Separable linear image processing

- Impulse response is separable in $[x,\alpha]$ and $[y,\beta]$, i.e., can be written as

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_x[x,\alpha]h_y[y,\beta]$$

- Processing can be carried out
  - row by row, then column by column

$$g[\alpha,\beta]=\sum_{y=0}^{L-1} h_y[y,\beta]\sum_{x=0}^{N-1} f[x,y]\cdot h_x[x,\alpha]$$

  - column by column, then row by row

$$g[\alpha,\beta]=\sum_{x=0}^{N-1} h_x[x,\alpha]\sum_{y=0}^{L-1} f[x,y]\cdot h_y[y,\beta]$$

Separable linear image processing (cont.)

- If the digital input and output images are written as a matrices **g** and **f**, we can conveniently write

$$\mathbf{g} = \mathbf{H}_\mathbf{y}^\mathrm{T} \cdot \mathbf{f} \cdot \mathbf{H}_\mathbf{x}$$

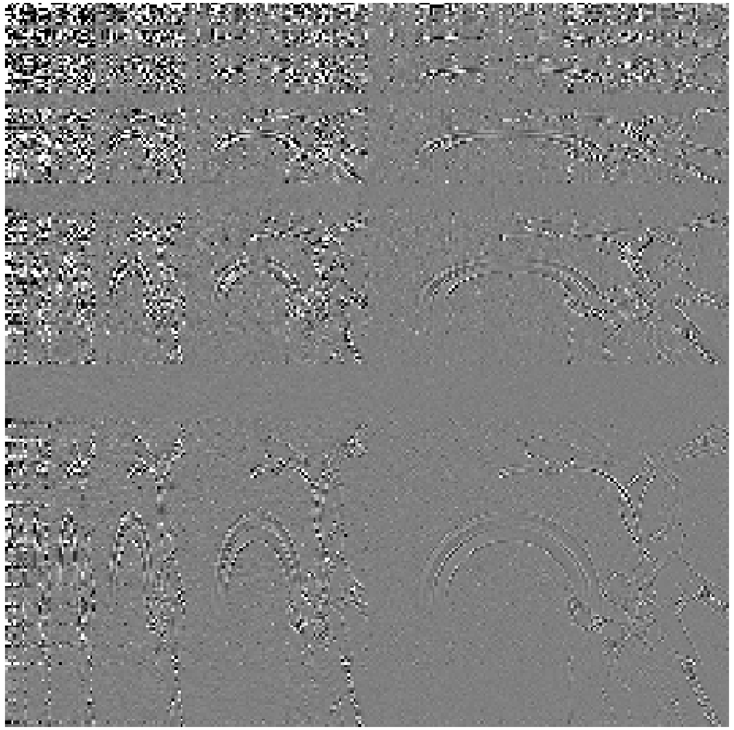
$$\mathbf{H}_y = \begin{bmatrix} h_y[0,0] & h_y[0,1] & \cdots & h_y[0,L_g-1] \\ h_y[1,0] & h_y[1,1] & \cdots & h_y[1,L_g-1] \\ \vdots & \vdots & & \vdots \\ h_y[L-1,0] & h_y[L-1,1] & \cdots & h_y[L-1,L_g-1] \end{bmatrix}$$

$$\mathbf{H}_x = \begin{bmatrix} h_x[0,0] & h_x[0,1] & \cdots & h_x[0,N_g-1] \\ h_x[1,0] & h_x[1,1] & \cdots & h_x[1,N_g-1] \\ \vdots & \vdots & & \vdots \\ h_x[N-1,0] & h_x[N-1,1] & \cdots & h_x[N-1,N_g-1] \end{bmatrix}$$

- Output image **g** has size L_g x N_g
- If the operator does not change image size, $\mathbf{H_x}$ and $\mathbf{H_y}$ are square matrices

Example: Separable Haar transform

Original *Bike*
256x256



256x256
Haar transform
$H_x = H_y = Hr_{256}$

Haar transform

- Haar transform matrix for sizes $N=2,4,8$

$$\mathbf{Hr}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{Hr}_4 = \frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}$$

$$\mathbf{Hr}_8 = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{pmatrix}$$

- Can be computed by taking sums and differences
- Fast algorithms by recursively applying $\mathbf{Hr}_2$

Example: Subsampling

- Image subsampling 2:1 horizontally and vertically
- Small input image of size 8x8, output image size 4x4

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example: Subsampling (cont.)

- A somewhat better technique for 2:1 image size reduction

$$\mathbf{H_x} = \mathbf{H_y} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Side by side comparison

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Another example: filtering

- Each pixel is replaced by the average of two horizontally (vertically) neighboring pixels

$$\mathbf{H_x} = \mathbf{H_y} = \begin{bmatrix} 0.5 & 0 & & \cdots & & & & 0 \\ 0.5 & 0.5 & & & & & & \\ 0 & 0.5 & 0.5 & & & & & \\ & & 0.5 & 0.5 & \ddots & & & \vdots \\ & & & 0.5 & 0.5 & & & \\ \vdots & & & \ddots & 0.5 & 0.5 & & \\ & & & & & 0.5 & 0.5 & 0 \\ 0 & & & \cdots & & 0 & 0.5 & 0.5 \end{bmatrix}$$

- Shift-invariant operation (except for image boundary)