1-d discrete-time Fourier transform

- Given a 1-d sequence $s[k], \quad k \in \mathbf{Z} = \{\ldots,-1,0,1,2,3,\ldots\}$
- Fourier transform

$$S\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} s[k] e^{-j\omega k} \qquad \omega \in \Re$$

- Fourier transform is periodic with 2π
- Inverse Fourier transform

$$s[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S\left(e^{j\omega}\right) e^{j\omega k} d\omega$$

2-d discrete-space Fourier transform

- Given a 2-d array of image samples

$$s[m,n], \quad m,n \in \mathbf{Z}^2$$

- Fourier transform

$$S\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n] e^{-j\omega_x m - j\omega_y n} \qquad \omega_x, \omega_y \in \Re^2$$

- Fourier transform is 2π-periodic both in ω_x and ω_y
- Inverse Fourier transform

$$s[m,n] = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S\left(e^{j\omega_x}, e^{j\omega_y}\right) e^{j\omega_x m + j\omega_y n} d\omega_x d\omega_y$$

5x5 box filter revisited

Original
Bike

Bike blurred by convolution
Impulse response „box filter“

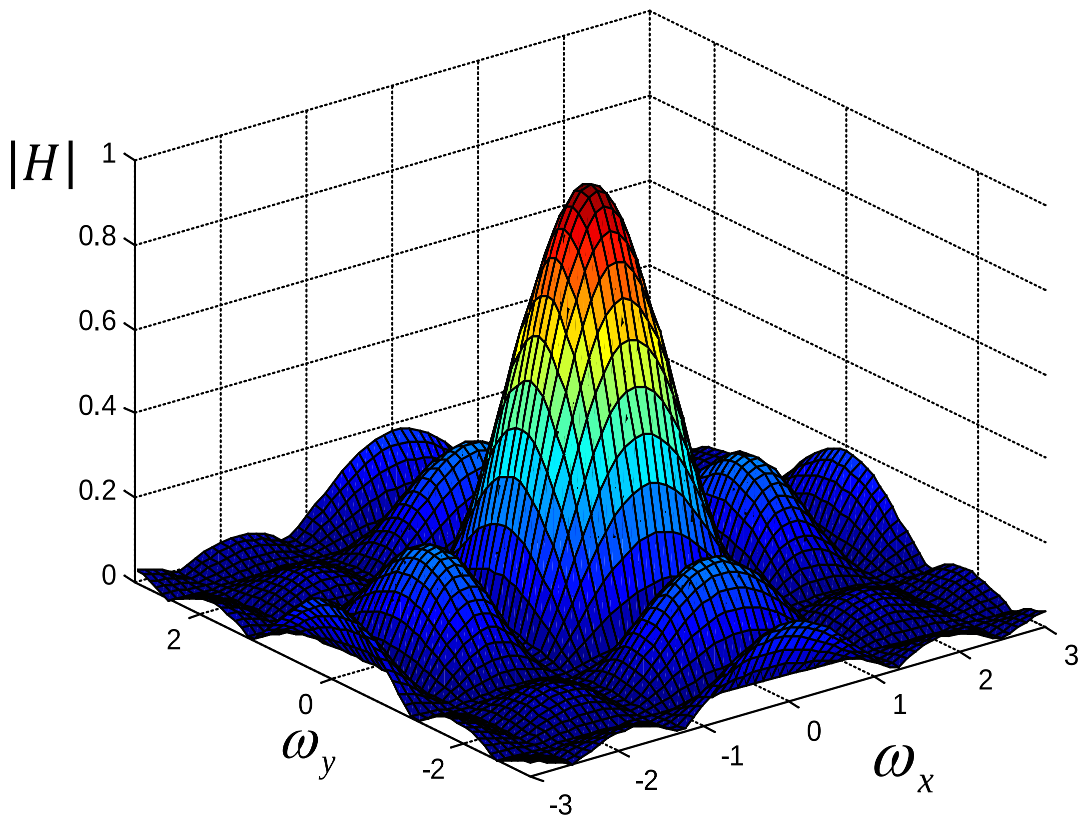
$$\frac{1}{25}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & [1] & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Frequency response of 5x5 lowpass filter

