Convex Optimization

Stephen Boyd   Lieven Vandenberghe

Revised slides by Stephen Boyd, Lieven Vandenberghe, and Parth Nobel

3. Convex functions

Outline

Convex functions

Operations that preserve convexity

Constructive convex analysis

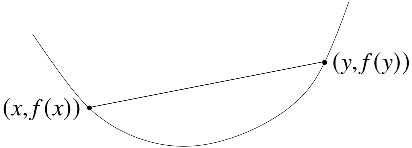
Perspective and conjugate

Quasiconvexity

Definition

- $f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and for all $x, y \in \mathbf{dom}\, f$, $0 \le \theta \le 1$,

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and for $x, y \in \mathbf{dom}\, f$, $x \ne y$, $0 < \theta < 1$,

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

Examples on $\mathbf{R}$

convex functions:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- exponential: e^{ax}, for any $a \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$
- positive part (relu): $\max\{0, x\}$

concave functions:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$
- entropy: $-x \log x$ on $\mathbf{R}_{++}$
- negative part: $\min\{0, x\}$

Examples on $\mathbf{R}^n$

convex functions:

- affine functions: $f(x) = a^T x + b$
- any norm, *e.g.*, the ℓ_p norms
  - $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ for $p \geq 1$
  - $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$
- sum of squares: $\|x\|_2^2 = x_1^2 + \cdots + x_n^2$
- max function: $\max(x) = \max\{x_1, x_2, \ldots, x_n\}$
- softmax or log-sum-exp function: $\log(\exp x_1 + \cdots + \exp x_n)$

Examples on $\mathbf{R}^{m\times n}$

- $X \in \mathbf{R}^{m\times n}$ ($m \times n$ matrices) is the variable
- general affine function has form

$$f(X) = \mathbf{tr}(A^T X) + b = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} X_{ij} + b$$

  for some $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}$
- spectral norm (maximum singular value) is convex

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

- log-determinant: for $X \in \mathbf{S}^n_{++}$, $f(X) = \log\det X$ is concave

Extended-value extension

- suppose f is convex on $\mathbf{R}^n$, with domain $\mathbf{dom}\, f$
- its extended-value extension $\tilde{f}$ is function $\tilde{f} : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbf{dom}\, f \\ \infty & x \notin \mathbf{dom}\, f \end{cases}$$

- often simplifies notation; for example, the condition

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad \tilde{f}(\theta x + (1-\theta)y) \leq \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

  (as an inequality in $\mathbf{R} \cup \{\infty\}$), means the same as the two conditions
  - $\mathbf{dom}\, f$ is convex
  - $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$

Restriction of a convex function to a line

- $f : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if the function $g : \mathbf{R} \to \mathbf{R}$,

$$g(t) = f(x + tv), \qquad \mathbf{dom}\, g = \{t \mid x + tv \in \mathbf{dom}\, f\}$$

  is convex (in t) for any $x \in \mathbf{dom}\, f$, $v \in \mathbf{R}^n$

- can check convexity of f by checking convexity of functions of one variable

Example

- $f : \mathbf{S}^n \to \mathbf{R}$ with $f(X) = \log\det X$, $\mathbf{dom}\, f = \mathbf{S}^n_{++}$
- consider line in $\mathbf{S}^n$ given by $X + tV$, $X \in \mathbf{S}^n_{++}$, $V \in \mathbf{S}^n$, $t \in \mathbf{R}$

$$\begin{aligned} g(t) &= \log\det(X + tV) \\ &= \log\det\left(X^{1/2}\left(I + tX^{-1/2}VX^{-1/2}\right)X^{1/2}\right) \\ &= \log\det X + \log\det\left(I + tX^{-1/2}VX^{-1/2}\right) \\ &= \log\det X + \sum_{i=1}^{n}\log(1 + t\lambda_i) \end{aligned}$$

  where λ_i are the eigenvalues of $X^{-1/2}VX^{-1/2}$
- g is concave in t (for any choice of $X \in \mathbf{S}^n_{++}$, $V \in \mathbf{S}^n$); hence f is concave

First-order condition

- f is **differentiable** if $\mathbf{dom}\, f$ is open and the gradient

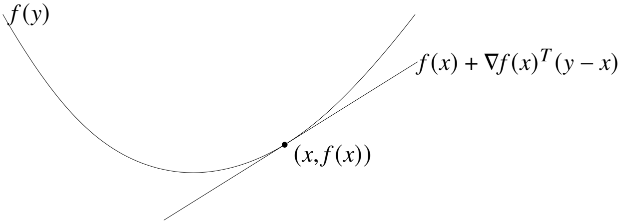
$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right) \in \mathbf{R}^n$$

  exists at each $x \in \mathbf{dom}\, f$

- **1st-order condition:** differentiable f with convex domain is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$

