Convex Optimization

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4. Convex optimization problems

Outline

Optimization problems

Some standard convex problems

Transforming problems

Disciplined convex programming

Geometric programming

Quasiconvex optimization

Multicriterion optimization

Optimization problem in standard form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

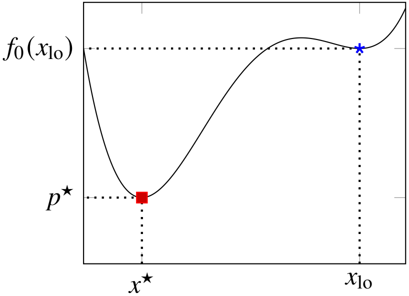
Feasible and optimal points

- $x \in \mathbf{R}^n$ is **feasible** if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints
- **optimal value** is $p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$
- $p^\star = \infty$ if problem is infeasible
- $p^\star = -\infty$ if problem is **unbounded below**
- a feasible x is **optimal** if $f_0(x) = p^\star$
- X_{opt} is the set of optimal points

Locally optimal points

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

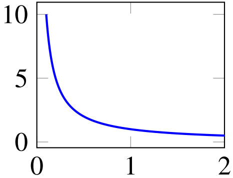
$$\begin{array}{ll} \text{minimize (over } z\text{)} & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \leq R \end{array}$$



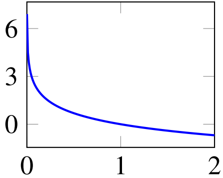
Examples

examples with $n = 1$, $m = p = 0$

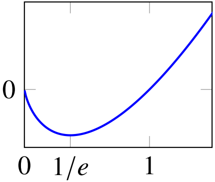
- $f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -\infty$
- $f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$: $p^\star = -\infty$, $x = 1$ is locally optimal



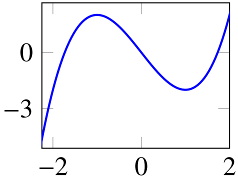
$f_0(x) = 1/x$



$f_0(x) = -\log x$



$f_0(x) = x \log x$



$f_0(x) = x^3 - 3x$

Implicit and explicit constraints

standard form optimization problem has **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- we call $\mathcal{D}$ the **domain** of the problem
- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the **explicit constraints**
- a problem is **unconstrained** if it has no explicit constraints ($m = p = 0$)

example:

$$\text{minimize} \quad f_0(x) = -\textstyle\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Standard form convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array}$$

- objective and inequality constraints $f_0, f_1, \ldots, f_m$ are convex
- equality constraints are affine, often written as $Ax = b$
- feasible and optimal sets of a convex optimization problem are convex

- problem is **quasiconvex** if f_0 is quasiconvex, $f_1, \ldots, f_m$ are convex, $h_1, \ldots, h_p$ are affine

Example

- standard form problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1+x_2)^2 = 0 \end{array}$$

- f_0 is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (by our definition) since f_1 is not convex, h_1 is not affine
- equivalent (but not identical) to the convex problem

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{array}$$

Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof:

- suppose x is locally optimal, but there exists a feasible y with $f_0(y) < f_0(x)$
- x locally optimal means there is an $R > 0$ such that

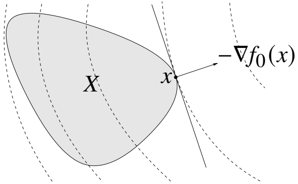
$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Longrightarrow \quad f_0(z) \geq f_0(x)$$

- consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$
- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and $f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x) < f_0(x)$, which contradicts our assumption that x is locally optimal

Optimality criterion for differentiable f_0

- x is optimal for a convex problem if and only if it is feasible and

$$\nabla f_0(x)^T(y-x) \geq 0 \quad \text{for all feasible } y$$



- if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

Examples

- **unconstrained problem**: x minimizes $f_0(x)$ if and only if $\nabla f_0(x) = 0$
- **equality constrained problem**: x minimizes $f_0(x)$ subject to $Ax = b$ if and only if there exists a ν such that

$$Ax = b, \qquad \nabla f_0(x) + A^T \nu = 0$$

- **minimization over nonnegative orthant**: x minimizes $f_0(x)$ over $\mathbf{R}^n_+$ if and only if

$$x \succeq 0, \qquad \left\{ \begin{array}{ll} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{array} \right.$$