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m,n] e^{-j\omega_x m - j\omega_y n}$$

$$= \frac{1}{25} \sum_{m=-2}^{2} \sum_{n=-2}^{2} e^{-j\omega_x m - j\omega_y n} = \frac{1}{25} \sum_{m=-2}^{2} e^{-j\omega_x m} \sum_{n=-2}^{2} e^{-j\omega_y n}$$

$$= \frac{1}{25} \left(1 + 2\cos\omega_x + 2\cos(2\omega_x)\right)\left(1 + 2\cos\omega_y + 2\cos(2\omega_y)\right)$$



Separable filter:
1-d frequency responses are multiplied

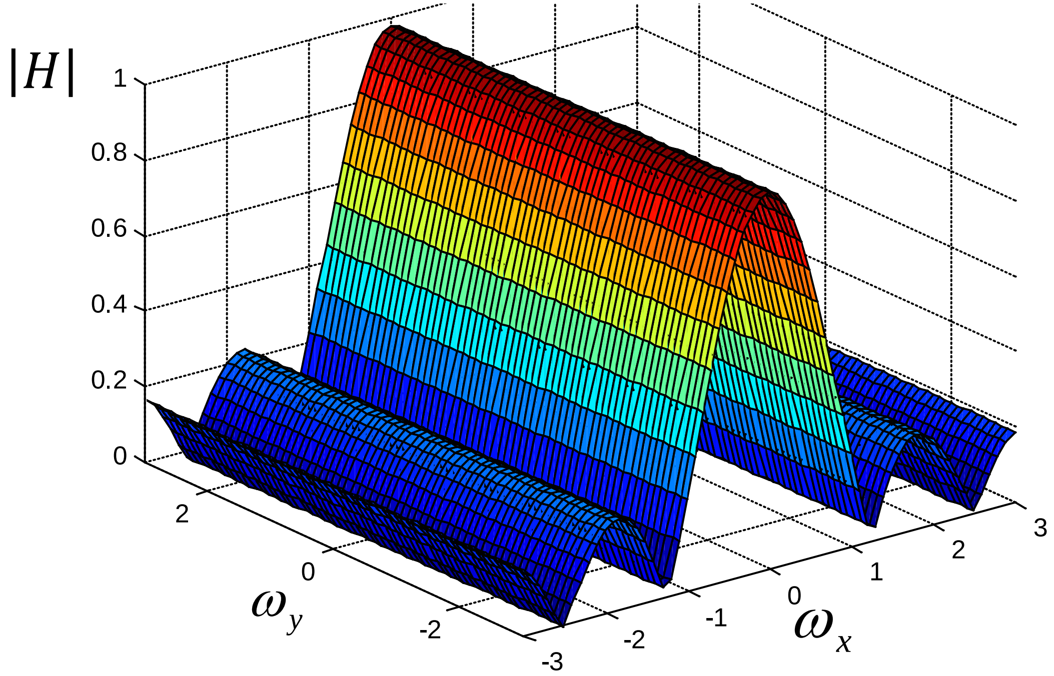
Horizontal lowpass filter

Bike blurred horizontally

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 & 1 & [1] & 1 & 1 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{1}{5}\left(1 + 2\cos\omega_x + 2\cos(2\omega_x)\right)$$



