Convex Optimization

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8. Geometric problems

Outline

Extremal volume ellipsoids

Centering

Classification

Placement and facility location

Minimum volume ellipsoid around a set

- **Löwner-John ellipsoid** of a set C: minimum volume ellipsoid $\mathcal{E}$ with $C \subseteq \mathcal{E}$
- parametrize $\mathcal{E}$ as $\mathcal{E} = \{v \mid \|Av + b\|_2 \leq 1\}$; can assume $A \in \mathbf{S}^n_{++}$
- $\mathbf{vol}\,\mathcal{E}$ is proportional to $\det A^{-1}$; to find Löwner-John ellipsoid, solve problem

$$\begin{array}{ll} \text{minimize (over } A,\, b) & \log\det A^{-1} \\ \text{subject to} & \sup_{v \in C} \|Av + b\|_2 \leq 1 \end{array}$$

  convex, but evaluating the constraint can be hard (for general C)

- **finite set** $C = \{x_1, \ldots, x_m\}$:

$$\begin{array}{ll} \text{minimize (over } A,\, b) & \log\det A^{-1} \\ \text{subject to} & \|Ax_i + b\|_2 \leq 1, \quad i = 1, \ldots, m \end{array}$$

  also gives Löwner-John ellipsoid for polyhedron $\mathbf{conv}\{x_1, \ldots, x_m\}$

Maximum volume inscribed ellipsoid

- maximum volume ellipsoid $\mathcal{E}$ with $\mathcal{E} \subseteq C$, $C \subseteq \mathbf{R}^n$ convex
- parametrize $\mathcal{E}$ as $\mathcal{E} = \{Bu + d \mid \|u\|_2 \leq 1\}$; can assume $B \in \mathbf{S}^n_{++}$
- $\mathbf{vol}\, \mathcal{E}$ is proportional to $\det B$; can find $\mathcal{E}$ by solving

$$
\begin{array}{ll}
\text{maximize} & \log\det B \\
\text{subject to} & \sup_{\|u\|_2 \leq 1} I_C(Bu + d) \leq 0
\end{array}
$$

  (where $I_C(x) = 0$ for $x \in C$ and $I_C(x) = \infty$ for $x \notin C$)
  convex, but evaluating the constraint can be hard (for general C)

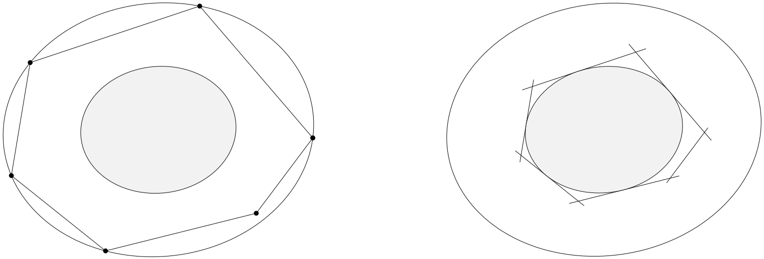
- **polyhedron** $\{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$:

$$
\begin{array}{ll}
\text{maximize} & \log\det B \\
\text{subject to} & \|Ba_i\|_2 + a_i^T d \leq b_i, \quad i = 1, \ldots, m
\end{array}
$$

  (constraint follows from $\sup_{\|u\|_2 \leq 1} a_i^T (Bu + d) = \|Ba_i\|_2 + a_i^T d$)

Efficiency of ellipsoidal approximations

- $C \subseteq \mathbf{R}^n$ convex, bounded, with nonempty interior
- Löwner-John ellipsoid, shrunk by a factor n (around its center), lies inside C
- maximum volume inscribed ellipsoid, expanded by a factor n (around its center) covers C
- **example** (for polyhedra in $\mathbf{R}^2$)