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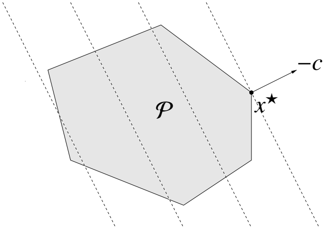
Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Example: Diet problem

- choose nonnegative quantities $x_1, \ldots, x_n$ of n foods
- one unit of food j costs c_j and contains amount A_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i
- to find cheapest healthy diet, solve

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \quad x \geq 0 \end{array}$$

- express in standard LP form as

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \left[\begin{array}{c} -A \\ -I \end{array}\right] x \leq \left[\begin{array}{c} -b \\ 0 \end{array}\right] \end{array}$$

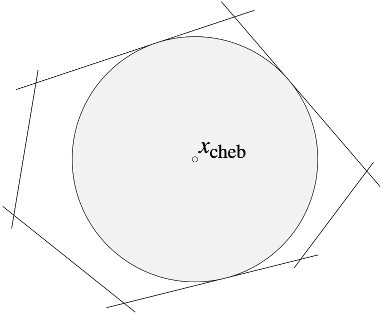
Example: Piecewise-linear minimization

- minimize convex piecewise-linear function $f_0(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$, $x \in \mathbf{R}^n$
- equivalent to LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

  with variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$
- constraints describe $\mathbf{epi}\, f_0$

Example: Chebyshev center of a polyhedron



Chebyshev center of $\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$ is center of largest inscribed ball $\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$

- $a_i^T x \leq b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) \mid \|u\|_2 \leq r\} = a_i^T x_c + r\|a_i\|_2 \leq b_i$$

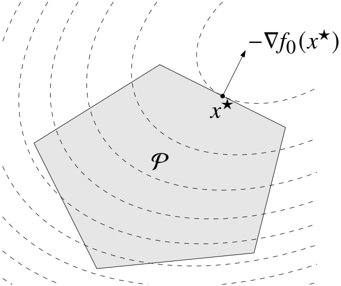
- hence, x_c, r can be determined by solving LP with variables x_c, r

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Example: Least squares

- **least squares** problem: minimize $\|Ax - b\|_2^2$
- analytical solution $x^\star = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)
- can add linear constraints, *e.g.*,
  - $x \geq 0$ (**nonnegative least squares**)
  - $x_1 \leq x_2 \leq \cdots \leq x_n$ (**isotonic regression**)

Example: Linear program with random cost

- LP with random cost c, with mean $\bar{c}$ and covariance Σ
- hence, LP objective $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- **risk-averse** problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c^T x + \gamma\, \mathbf{var}(c^T x) \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

- $\gamma > 0$ is **risk aversion parameter**; controls the trade-off between expected cost and variance (risk)
- express as QP

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

Quadratically constrained quadratic program (QCQP)

$$\begin{array}{ll}\text{minimize} & (1/2)x^TP_0x + q_0^Tx + r_0 \\ \text{subject to} & (1/2)x^TP_ix + q_i^Tx + r_i \leq 0, \quad i = 1, \ldots, m \\ & Ax = b\end{array}$$

- $P_i \in \mathbf{S}^n_+$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}^n_{++}$, feasible region is intersection of m ellipsoids and an affine set

Second-order cone programming

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

($A_i \in \mathbf{R}^{n_i \times n}$, $F \in \mathbf{R}^{p \times n}$)

- inequalities are called second-order cone (SOC) constraints:

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbf{R}^{n_i+1}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP
- more general than QCQP and LP

Example: Robust linear programming

suppose constraint vectors a_i are uncertain in the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m, \end{array}$$

two common approaches to handling uncertainty

- **deterministic worst-case**: constraints must hold for all $a_i \in \mathcal{E}_i$ (uncertainty ellipsoids)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m, \end{array}$$

- **stochastic**: a_i is random variable; constraints must hold with probability η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \le b_i) \ge \eta, \quad i = 1, \ldots, m \end{array}$$

