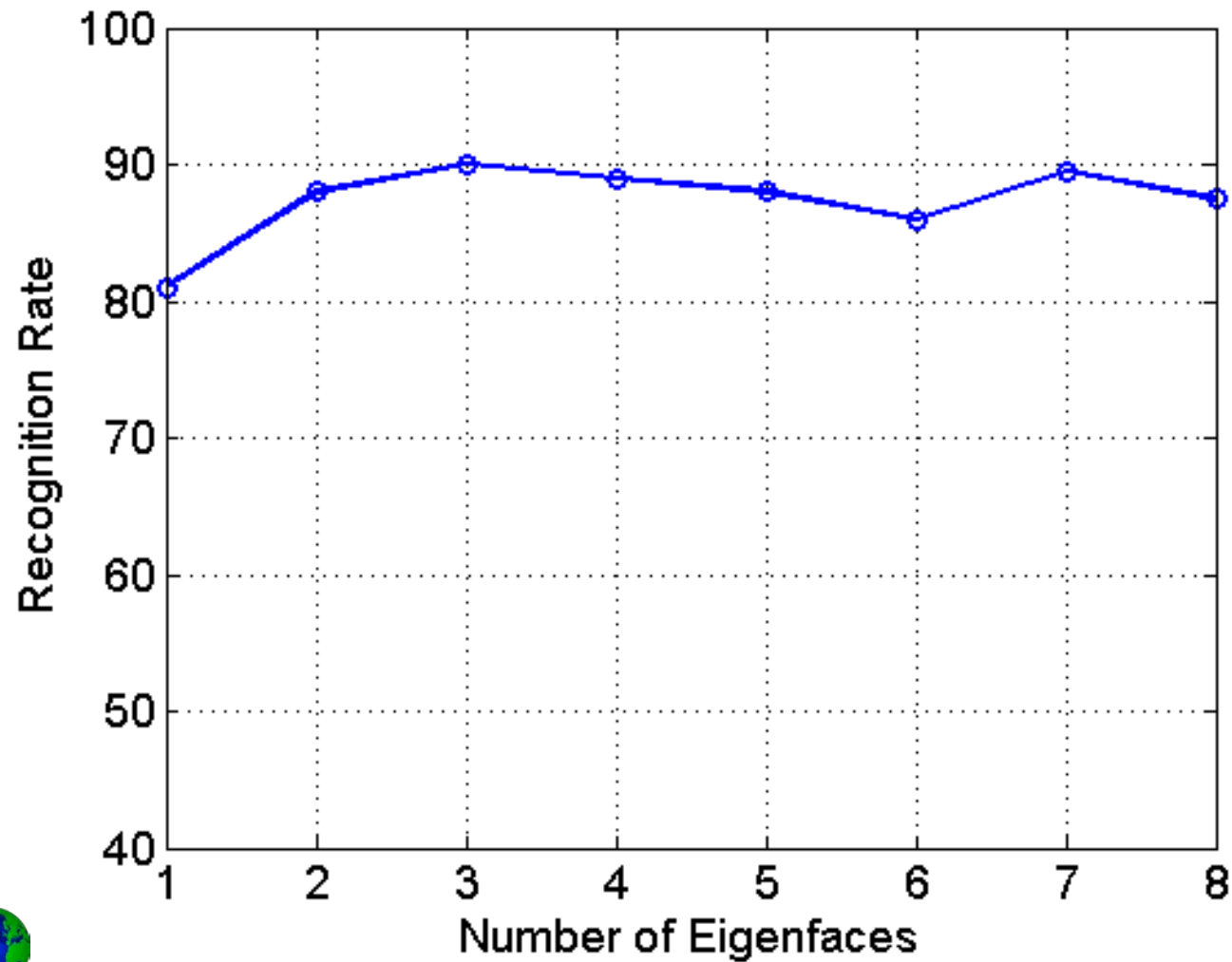
Eigenface 8

- Can be used to generate faces by adjusting 8 coefficients.
- Can be used for face recognition by nearest-neighbor search in 8-d „face space.“



Gender recognition using eigenfaces

Nearest neighbor search "face space"



Female face samples



Male face samples



Fisher linear discriminant analysis

- Eigenimage method maximizes “scatter” within the linear subspace over the entire image set – regardless of classification task

$$W_{opt} = \arg \max_W \left(\det(WR_W W^H) \right)$$

- Fisher linear discriminant analysis (1936): maximize between-class scatter, while minimizing within-class scatter

$$W_{opt} = \arg \max_W \left(\frac{\det(WR_B W^H)}{\det(WR_W W^H)} \right)$$

$$R_B = \sum_{i=1}^c N_i (\bar{\mu}_i - \bar{\mu})(\bar{\mu}_i - \bar{\mu})^H$$

Samples
in class i

Mean in class i

$$R_W = \sum_{i=1}^c \sum_{\Gamma_l \in \text{Class}(i)} (\bar{\Gamma}_l - \bar{\mu}_i)(\bar{\Gamma}_l - \bar{\mu}_i)^H$$

Fisher linear discriminant analysis (cont.)