- factor n can be improved to $\sqrt{n}$ if C is symmetric

Outline

Extremal volume ellipsoids

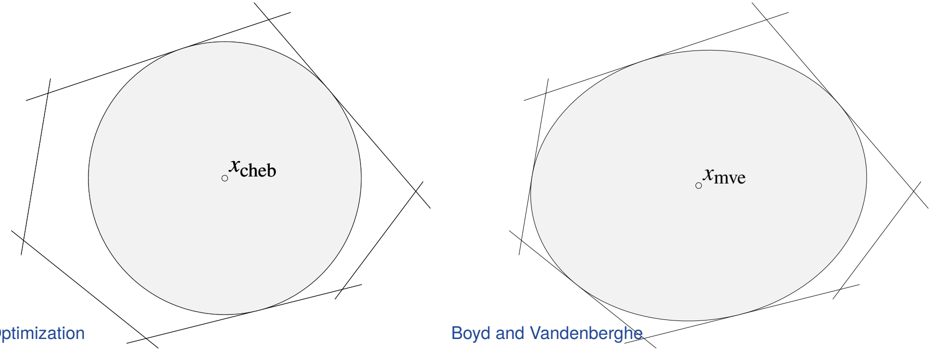
Centering

Classification

Placement and facility location

Centering

- many possible definitions of 'center' of a convex set C
- Chebyshev center: center of largest inscribed ball
  - for polyhedron, can be found via linear programming
- center of maximum volume inscribed ellipsoid
  - invariant under affine coordinate transformations



Analytic center of a set of inequalities

- the **analytic center** of set of convex inequalities and linear equations

$$f_i(x) \leq 0, \quad i = 1, \ldots, m, \qquad Fx = g$$

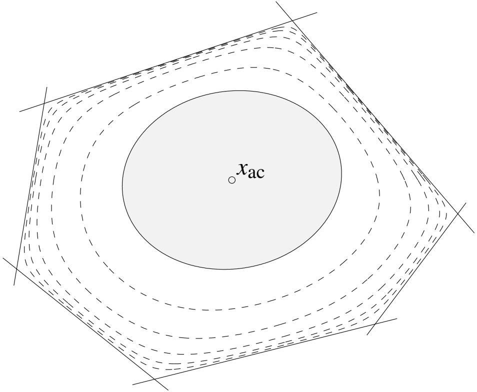
  is defined as solution of

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Fx = g \end{array}$$

- objective is called the **log-barrier** for the inequalities
- (we'll see later) analytic center more easily computed than MVE or Chebyshev center
- two sets of inequalities can describe the same set, but have different analytic centers

Analytic center of linear inequalities

- $a_i^T x \leq b_i$, $i = 1, \ldots, m$
- x_{ac} minimizes $\phi(x) = -\sum_{i=1}^m \log(b_i - a_i^T x)$
- dashed lines are level curves of ϕ



Inner and outer ellipsoids from analytic center

- we have

$$\mathcal{E}_{\text{inner}} \subseteq \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\} \subseteq \mathcal{E}_{\text{outer}}$$

  where

$$\begin{aligned} \mathcal{E}_{\text{inner}} &= \{x \mid (x - x_{\text{ac}})^T \nabla^2 \phi(x_{\text{ac}})(x - x_{\text{ac}}) \leq 1\} \\ \mathcal{E}_{\text{outer}} &= \{x \mid (x - x_{\text{ac}})^T \nabla^2 \phi(x_{\text{ac}})(x - x_{\text{ac}}) \leq m(m-1)\} \end{aligned}$$

- ellipsoid expansion/shrinkage factor is $\sqrt{m(m-1)}$
  (cf. n for Löwner-John or max volume inscribed ellpsoids)

Outline

Linear discrimination

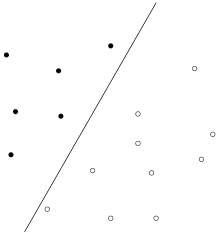
- separate two sets of points $\{x_1, \ldots, x_N\}$, $\{y_1, \ldots, y_M\}$ by a hyperplane
- *i.e.*, find $a \in \mathbf{R}^n$, $b \in \mathbf{R}$ with

$$a^T x_i + b > 0, \quad i = 1, \ldots, N, \qquad a^T y_i + b < 0, \quad i = 1, \ldots, M$$