Deterministic worst-case approach

- uncertainty ellipsoids are $\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$, ($\bar{a}_i \in \mathbf{R}^n$, $P_i \in \mathbf{R}^{n \times n}$)
- center of $\mathcal{E}_i$ is $\bar{a}_i$; semi-axes determined by singular values/vectors of P_i
- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

- equivalent to SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

(follows from $\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

Stochastic approach

- assume $a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$
- $a_i^T x \sim \mathcal{N}(\bar{a}_i^T x, x^T \Sigma_i x)$, so

$$\mathbf{prob}(a_i^T x \leq b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2}\right)$$

where $\Phi(u) = (1/\sqrt{2\pi}) \int_{-\infty}^{u} e^{-t^2/2}\, dt$ is $\mathcal{N}(0,1)$ CDF

- $\mathbf{prob}(a_i^T x \leq b_i) \geq \eta$ can be expressed as $\bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i$
- for $\eta \geq 1/2$, robust LP equivalent to SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Conic form problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b \end{array}$$

- constraint $Fx + g \preceq_K 0$ involves a generalized inequality with respect to a proper cone K
- linear programming is a conic form problem with $K = \mathbf{R}^m_+$
- as with standard convex problem
  - feasible and optimal sets are convex
  - any local optimum is global

Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

with $F_i, G \in \mathbf{S}^k$

- inequality constraint is called **linear matrix inequality** (LMI)
- includes problems with multiple LMI constraints: for example,

$$x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \qquad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

  is equivalent to single LMI

$$x_1 \left[\begin{array}{cc} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{array} \right] + x_2 \left[\begin{array}{cc} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{array} \right] + \cdots + x_n \left[\begin{array}{cc} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{array} \right] + \left[\begin{array}{cc} \hat{G} & 0 \\ 0 & \tilde{G} \end{array} \right] \preceq 0$$

Example: Matrix norm minimization

$$\text{minimize} \quad \|A(x)\|_2 = \left(\lambda_{\max}(A(x)^T A(x))\right)^{1/2}$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{R}^{p\times q}$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} tI & A(x) \\ A(x)^T & tI \end{array}\right] \succeq 0 \end{array}$$

- variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$
- constraint follows from

$$\begin{aligned} \|A\|_2 \leq t \quad & \Longleftrightarrow \quad A^T A \preceq t^2 I, \quad t \geq 0 \\ & \Longleftrightarrow \quad \left[\begin{array}{cc} tI & A \\ A^T & tI \end{array}\right] \succeq 0 \end{aligned}$$

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Change of variables

- $\phi : \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one with $\phi(\mathbf{dom}\,\phi) \supseteq \mathcal{D}$
- consider (possibly non-convex) problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- change variables to z with $x = \phi(z)$
- can solve equivalent problem

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(z) \\ \text{subject to} & \tilde{f}_i(z) \leq 0, \quad i = 1, \ldots, m \\ & \tilde{h}_i(z) = 0, \quad i = 1, \ldots, p \end{array}$$

  where $\tilde{f}_i(z) = f_i(\phi(z))$ and $\tilde{h}_i(z) = h_i(\phi(z))$
- recover original optimal point as $x^\star = \phi(z^\star)$

Example

- **non-convex** problem

$$\begin{array}{ll} \text{minimize} & x_1/x_2 + x_3/x_1 \\ \text{subject to} & x_2/x_3 + x_1 \leq 1 \end{array}$$

  with implicit constraint $x \succ 0$

- change variables using $x = \phi(z) = \exp z$ to get

$$\begin{array}{ll} \text{minimize} & \exp(z_1 - z_2) + \exp(z_3 - z_1) \\ \text{subject to} & \exp(z_2 - z_3) + \exp(z_1) \leq 1 \end{array}$$

  which is **convex**

Transformation of objective and constraint functions

suppose

- ϕ_0 is monotone increasing
- $\psi_i(u) \leq 0$ if and only if $u \leq 0$, $i = 1, \dots, m$
- $\varphi_i(u) = 0$ if and only if $u = 0$, $i = 1, \dots, p$

standard form optimization problem is equivalent to

$$\begin{array}{lll} \text{minimize} & \phi_0(f_0(x)) & \\ \text{subject to} & \psi_i(f_i(x)) \leq 0, & i = 1, \dots, m \\ & \varphi_i(h_i(x)) = 0, & i = 1, \dots, p \end{array}$$

example: minimizing $\|Ax - b\|$ is equivalent to minimizing $\|Ax - b\|^2$

Converting maximization to minimization

- suppose ϕ_0 is monotone decreasing
- the maximization problem

$$\begin{array}{lll} \text{maximize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

  is equivalent to the minimization problem

$$\begin{array}{lll} \text{minimize} & \phi_0(f_0(x)) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