- Solution: Generalized eigenvectors \vec{w}_i corresponding to the J largest eigenvalues $\{\lambda_i \mid i = 1, 2, \dots, J\}$, i.e.

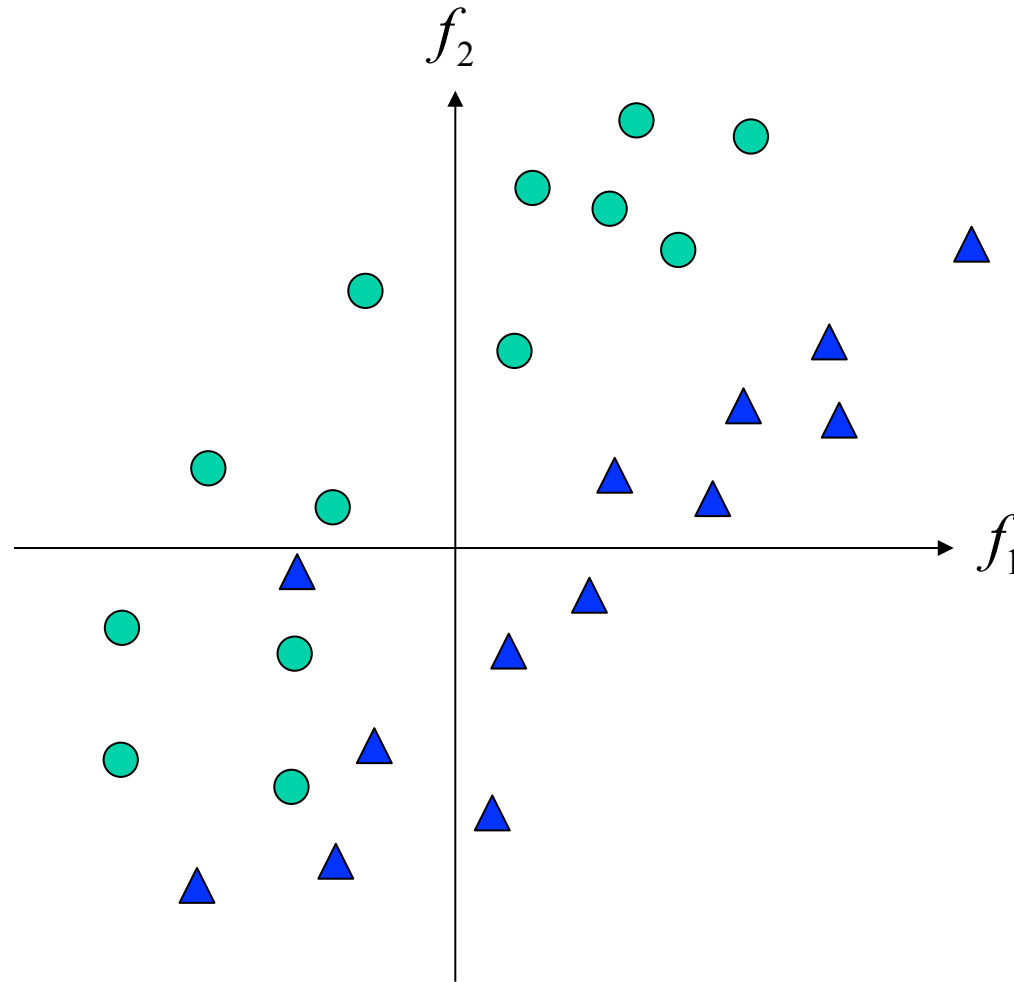
$$R_B \vec{w}_i = \lambda_i R_W \vec{w}_i, \quad i = 1, 2, \dots, J$$

- Problem: within-class scatter matrix R_W at most of rank $L-c$, hence usually singular.
- Apply KLT first to reduce dimensionality of feature space to $L-c$ (or less), proceed with Fisher LDA in lower-dimensional space

Eigenimages vs. Fisherimages

2-d example:

Goal: project samples on a 1-d subspace, then perform classification.

