Shift-invariant systems and Toeplitz matrices

- For a separable, shift-invariant, linear system

$$h_x[x,\alpha] = h_{siv/x}[\alpha - x] \quad \text{and} \quad h_y[y,\beta] = h_{siv/y}[\beta - y]$$

- Matrices $\mathbf{H_x}$ and $\mathbf{H_y}$ are square, and Toeplitz matrices, e.g.,

$$\mathbf{H_x} = \begin{bmatrix} h_{siv/x}[0] & h_{siv/x}[1] & \cdots & h_{siv/x}[N-1] \\ h_{siv/x}[-1] & h_{siv/x}[0] & \cdots & h_{siv/x}[N-2] \\ \vdots & \vdots & & \vdots \\ h_{siv/x}[1-N] & h_{siv/x}[2-N] & \cdots & h_{siv/x}[0] \end{bmatrix}$$

- Operation is a 2-d separable convolution („filtering“)

$$g[\alpha,\beta] = \sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y] \cdot h_{siv/x}[\alpha - x] h_{siv/y}[\beta - y]$$

Non-separable 2-d convolution

- Convolution kernel of linear shift-invariant system („filter“) can also be non-separable

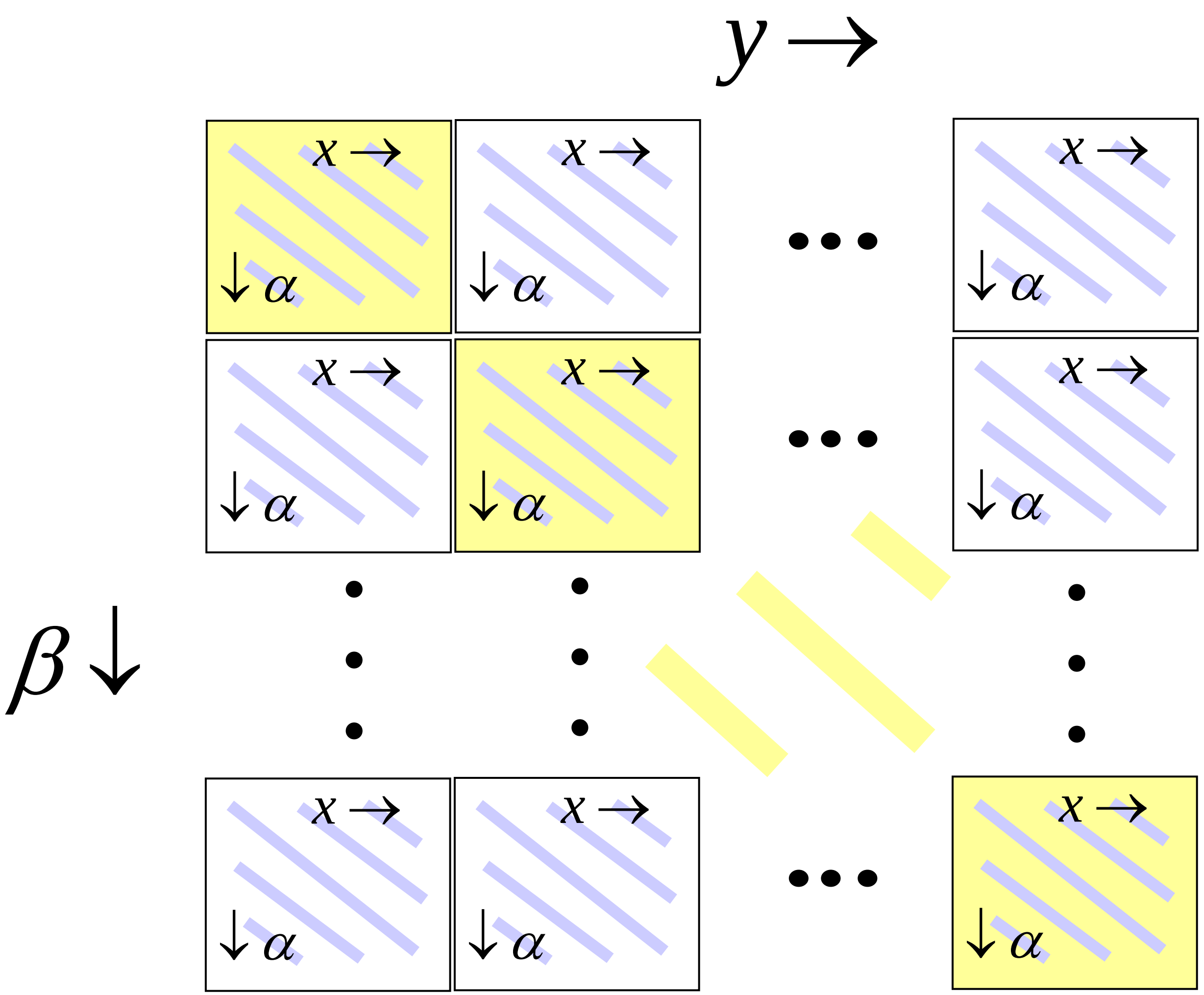
$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_{siv}[\alpha-x,\beta-y]$$