- **examples**:
  - $\phi_0(u) = -u$ transforms maximizing a concave function to minimizing a convex function
  - $\phi_0(u) = 1/u$ transforms maximizing a concave positive function to minimizing a convex function

Eliminating equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } z\text{)} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m \end{array}$$

where F and x_0 are such that $Ax = b \iff x = Fz + x_0$ for some z

Introducing equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(A_0x+b_0) \\ \text{subject to} & f_i(A_ix+b_i) \leq 0, \quad i=1,\ldots,m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x,\, y_i\text{)} & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i=1,\ldots,m \\ & y_i = A_ix+b_i, \quad i=0,1,\ldots,m \end{array}$$

Introducing slack variables for linear inequalities

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{minimize (over } x, s) & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\ & s_i \geq 0, \quad i = 1, \ldots m\end{array}$$

Epigraph form

standard form convex problem is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, t) & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

Minimizing over some variables

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{array}$$

where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

LP and SOCP as SDP

LP and equivalent SDP

LP: $\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$ SDP: $\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{diag}(Ax - b) \preceq 0 \end{array}$

(note different interpretation of generalized inequalities $\preceq$ in LP and SDP)

SOCP and equivalent SDP

SOCP:
$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

SDP:
$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \left[\begin{array}{cc} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{array} \right] \succeq 0, \quad i = 1, \ldots, m \end{array}$$

Convex relaxation

- start with **nonconvex problem**: minimize $h(x)$ subject to $x \in C$
- find convex function $\hat{h}$ with $\hat{h}(x) \leq h(x)$ for all $x \in \mathbf{dom}\, h$ (*i.e.*, a pointwise lower bound on h)
- find set $\hat{C} \supseteq C$ (*e.g.*, $\hat{C} = \mathbf{conv}\, C$) described by linear equalities and convex inequalities

$$\hat{C} = \{x \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ f_m(x) \leq 0,\ Ax = b\}$$

- convex problem

$$\begin{array}{ll} \text{minimize} & \hat{h}(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b \end{array}$$

  is a **convex relaxation** of the original problem
- optimal value of relaxation is lower bound on optimal value of original problem

Example: Boolean LP

- **mixed integer linear program (MILP)**:

$$\begin{array}{ll} \text{minimize} & c^T(x,z) \\ \text{subject to} & F(x,z) \preceq g, \quad A(x,z) = b, \quad z \in \{0,1\}^q \end{array}$$

  with variables $x \in \mathbf{R}^n$, $z \in \mathbf{R}^q$

- z_i are called **Boolean variables**
- this problem is in general hard to solve

- **LP relaxation**: replace $z \in \{0,1\}^q$ with $z \in [0,1]^q$
- optimal value of relaxation LP is lower bound on MILP
- can use as heuristic for approximately solving MILP, *e.g.*, **relax and round**

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Some standard convex problems

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Disciplined convex programming

Geometric programming

Quasiconvex optimization

Multicriterion optimization

Disciplined convex program

- specify objective as
  - minimize {scalar convex expression}, or
  - maximize {scalar concave expression}
- specify constraints as
  - {convex expression} <= {concave expression} or
  - {concave expression} >= {convex expression} or
  - {affine expression} == {affine expression}
- curvature of expressions are DCP certified, *i.e.*, follow composition rule
- DCP-compliant problems can be automatically transformed to standard forms, then solved

CVXPY example

math:

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & Ax = b \\ & \|x\|_\infty \leq 1 \end{array}$$

- x is the variable
- A, b are given

CVXPY code:

```
import cvxpy as cp

A, b = ...

x = cp.Variable(n)
obj = cp.norm(x, 1)
constr = [
  A @ x == b,
  cp.norm(x, 'inf') <= 1,
]
prob = cp.Problem(cp.Minimize(obj), constr)
prob.solve()
```