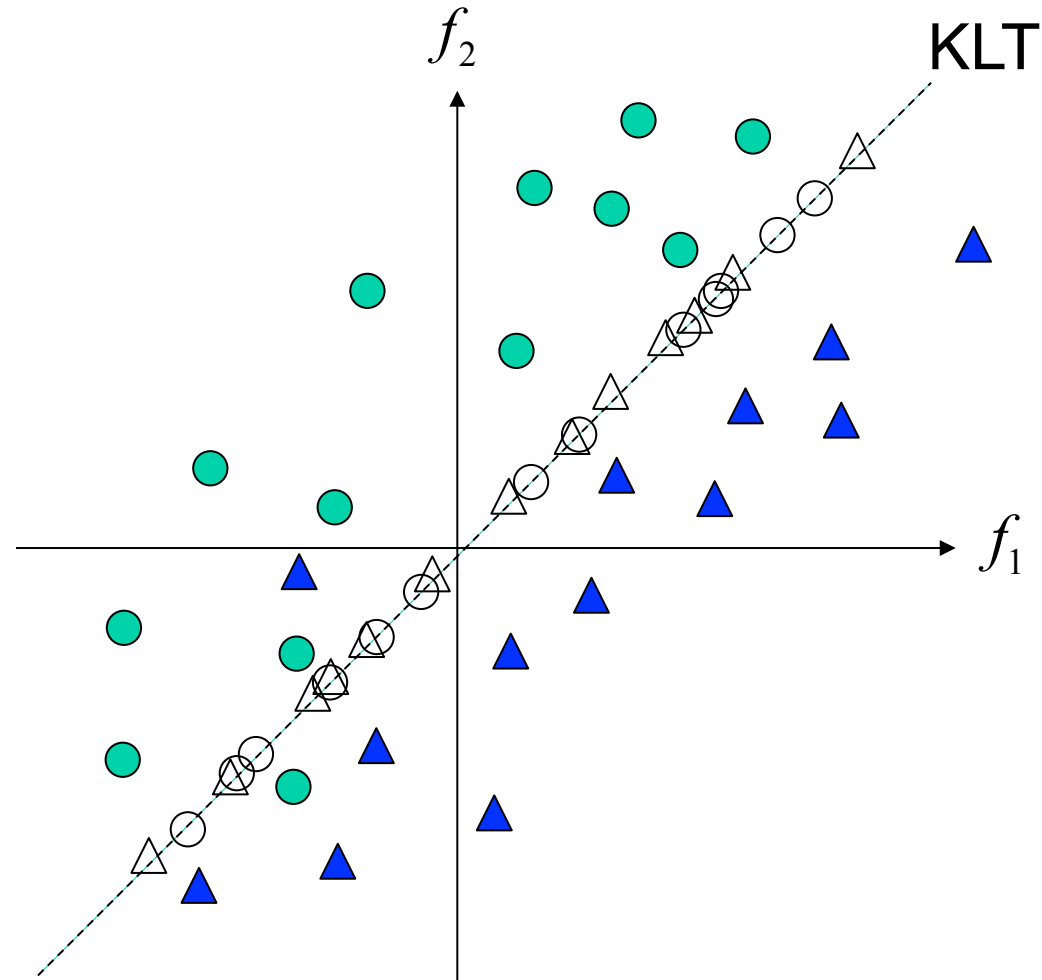


Eigenimages vs. Fisherimages

2-d example:

Goal: project samples on a 1-d subspace, then perform classification.

The KLT preserves maximum energy, but the 2 classes are no longer distinguishable.