- first order Taylor approximation of convex f is a **global underestimator** of f



Second-order conditions

- f is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

  $$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

  exists at each $x \in \mathbf{dom}\, f$

- **2nd-order conditions:** for twice differentiable f with convex domain
  - f is convex if and only if $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbf{dom}\, f$
  - if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then f is strictly convex

Examples

- **quadratic function:** $f(x) = (1/2)x^T Px + q^T x + r$ (with $P \in \mathbf{S}^n$)

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

  convex if $P \succeq 0$ (concave if $P \preceq 0$)

- **least-squares objective**: $f(x) = \|Ax - b\|_2^2$

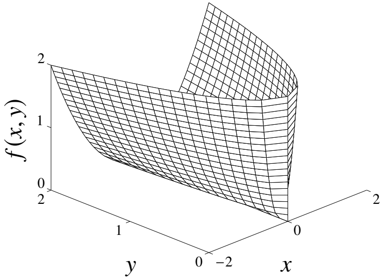
$$\nabla f(x) = 2A^T(Ax - b), \qquad \nabla^2 f(x) = 2A^T A$$

  convex (for any A)

- **quadratic-over-linear:** $f(x, y) = x^2/y$, $y > 0$

$$\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

  convex for $y > 0$



More examples

- **log-sum-exp**: $f(x) = \log \sum_{k=1}^{n} \exp x_k$ is convex

$$\nabla^2 f(x) = \frac{1}{\mathbf{1}^T z} \mathbf{diag}(z) - \frac{1}{(\mathbf{1}^T z)^2} zz^T \qquad (z_k = \exp x_k)$$

- to show $\nabla^2 f(x) \succeq 0$, we must verify that $v^T \nabla^2 f(x) v \geq 0$ for all v:

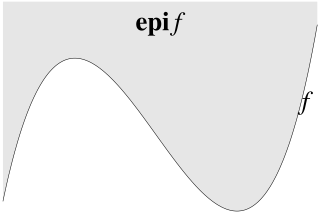
$$v^T \nabla^2 f(x) v = \frac{(\sum_k z_k v_k^2)(\sum_k z_k) - (\sum_k v_k z_k)^2}{(\sum_k z_k)^2} \geq 0$$

  since $(\sum_k v_k z_k)^2 \leq (\sum_k z_k v_k^2)(\sum_k z_k)$ (from Cauchy-Schwarz inequality)

- **geometric mean**: $f(x) = (\prod_{k=1}^{n} x_k)^{1/n}$ on $\mathbf{R}_{++}^n$ is concave (similar proof as above)

Epigraph and sublevel set

- α**-sublevel set** of $f : \mathbf{R}^n \to \mathbf{R}$ is $C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \le \alpha\}$
- sublevel sets of convex functions are convex sets (but converse is false)
- **epigraph** of $f : \mathbf{R}^n \to \mathbf{R}$ is $\mathbf{epi}\, f = \{(x,t) \in \mathbf{R}^{n+1} \mid x \in \mathbf{dom}\, f,\ f(x) \le t\}$



- f is convex if and only if $\mathbf{epi}\, f$ is a convex set

Jensen's inequality

- **basic inequality:** if f is convex, then for $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

- **extension:** if f is convex and z is a random variable on $\mathbf{dom}\, f$,