How CVXPY works

- starts with your optimization problem $\mathcal{P}_1$
- finds a sequence of equivalent problems $\mathcal{P}_2, \ldots, \mathcal{P}_N$
- final problem $\mathcal{P}_N$ matches a standard form (*e.g.*, LP, QP, SOCP, or SDP)
- calls a specialized solver on $\mathcal{P}_N$
- retrieves solution of original problem by reversing the transformations

$$\begin{tikzcd}
\mathcal{P}_1 \arrow[r, Leftrightarrow] & \mathcal{P}_2 \arrow[r, Leftrightarrow] & \cdots \arrow[r, Leftrightarrow] & \mathcal{P}_{N-1} \arrow[r, Leftrightarrow] & \mathcal{P}_N \\
\text{your problem} \arrow[u] & & & & \text{standard problem} \arrow[u]
\end{tikzcd}$$

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- **monomial function**:
$$f(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$
with $c > 0$; exponent a_i can be any real number
- **posynomial function**: sum of monomials
$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$
- **geometric program (GP)**
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \ldots, m \\ & h_i(x) = 1, \quad i = 1, \ldots, p \end{array}$$
with f_i posynomial, h_i monomial

Geometric program in convex form

- change variables to $y_i = \log x_i$, and take logarithm of cost, constraints
- monomial $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \qquad (b = \log c)$$

- posynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = \log \left(\sum_{k=1}^{K} e^{a_k^T y + b_k} \right) \qquad (b_k = \log c_k)$$

- geometric program transforms to convex problem

$$\begin{array}{ll} \text{minimize} & \log \left(\sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k}) \right) \\ \text{subject to} & \log \left(\sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \\ & Gy + d = 0 \end{array}$$

Examples: Frobenius norm diagonal scaling

- we seek diagonal matrix $D = \mathbf{diag}(d)$, $d \succ 0$, to minimize $\|DMD^{-1}\|_F^2$
- express as

$$\|DMD^{-1}\|_F^2 = \sum_{i,j=1}^{n} \left(DMD^{-1}\right)_{ij}^2 = \sum_{i,j=1}^{n} M_{ij}^2 d_i^2 / d_j^2$$

- a posynomial in d (with exponents 0, 2, and -2)
- in convex form, with $y = \log d$,

$$\log \|DMD^{-1}\|_F^2 = \log\left(\sum_{i,j=1}^{n} \exp\left(2(y_i - y_j + \log |M_{ij}|)\right)\right)$$

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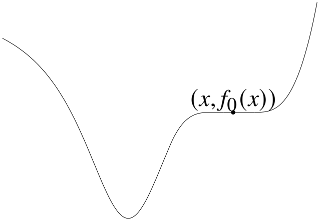
Quasiconvex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

with $f_0 : \mathbf{R}^n \to \mathbf{R}$ quasiconvex, $f_1, \ldots, f_m$ convex

can have locally optimal points that are not (globally) optimal



Linear-fractional program

- **linear-fractional program**

$$\begin{array}{ll} \text{minimize} & (c^T x + d)/(e^T x + f) \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

  with variable x and implicit constraint $e^T x + f > 0$

- equivalent to the LP (with variables y, z)

$$\begin{array}{ll} \text{minimize} & c^T y + dz \\ \text{subject to} & Gy \preceq hz, \quad Ay = bz \\ & e^T y + fz = 1, \quad z \geq 0 \end{array}$$

- recover $x^\star = y^\star / z^\star$

Von Neumann model of a growing economy

- $x, x^+ \in \mathbf{R}^n_{++}$: activity levels of n economic sectors, in current and next period
- $(Ax)_i$: amount of good i produced in current period
- $(Bx^+)_i$: amount of good i consumed in next period
- $Bx^+ \preceq Ax$: goods consumed next period no more than produced this period
- x_i^+/x_i: growth rate of sector i
- allocate activity to maximize growth rate of slowest growing sector

$$\begin{array}{ll} \text{maximize (over } x, x^+) & \min_{i=1,\ldots,n} x_i^+/x_i \\ \text{subject to} & x^+ \succeq 0, \quad Bx^+ \preceq Ax \end{array}$$