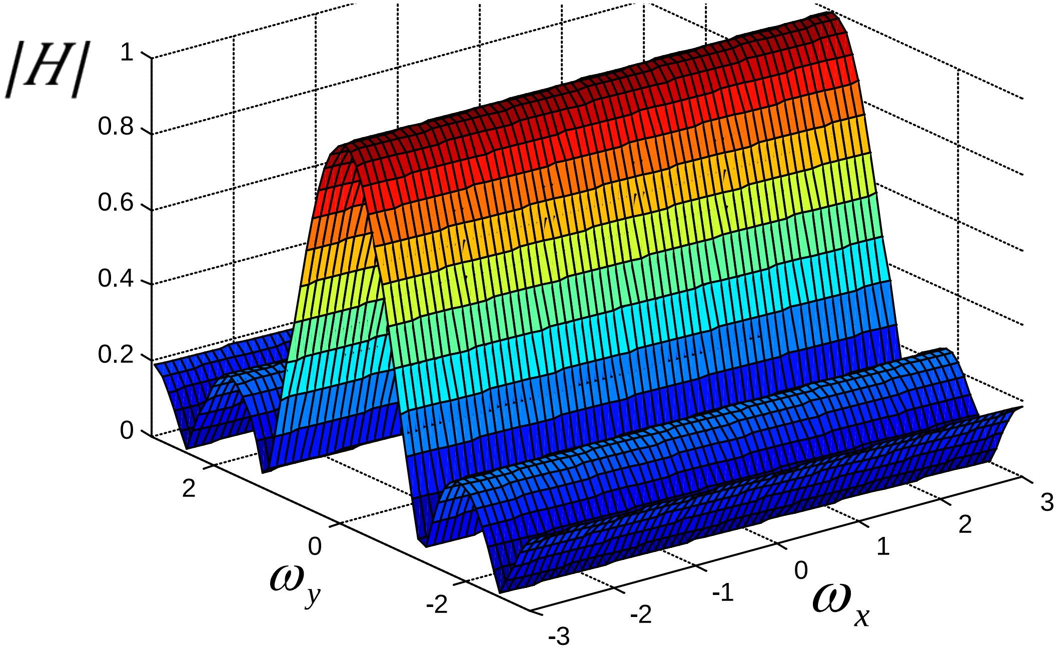
Vertical lowpass filter

Bike blurred vertically

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ [1] \\ 1 \\ 1 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{1}{5}\left(1 + 2\cos\omega_y + 2\cos(2\omega_y)\right)$$



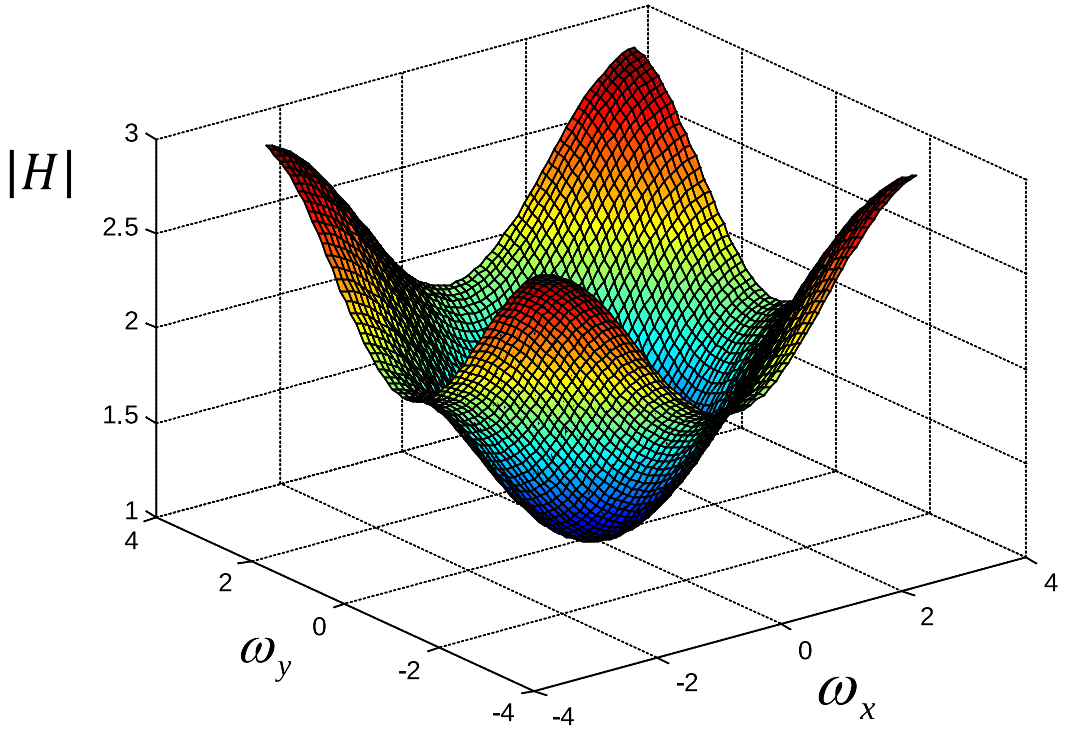
Sharpening filter

Original
Bike

Bike sharpened
Filter impulse response

$$\frac{1}{4}\begin{pmatrix} 0 & -1 & 0 \\ -1 & [8] & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Frequency response of sharpening filter



$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m,n] e^{-j\omega_x m - j\omega_y n}$$

$$= \frac{1}{4}\left(8 - e^{-j\omega_x} - e^{j\omega_x} - e^{-j\omega_y} - e^{j\omega_y}\right)$$

$$= 2 - \frac{1}{2}\cos\omega_x - \frac{1}{2}\cos\omega_y$$

More aggressive sharpening

Bike sharpened

Filter impulse response

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & [5] & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = 5 - 2\cos\omega_x - 2\cos\omega_y$$