Convex Optimization

Stephen Boyd   Lieven Vandenberghe

Revised slides by Stephen Boyd, Lieven Vandenberghe, and Parth Nobel

1. Introduction

Outline

Mathematical optimization

Convex optimization

Optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & g_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $x \in \mathbf{R}^n$ is (vector) variable to be chosen (n scalar variables $x_1, \ldots, x_n$)
- f_0 is the **objective function**, to be minimized
- $f_1, \ldots, f_m$ are the **inequality constraint functions**
- $g_1, \ldots, g_p$ are the **equality constraint functions**

- variations: maximize objective, multiple objectives, ...

Finding good (or best) actions

- x represents some **action**, *e.g.*,
  - trades in a portfolio
  - airplane control surface deflections
  - schedule or assignment
  - resource allocation
- constraints limit actions or impose conditions on outcome
- the smaller the objective $f_0(x)$, the better
  - total cost (or negative profit)
  - deviation from desired or target outcome
  - risk
  - fuel use

Finding good models

- x represents the **parameters** in a model
- constraints impose requirements on model parameters (*e.g.*, nonnegativity)
- objective $f_0(x)$ is sum of two terms:
  - a prediction error (or loss) on some observed data
  - a (regularization) term that penalizes model complexity

Worst-case analysis (pessimization)

- variables are actions or parameters out of our control
  (and possibly under the control of an adversary)
- constraints limit the possible values of the parameters
- minimizing $-f_0(x)$ finds **worst possible parameter values**

- if the worst possible value of $f_0(x)$ is tolerable, you're OK
- it's good to know what the worst possible scenario can be

Optimization-based models

- model an entity as taking actions that solve an optimization problem
  - an individual makes choices that maximize expected utility
  - an organism acts to maximize its reproductive success
  - reaction rates in a cell maximize growth
  - currents in a circuit minimize total power
- (except the last) these are **very crude** models
- and yet, they often work very well

Basic use model for mathematical optimization

- instead of saying how to choose (action, model) x
- you articulate what you want (by stating the problem)
- then let an algorithm decide on (action, model) x

Can you solve it?

- generally, no
- but you can try to solve it approximately, and it often doesn't matter

- the exception: **convex optimization**
  - includes linear programming (LP), quadratic programming (QP), many others
  - we can solve these problems reliably and efficiently
  - come up in many applications across many fields

Nonlinear optimization

traditional techniques for general nonconvex problems involve compromises

local optimization methods (nonlinear programming)

- find a point that minimizes f_0 among feasible points near it
- can handle large problems, *e.g.*, neural network training
- require initial guess, and often, algorithm parameter tuning
- provide no information about how suboptimal the point found is

global optimization methods

- find the (global) solution
- worst-case complexity grows exponentially with problem size
- often based on solving convex subproblems

Outline

Mathematical optimization

Convex optimization

Convex optimization

convex optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- equality constraints are linear
- $f_0, \ldots, f_m$ are **convex**: for $\theta \in [0, 1]$,

$$f_i(\theta x + (1 - \theta)y) \leq \theta f_i(x) + (1 - \theta) f_i(y)$$

i.e., f_i have nonnegative (upward) curvature

When is an optimization problem hard to solve?

- classical view:
  - linear (zero curvature) is easy
  - nonlinear (nonzero curvature) is hard

- the classical view is **wrong**

- the correct view:
  - convex (nonnegative curvature) is easy
  - nonconvex (negative curvature) is hard

Solving convex optimization problems

- many different algorithms (that run on many platforms)
  - interior-point methods for up to 10000s of variables
  - first-order methods for larger problems
  - do not require initial point, babysitting, or tuning
- can develop and deploy quickly using modeling languages such as CVXPY
- solvers are reliable, so can be embedded
- code generation yields real-time solvers that execute in milliseconds (*e.g.*, on Falcon 9 and Heavy for landing)

Modeling languages for convex optimization

- domain specific languages (DSLs) for convex optimization
  - describe problem in high level language, close to the math
  - can automatically transform problem to standard form, then solve

- enables rapid prototyping
- it's now much easier to develop an optimization-based application
- ideal for teaching and research (can do a lot with short scripts)

- gets close to the basic idea: **say what you want, not how to get it**

CVXPY example: non-negative least squares

math:

$$
\begin{array}{ll}
\text{minimize} & \|Ax - b\|_2^2 \\
\text{subject to} & x \geq 0
\end{array}
$$

- variable is x
- A, b given
- $x \geq 0$ means $x_1 \geq 0, \ldots, x_n \geq 0$

CVXPY code:

```
import cvxpy as cp

A, b = ...

x = cp.Variable(n)
obj = cp.norm2(A @ x - b)**2
constr = [x >= 0]
prob = cp.Problem(cp.Minimize(obj), constr)
prob.solve()
```

Brief history of convex optimization

- **theory (convex analysis)**: 1900–1970
- **algorithms**
  - 1947: simplex algorithm for linear programming (Dantzig)
  - 1960s: early interior-point methods (Fiacco & McCormick, Dikin, ...)
  - 1970s: ellipsoid method and other subgradient methods
  - 1980s & 90s: interior-point methods (Karmarkar, Nesterov & Nemirovski)
  - since 2000s: many methods for large-scale convex optimization
- **applications**
  - before 1990: mostly in operations research, a few in engineering
  - since 1990: many applications in engineering (control, signal processing, communications, circuit design, ...)
  - since 2000s: machine learning and statistics, finance

Summary

convex optimization problems

- are optimization problems of a special form
- arise in many applications
- can be solved effectively
- are easy to specify using DSLs