$$f(\mathbf{E}\, z) \leq \mathbf{E}\, f(z)$$

- basic inequality is special case with discrete distribution

$$\mathbf{prob}(z = x) = \theta, \qquad \mathbf{prob}(z = y) = 1 - \theta$$

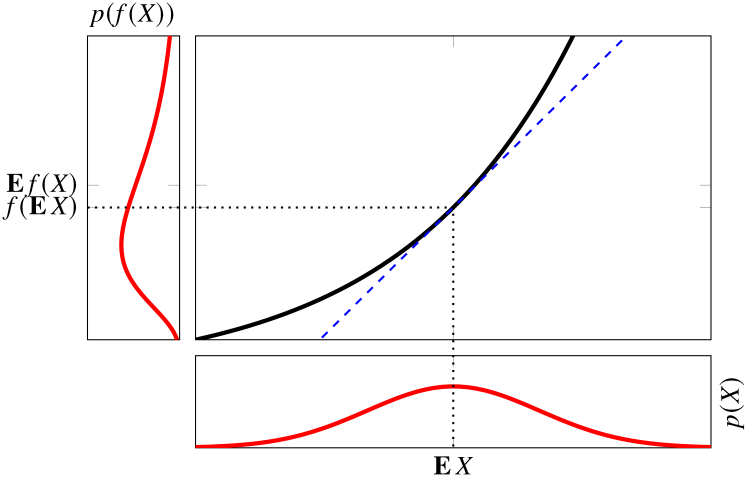
Example: log-normal random variable

- suppose $X \sim \mathcal{N}(\mu, \sigma^2)$
- with $f(u) = \exp u$, $Y = f(X)$ is log-normal
- we have $\mathbf{E} f(X) = \exp(\mu + \sigma^2/2)$
- Jensen's inequality is

$$f(\mathbf{E}\, X) = \exp \mu \leq \mathbf{E} f(X) = \exp(\mu + \sigma^2/2)$$

which indeed holds since $\exp \sigma^2/2 > 1$

Example: log-normal random variable



Outline

Convex functions

Operations that preserve convexity

Constructive convex analysis

Perspective and conjugate

Quasiconvexity

Showing a function is convex

methods for establishing convexity of a function f

1. verify definition (often simplified by restricting to a line)
2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
   - recommended only for **very simple** functions
3. show that f is obtained from simple convex functions by operations that preserve convexity
   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective

you'll mostly use methods 2 and 3

Nonnegative scaling, sum, and integral

- **nonnegative multiple:** αf is convex if f is convex, $\alpha \geq 0$
- **sum:** $f_1 + f_2$ convex if f_1, f_2 convex
- **infinite sum:** if $f_1, f_2, \ldots$ are convex functions, infinite sum $\sum_{i=1}^{\infty} f_i$ is convex
- **integral:** if $f(x, \alpha)$ is convex in x for each $\alpha \in \mathcal{A}$, then $\int_{\alpha \in \mathcal{A}} f(x, \alpha)\, d\alpha$ is convex

- there are analogous rules for concave functions

Composition with affine function

(pre-)composition with affine function: $f(Ax+b)$ is convex if f is convex

examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \textbf{dom} f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$$

- norm approximation error: $f(x) = \|Ax - b\|$ (any norm)

Pointwise maximum

if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$
- sum of r largest components of $x \in \mathbf{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

  ($x_{[i]}$ is ith largest component of x)

  proof: $f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$

Pointwise supremum

if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then $g(x) = \sup_{y \in \mathcal{A}} f(x, y)$ is convex

examples

- distance to farthest point in a set C: $f(x) = \sup_{y \in C} \|x - y\|$
- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$, $\lambda_{\max}(X) = \sup_{\|y\|_2=1} y^T X y$ is convex
- support function of a set C: $S_C(x) = \sup_{y \in C} y^T x$ is convex

Partial minimization

- the function $g(x) = \inf_{y \in C} f(x, y)$ is called the **partial minimization** of f (w.r.t. y)
- if $f(x, y)$ is convex in (x, y) and C is a convex set, then partial minimization g is convex

examples

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$ with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \qquad C \succ 0$$

  minimizing over y gives $g(x) = \inf_y f(x, y) = x^T (A - BC^{-1} B^T) x$
  g is convex, hence Schur complement $A - BC^{-1} B^T \succeq 0$

- distance to a set: $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$ is convex if S is convex

Composition with scalar functions

- composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ is $f(x) = h(g(x))$ (written as $f = h \circ g$)
- composition f is convex if
  - g convex, h convex, $\tilde{h}$ nondecreasing
  - or g concave, h convex, $\tilde{h}$ nonincreasing

  (monotonicity must hold for extended-value extension $\tilde{h}$)
- proof (for $n = 1$, differentiable g, h)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

examples

- $f(x) = \exp g(x)$ is convex if g is convex
- $f(x) = 1/g(x)$ is convex if g is concave and positive

General composition rule

- composition of $g : \mathbf{R}^n \to \mathbf{R}^k$ and $h : \mathbf{R}^k \to \mathbf{R}$ is $f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$
- f is convex if h is convex and for each i one of the following holds
  - g_i convex, $\tilde{h}$ nondecreasing in its ith argument
  - g_i concave, $\tilde{h}$ nonincreasing in its ith argument
  - g_i affine