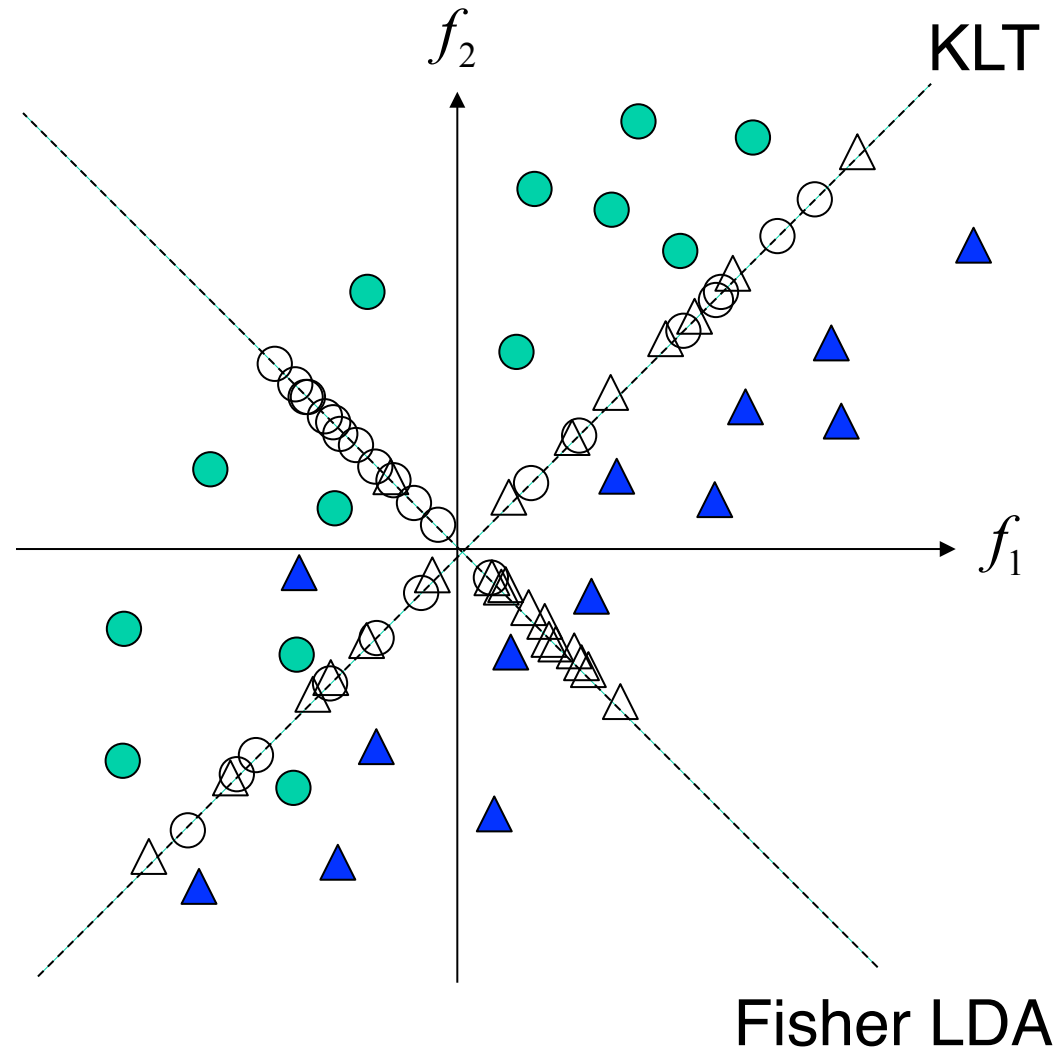
Eigenimages vs. Fisherimages

2-d example:

Goal: project samples on a 1-d subspace, then perform classification.

The KLT preserves maximum energy, but the 2 classes are no longer distinguishable.

Fisher LDA separates the classes by choosing a better 1-d subspace.