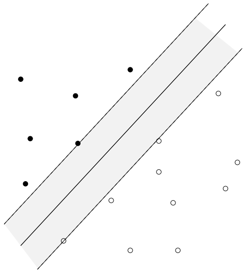
- homogeneous in a, b, hence equivalent to

$$a^T x_i + b \geq 1, \quad i = 1, \ldots, N, \qquad a^T y_i + b \leq -1, \quad i = 1, \ldots, M$$

a set of linear inequalities in a, b, *i.e.*, an LP feasibility problem



Robust linear discrimination



(Euclidean) distance between hyperplanes

$$
\begin{aligned}
\mathcal{H}_1 &= \{z \mid a^T z + b = 1\} \\
\mathcal{H}_2 &= \{z \mid a^T z + b = -1\}
\end{aligned}
$$

is $\mathbf{dist}(\mathcal{H}_1, \mathcal{H}_2) = 2/\|a\|_2$

to separate two sets of points by maximum margin,

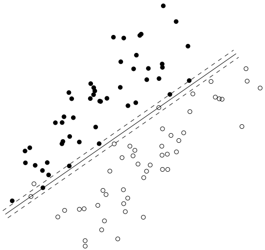
$$
\begin{array}{ll}
\text{minimize} & (1/2)\|a\|_2^2 \\
\text{subject to} & a^T x_i + b \geq 1, \quad i = 1, \ldots, N \\
& a^T y_i + b \leq -1, \quad i = 1, \ldots, M
\end{array} \qquad (2)
$$

a QP in a, b

Approximate linear separation of non-separable sets

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u + \mathbf{1}^T v \\ \text{subject to} & a^T x_i + b \geq 1 - u_i, \quad i = 1, \ldots, N, \quad a^T y_i + b \leq -1 + v_i, \quad i = 1, \ldots, M \\ & u \succeq 0, \quad v \succeq 0 \end{array}$$

- an LP in a, b, u, v
- at optimum, $u_i = \max\{0, 1 - a^T x_i - b\}$, $v_i = \max\{0, 1 + a^T y_i + b\}$
- equivalent to minimizing the sum of violations of the original inequalities

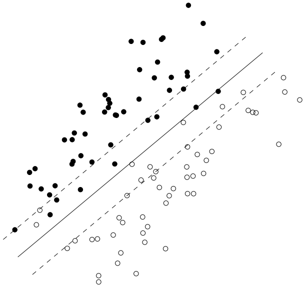


Support vector classifier

$$\begin{array}{ll} \text{minimize} & \|a\|_2 + \gamma(\mathbf{1}^T u + \mathbf{1}^T v) \\ \text{subject to} & a^T x_i + b \geq 1 - u_i, \quad i = 1, \ldots, N \\ & a^T y_i + b \leq -1 + v_i, \quad i = 1, \ldots, M \\ & u \succeq 0, \quad v \succeq 0 \end{array}$$

produces point on trade-off curve between inverse of margin $2/\|a\|_2$ and classification error, measured by total slack $\mathbf{1}^T u + \mathbf{1}^T v$

example on previous slide, with $\gamma = 0.1$:



Nonlinear discrimination

- separate two sets of points by a nonlinear function f: find $f : \mathbf{R}^n \to \mathbf{R}$ with

$$f(x_i) > 0, \quad i = 1, \ldots, N, \qquad f(y_i) < 0, \quad i = 1, \ldots, M$$

- choose a linearly parametrized family of functions $f(z) = \theta^T F(z)$
  - $\theta \in \mathbf{R}^k$ is parameter
  - $F = (F_1, \ldots, F_k) : \mathbf{R}^n \to \mathbf{R}^k$ are basis functions