- you will use this composition rule **constantly** throughout this course
- you need to commit this rule to memory

Examples

- $\log \sum_{i=1}^{m} \exp g_i(x)$ is convex if g_i are convex
- $f(x) = p(x)^2/q(x)$ is convex if
  - p is nonnegative and convex
  - q is positive and concave

- composition rule subsumes others, *e.g.*,
  - αf is convex if f is, and $\alpha \geq 0$
  - sum of convex (concave) functions is convex (concave)
  - max of convex functions is convex
  - min of concave functions is concave

Outline

Convex functions

Operations that preserve convexity

Constructive convex analysis

Perspective and conjugate

Quasiconvexity

Constructive convexity verification

- start with function f given as **expression**
- build parse tree for expression
  - leaves are variables or constants
  - nodes are functions of child expressions
- use composition rule to tag subexpressions as convex, concave, affine, or none
- if root node is labeled convex (concave), then f is convex (concave)
- extension: tag sign of each expression, and use sign-dependent monotonicity

- this is sufficient to show f is convex (concave), but not necessary
- this method for checking convexity (concavity) is readily automated

Example

the function

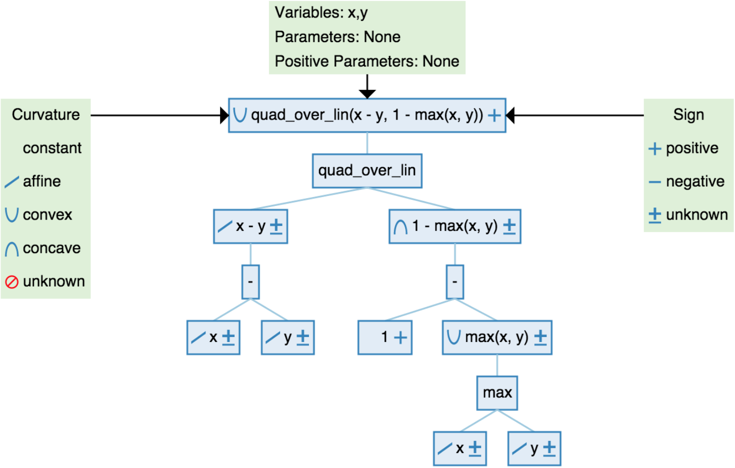
$$f(x, y) = \frac{(x - y)^2}{1 - \max(x, y)}, \qquad x < 1, \quad y < 1$$

is convex

constructive analysis:

- (leaves) x, y, and 1 are affine
- $\max(x, y)$ is convex; $x - y$ is affine
- $1 - \max(x, y)$ is concave
- function u^2/v is convex, monotone decreasing in v for $v > 0$
- f is composition of u^2/v with $u = x - y$, $v = 1 - \max(x, y)$, hence convex

Example (from dcp.stanford.edu)



Disciplined convex programming

in **disciplined convex programming** (DCP) users construct convex and concave functions as expressions using constructive convex analysis

- expressions formed from
  - **variables**,
  - **constants**,
  - and **atomic functions** from a library
- atomic functions have known convexity, monotonicity, and sign properties
- all subexpressions match general composition rule
- a valid DCP function is
  - convex-by-construction
  - 'syntactically' convex (can be checked 'locally')
- convexity depends only on attributes of atomic functions, not their meanings
  - *e.g.*, could swap $\sqrt{\cdot}$ and $\sqrt[4]{\cdot}$, or $\exp \cdot$ and $(\cdot)_+$, since their attributes match

CVXPY example

$$\frac{(x-y)^2}{1-\max(x,y)}, \qquad x < 1, \quad y < 1$$

```
import cvxpy as cp
x = cp.Variable()
y = cp.Variable()
expr = cp.quad_over_lin(x - y, 1 - cp.maximum(x, y))
expr.curvature # Convex
expr.sign # Positive
expr.is_dcp() # True
```

(atom `quad_over_lin(u,v)` includes domain constraint `v>0`)

DCP is only sufficient

- consider convex function $f(x) = \sqrt{1 + x^2}$
- expression `f1 = cp.sqrt(1+cp.square(x))` is **not** DCP
- expression `f2 = cp.norm2([1,x])` **is** DCP
- CVXPY will not recognize `f1` as convex, even though it represents a convex function