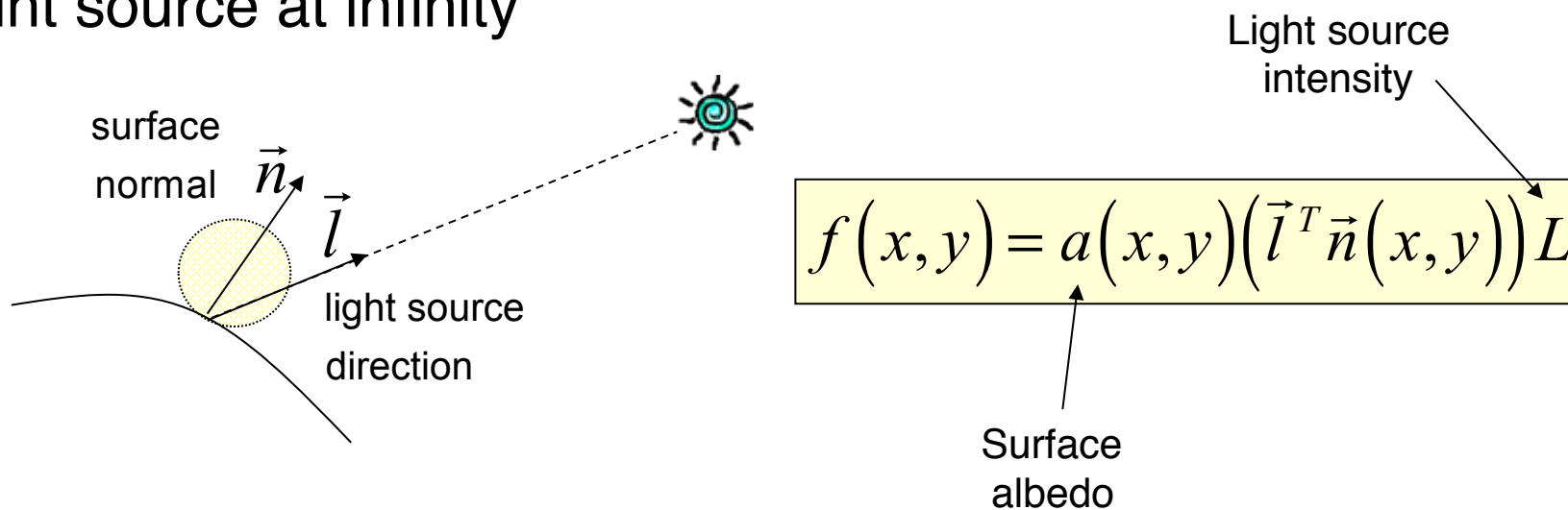
Fisherimages and varying illumination

Differences due to varying illumination can be much larger than differences among faces!



Fisherimages and varying illumination

- All images of same Lambertian surface with different illumination (without shadows) lie in a 3d linear subspace
- Single point source at infinity



- Superposition of arbitrary number of point sources at infinity still in same 3d linear subspace, due to linear superposition of each contribution to image
- Fisherimages can eliminate within-class scatter