- a quasiconvex problem with variables x, x^+

Convex representation of sublevel sets

- if f_0 is quasiconvex, there exists a family of functions ϕ_t such that:
  - $\phi_t(x)$ is convex in x for fixed t
  - t-sublevel set of f_0 is 0-sublevel set of ϕ_t, *i.e.*, $f_0(x) \leq t \iff \phi_t(x) \leq 0$

example:

- $f_0(x) = p(x)/q(x)$, with p convex and nonnegative, q concave and positive
- take $\phi_t(x) = p(x) - tq(x)$: for $t \geq 0$,
  - ϕ_t convex in x
  - $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$

Bisection method for quasiconvex optimization

- for fixed t, consider convex feasiblity problem

$$\phi_t(x) \leq 0, \qquad f_i(x) \leq 0, \quad i = 1, \ldots, m, \qquad Ax = b \tag{1}$$

  if feasible, we can conclude that $t \geq p^\star$; if infeasible, $t \leq p^\star$

- bisection method:

```
given $l \leq p^\star$, $u \geq p^\star$, tolerance $\epsilon > 0$.
repeat
  1. $t := (l + u)/2$.
  2. Solve the convex feasibility problem (1).
  3. if (1) is feasible, $u := t$;   else $l := t$.
until $u - l \leq \epsilon$.
```

- requires exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations

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- **multicriterion** or **multi-objective** problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) = (F_1(x), \ldots, F_q(x)) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b \end{array}$$

- objective is the **vector** $f_0(x) \in \mathbf{R}^q$
- q different objectives $F_1, \ldots, F_q$; roughly speaking we want all F_i's to be small
- feasible $x^\star$ is **optimal** if y feasible $\implies f_0(x^\star) \preceq f_0(y)$
- this means that $x^\star$ simultaneously minimizes each F_i; the objectives are **noncompeting**
- not surprisingly, this doesn't happen very often

Pareto optimality

- feasible x **dominates** another feasible $\tilde{x}$ if $f_0(x) \preceq f_0(\tilde{x})$ and for at least one i, $F_i(x) < F_i(\tilde{x})$
- *i.e.*, x meets $\tilde{x}$ on all objectives, and beats it on at least one

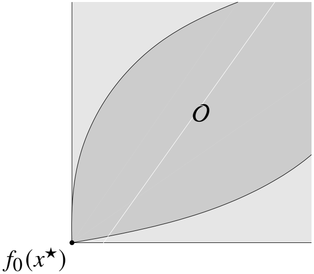
- feasible x^{po} is **Pareto optimal** if it is not dominated by any feasible point
- can be expressed as: y feasible, $f_0(y) \preceq f_0(x^{\mathrm{po}}) \implies f_0(x^{\mathrm{po}}) = f_0(y)$

- there are typically many Pareto optimal points
- for $q = 2$, set of Pareto optimal objective values is the **optimal trade-off curve**
- for $q = 3$, set of Pareto optimal objective values is the **optimal trade-off surface**

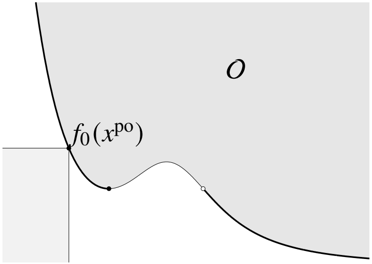
Optimal and Pareto optimal points

set of achievable objective values $\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$

- feasible x is **optimal** if $f_0(x)$ is the minimum value of $\mathcal{O}$
- feasible x is **Pareto optimal** if $f_0(x)$ is a minimal value of $\mathcal{O}$



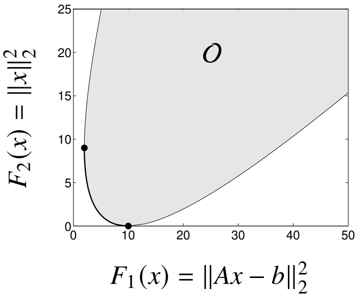
$x^\star$ is optimal



x^{po} is Pareto optimal

Regularized least-squares

- minimize $(\|Ax-b\|_2^2, \|x\|_2^2)$ (first objective is loss; second is regularization)
- example with $A \in \mathbf{R}^{100\times 10}$; heavy line shows Pareto optimal points