- Viewed as a matrix operation . . .

$$\vec{g}=H\vec{f}$$

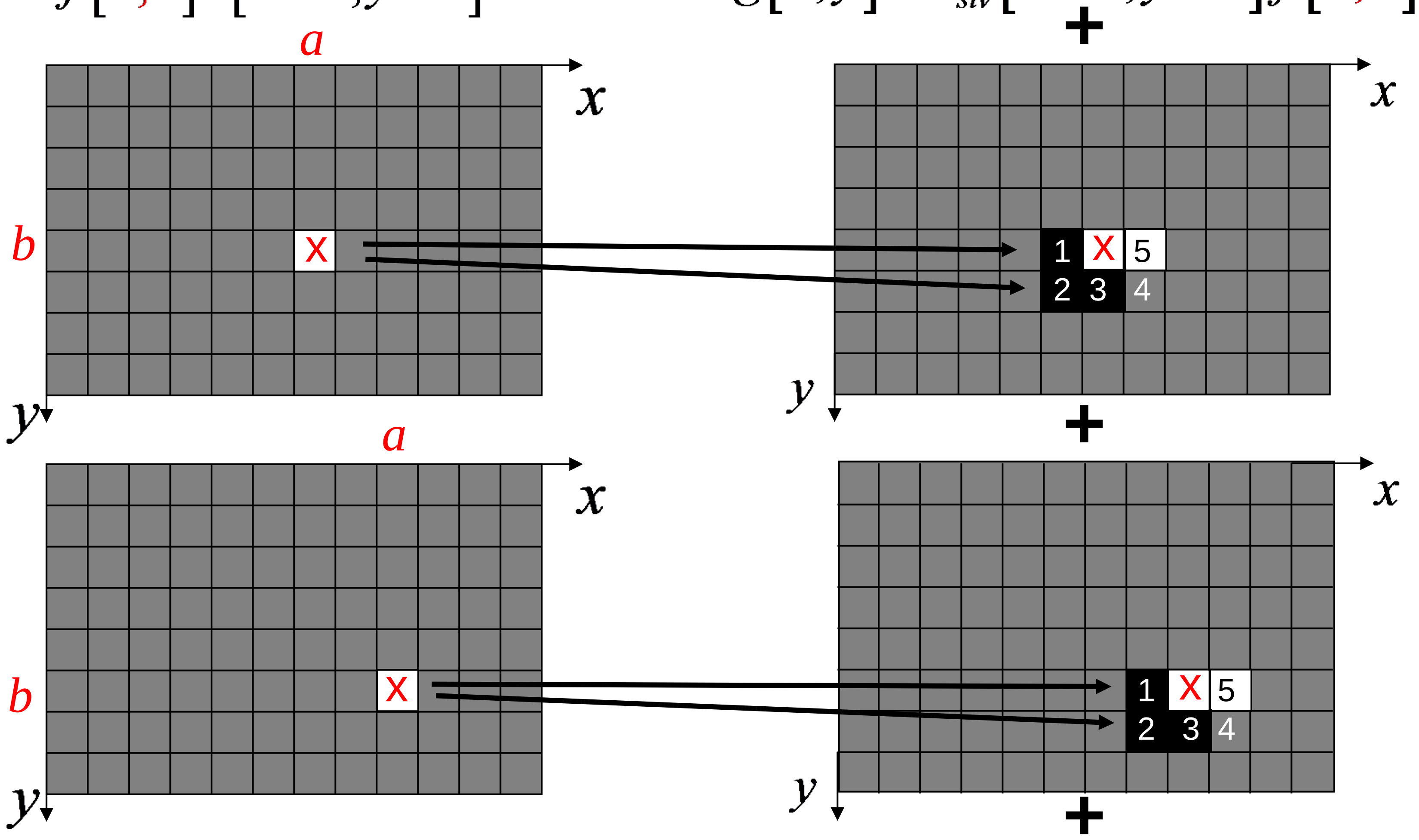
. . . ***H*** is a block Toeplitz matrix