Fisherface trained to recognize gender



Female face samples



Male face samples



Mean image

$$\vec{\mu}$$



Female mean

$$\vec{\mu}_1$$



Male mean

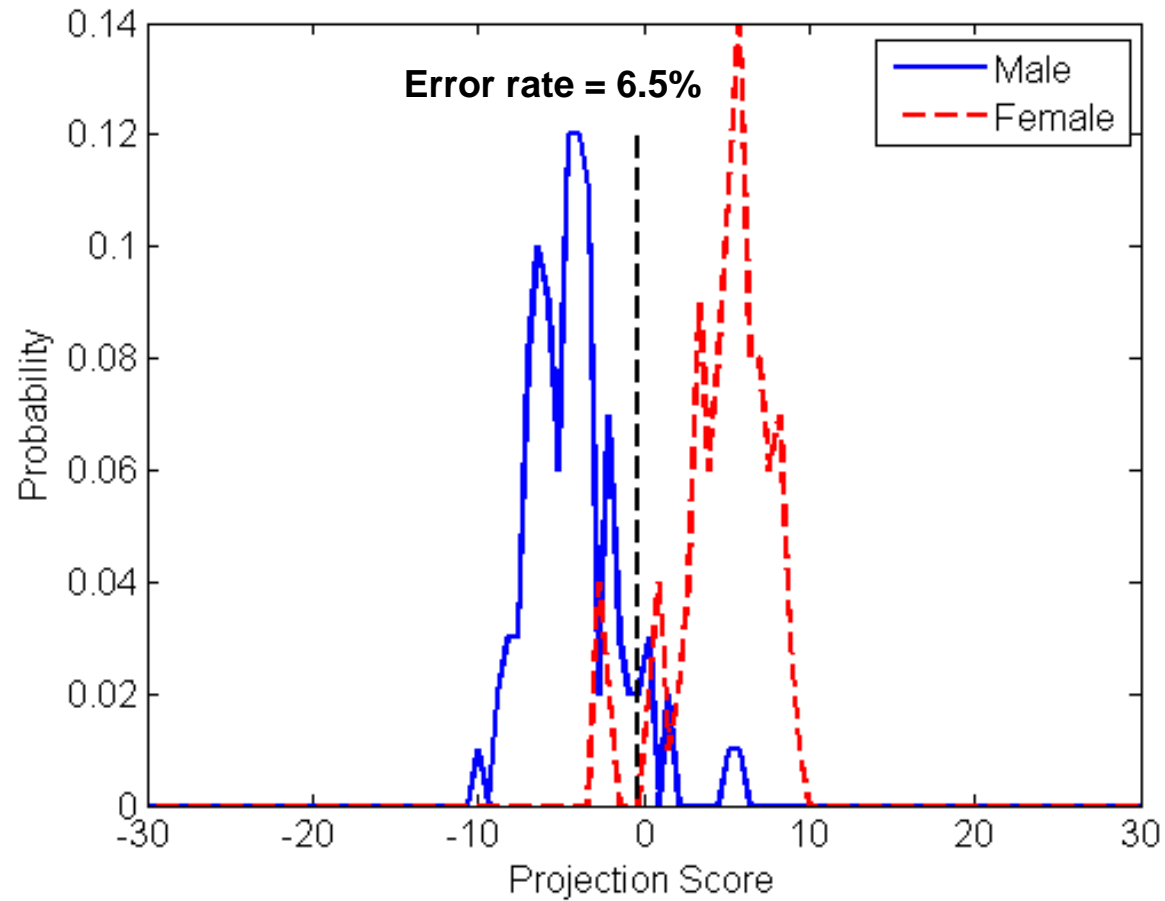
$$\vec{\mu}_2$$



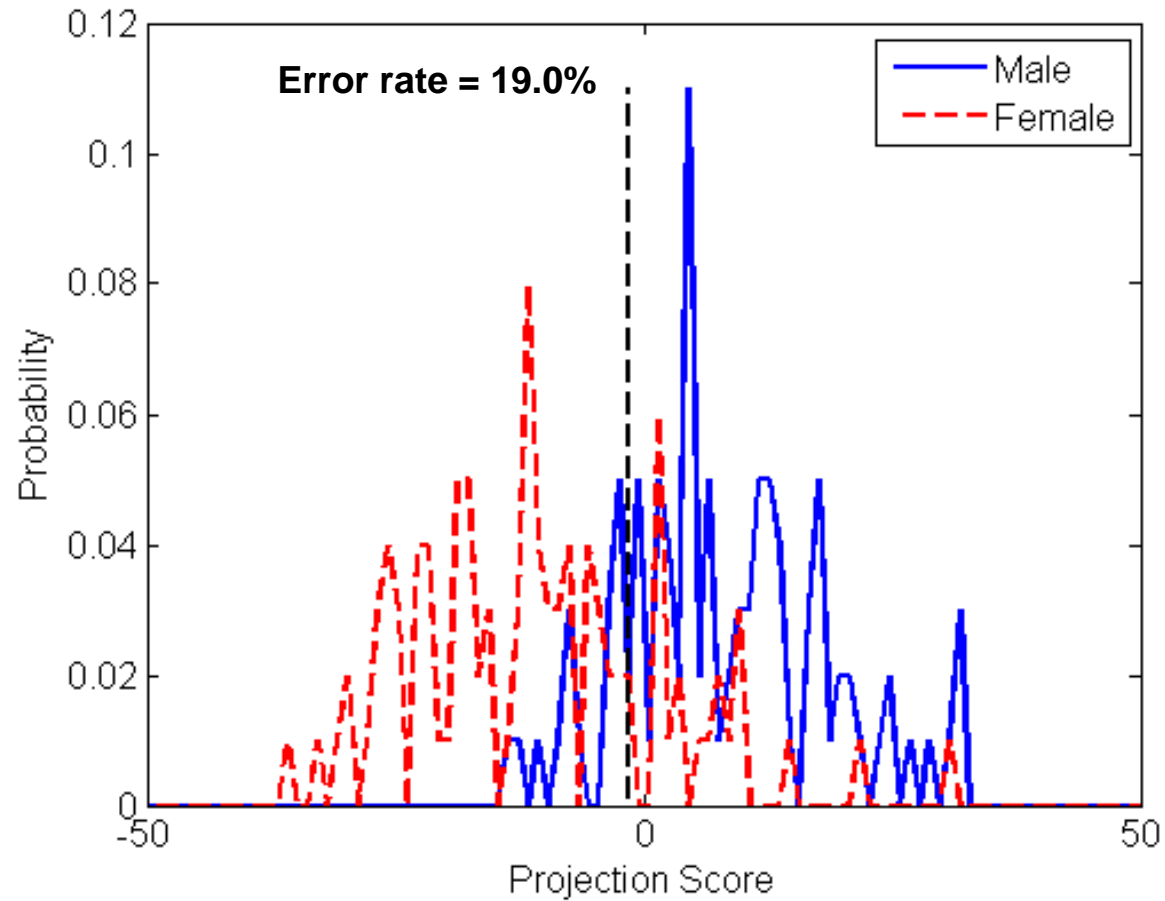
Fisherface



Gender recognition using 1st Fisherface



Gender recognition using 1st eigenface



Person identification with Fisherfaces and eigenfaces