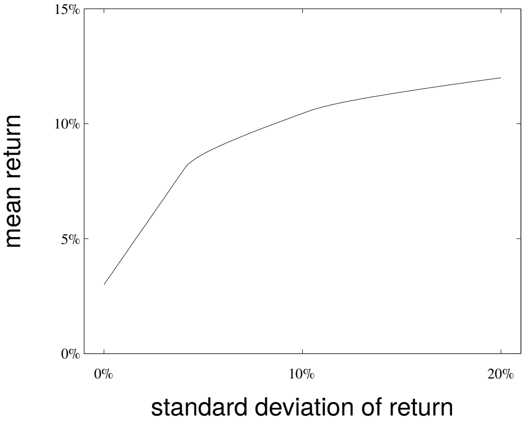


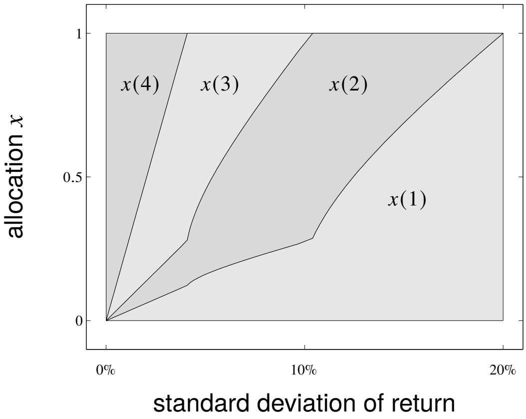
Risk return trade-off in portfolio optimization

- variable $x \in \mathbf{R}^n$ is investment portfolio, with x_i fraction invested in asset i
- $\bar{p} \in \mathbf{R}^n$ is mean, Σ is covariance of asset returns
- portfolio return has mean $\bar{p}^T x$, variance $x^T \Sigma x$

- minimize $(-\bar{p}^T x, x^T \Sigma x)$, subject to $\mathbf{1}^T x = 1$, $x \succeq 0$
- Pareto optimal portfolios trace out optimal risk-return curve

Example


15%
10%
5%
0%
0%
10%
20%
mean return
standard deviation of return


1
0.5
0
0%
10%
20%
$x(4)$
$x(3)$
$x(2)$
$x(1)$
allocation x
standard deviation of return

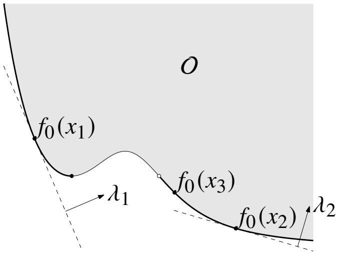
Scalarization

- **scalarization** combines the multiple objectives into one (scalar) objective
- a standard method for finding Pareto optimal points
- choose $\lambda \succ 0$ and solve scalar problem

$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

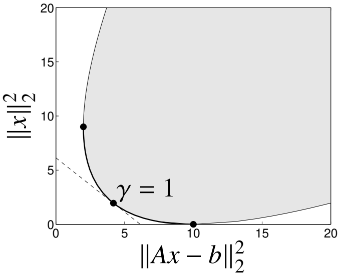
- λ_i are relative weights on the objectives
- if x is optimal for scalar problem, then it is Pareto-optimal for multicriterion problem
- for convex problems, can find (almost) all Pareto optimal points by varying $\lambda \succ 0$

Example



Example: Regularized least-squares

- regularized least-squares problem: minimize $(\|Ax - b\|_2^2, \|x\|_2^2)$
- take $\lambda = (1, \gamma)$ with $\gamma > 0$, and minimize $\|Ax - b\|_2^2 + \gamma\|x\|_2^2$



Example: Risk-return trade-off

- risk-return trade-off: minimize $(-\bar{p}^T x, x^T \Sigma x)$ subject to $\mathbf{1}^T x = 1$, $x \succeq 0$
- with $\lambda = (1, \gamma)$ we obtain scalarized problem

$$\begin{array}{ll} \text{minimize} & -\bar{p}^T x + \gamma x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

- objective is negative **risk-adjusted return**, $\bar{p}^T x - \gamma x^T \Sigma x$
- γ is called the **risk-aversion parameter**