Structure of $\boldsymbol{H}$ for non-separable convolution



Convolution: superposition of impulse responses

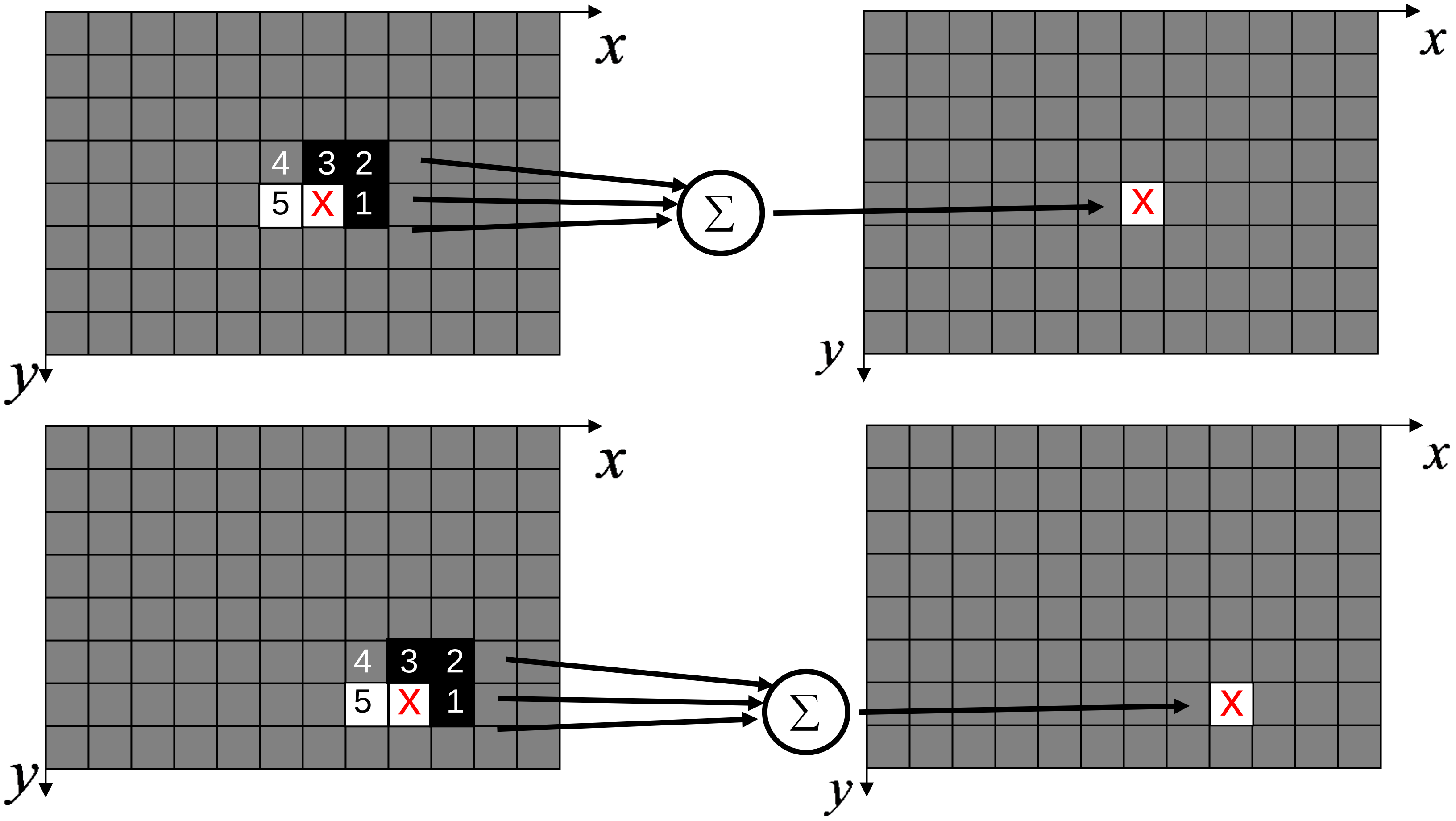
$$f[a,b]\,\delta[x-a,y-b] \qquad\qquad g[x,y]=h_{siv}[x-a,y-b]\,f[a,b]$$



Convolution: linear combination of neighboring pixel values

$$f[x,y] \cdot h_{siv}[\alpha - x, \beta - y]$$

$$g[\alpha,\beta] = \sum_{x=0}^{N-1} \sum_{y=0}^{L-1} f[x,y] \cdot h_{siv}[\alpha - x, \beta - y]$$



Convolution examples

Original
Bike

Bike blurred by convolution

Impulse response „box filter“

$$\frac{1}{25}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & [1] & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Convolution examples

Original
Bike

Bike blurred horizontally

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 & 1 & [1] & 1 & 1 \end{pmatrix}$$

Convolution examples

Original
Bike

Bike blurred vertically

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ [1] \\ 1 \\ 1 \end{pmatrix}$$