- solve a set of linear inequalities in θ:

$$\theta^T F(x_i) \geq 1, \quad i = 1, \ldots, N, \qquad \theta^T F(y_i) \leq -1, \quad i = 1, \ldots, M$$

Examples

- **quadratic discrimination**: $f(z) = z^T P z + q^T z + r$, $\theta = (P, q, r)$
- solve LP feasibility problem with variables $P \in \mathbf{S}^n$, $q \in \mathbf{R}^n$, $r \in \mathbf{R}$

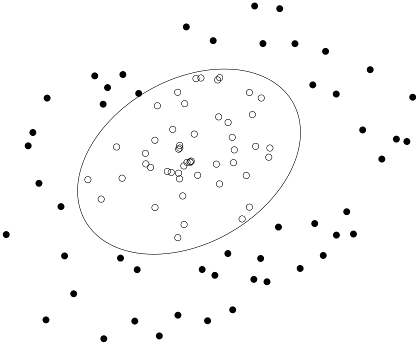
$$x_i^T P x_i + q^T x_i + r \geq 1, \qquad y_i^T P y_i + q^T y_i + r \leq -1$$

- can add additional constraints (*e.g.*, $P \preceq -I$ to separate by an ellipsoid)

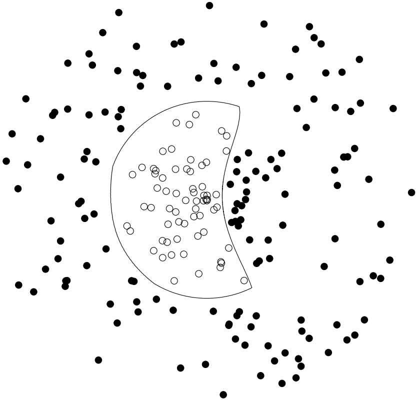
- **polynomial discrimination**: $F(z)$ are all monomials up to a given degree d
- *e.g.*, for $n = 2$, $d = 3$

$$F(z) = (1,\ z_1,\ z_2,\ z_1^2,\ z_1 z_2,\ z_2^2,\ z_1^3,\ z_1^2 z_2,\ z_1 z_2^2,\ z_2^3)$$

Example



separation by ellipsoid



separation by 4th degree polynomial

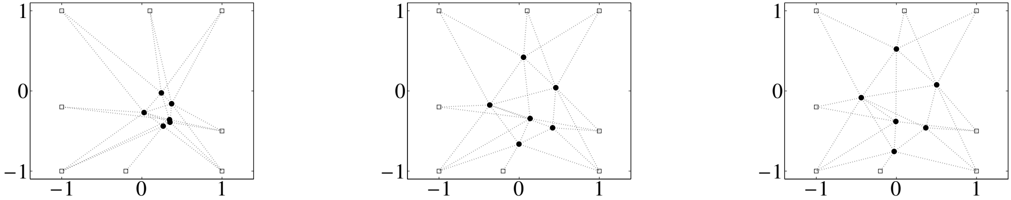
Outline

Placement and facility location

- N points with coordinates $x_i \in \mathbf{R}^2$ (or $\mathbf{R}^3$)
- some positions x_i are given; the other x_i's are variables
- for each pair of points, a cost function $f_{ij}(x_i, x_j)$
- **placement problem**: minimize $\sum_{i \neq j} f_{ij}(x_i, x_j)$
- **interpretations**
  - points are locations of plants or warehouses; f_{ij} is transportation cost between facilities i and j
  - points are locations of cells in an integrated circuit; f_{ij} represents wirelength

Example

- minimize $\sum_{(i,j)\in\mathcal{E}} h(\|x_i - x_j\|_2)$, with 6 free points, 27 edges
- optimal placements for $h(z) = z$, $h(z) = z^2$, $h(z) = z^4$



- histograms of edge lengths $\|x_i - x_j\|_2$, $(i,) \in \mathcal{E}$