Outline

Convex functions

Operations that preserve convexity

Constructive convex analysis

Perspective and conjugate

Quasiconvexity

Perspective

- the **perspective** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the function $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$,

$$g(x, t) = tf(x/t), \qquad \mathbf{dom}\, g = \{(x, t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}$$

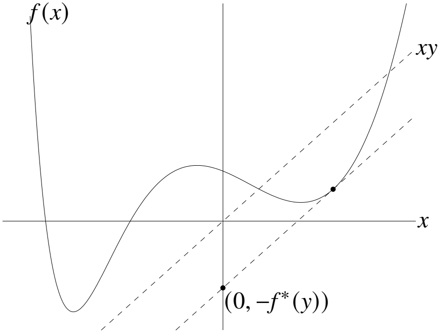
- g is convex if f is convex

examples

- $f(x) = x^T x$ is convex; so $g(x, t) = x^T x/t$ is convex for $t > 0$
- $f(x) = -\log x$ is convex; so relative entropy $g(x, t) = t \log t - t \log x$ is convex on $\mathbf{R}^2_{++}$

Conjugate function

- the **conjugate** of a function f is $f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$



- f^* is convex (even if f is not)
- will be useful in chapter 5

Examples

- negative logarithm $f(x) = -\log x$

$$f^*(y) = \sup_{x>0}(xy + \log x) = \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases}$$

- strictly convex quadratic, $f(x) = (1/2)x^TQx$ with $Q \in \mathbf{S}^n_{++}$

$$f^*(y) = \sup_x (y^Tx - (1/2)x^TQx) = \frac{1}{2}y^TQ^{-1}y$$

Outline

Convex functions

Operations that preserve convexity

Constructive convex analysis

Perspective and conjugate

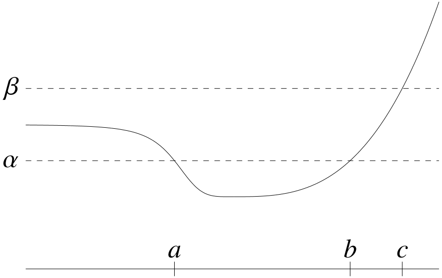
Quasiconvexity

Quasiconvex functions

- $f : \mathbf{R}^n \to \mathbf{R}$ is **quasiconvex** if $\mathbf{dom}\, f$ is convex and the sublevel sets

$$S_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$$

are convex for all α



- f is **quasiconcave** if $-f$ is quasiconvex
- f is **quasilinear** if it is quasiconvex and quasiconcave

Examples

- $\sqrt{|x|}$ is quasiconvex on $\mathbf{R}$
- $\text{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$ is quasilinear
- $\log x$ is quasilinear on $\mathbf{R}_{++}$
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbf{R}^2_{++}$
- linear-fractional function

$$f(x) = \frac{a^T x + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

  is quasilinear

Example: Internal rate of return

- cash flow $x = (x_0, \ldots, x_n)$; x_i is payment in period i (to us if $x_i > 0$)
- we assume $x_0 < 0$ (*i.e.*, an initial investment) and $x_0 + x_1 + \cdots + x_n > 0$
- **net present value** (NPV) of cash flow x, for interest rate r, is $\mathrm{PV}(x, r) = \sum_{i=0}^{n} (1+r)^{-i} x_i$
- **internal rate of return** (IRR) is smallest interest rate for which $\mathrm{PV}(x, r) = 0$:

$$\mathrm{IRR}(x) = \inf\{r \geq 0 \mid \mathrm{PV}(x, r) = 0\}$$

- IRR is quasiconcave: superlevel set is intersection of open halfspaces

$$\mathrm{IRR}(x) \geq R \quad \Longleftrightarrow \quad \sum_{i=0}^{n} (1+r)^{-i} x_i > 0 \text{ for } 0 \leq r < R$$

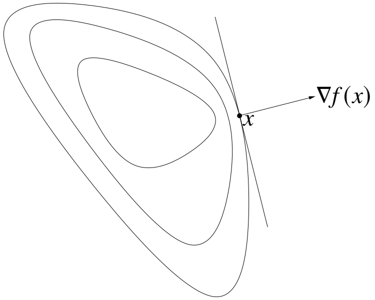
Properties of quasiconvex functions

- **modified Jensen inequality:** for quasiconvex f

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$$

- **first-order condition:** differentiable f with convex domain is quasiconvex if and only if

$$f(y) \leq f(x) \quad \Longrightarrow \quad \nabla f(x)^T (y - x) \leq 0$$



- **sum** of quasiconvex functions is not necessarily quasiconvex