Convex Optimization

Stephen Boyd   Lieven Vandenberghe

Revised slides by Stephen Boyd, Lieven Vandenberghe, and Parth Nobel

5. Duality

Outline

Lagrangian and dual function

Lagrange dual problem

KKT conditions

Sensitivity analysis

Problem reformulations

Theorems of alternatives

Lagrangian

- **standard form problem** (not necessarily convex)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

  variable $x \in \mathbf{R}^n$, domain $\mathcal{D}$, optimal value $p^\star$

- **Lagrangian:** $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$, with $\mathbf{dom}\, L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

  - weighted sum of objective and constraint functions
  - λ_i is **Lagrange multiplier** associated with $f_i(x) \leq 0$
  - ν_i is Lagrange multiplier associated with $h_i(x) = 0$

Lagrange dual function

- **Lagrange dual function:** $g : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$,

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

- g is concave, can be $-\infty$ for some λ, ν
- **lower bound property:** if $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^\star$
- proof: if $\tilde{x}$ is feasible and $\lambda \succeq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)$$

minimizing over all feasible $\tilde{x}$ gives $p^\star \geq g(\lambda, \nu)$

Least-norm solution of linear equations

$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax = b \end{array}$$

- Lagrangian is $L(x, \nu) = x^T x + \nu^T (Ax - b)$
- to minimize L over x, set gradient equal to zero:

$$\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \quad \Longrightarrow \quad x = -(1/2) A^T \nu$$

- plug x into L to obtain

$$g(\nu) = L((-1/2)A^T \nu, \nu) = -\frac{1}{4} \nu^T A A^T \nu - b^T \nu$$

- lower bound property: $p^\star \geq -(1/4)\nu^T A A^T \nu - b^T \nu$ for all ν

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$

- Lagrangian is

$$L(x, \lambda, \nu) = c^T x + \nu^T (Ax - b) - \lambda^T x = -b^T \nu + (c + A^T \nu - \lambda)^T x$$

- L is affine in x, so

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

- g is linear on affine domain $\{(\lambda, \nu) \mid A^T \nu - \lambda + c = 0\}$, hence concave
- lower bound property: $p^\star \geq -b^T \nu$ if $A^T \nu + c \succeq 0$

Equality constrained norm minimization

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = b \end{array}$$

- dual function is

$$g(\nu) = \inf_x (\|x\| - \nu^T Ax + b^T \nu) = \begin{cases} b^T \nu & \|A^T \nu\|_* \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

  where $\|v\|_* = \sup_{\|u\| \leq 1} u^T v$ is dual norm of $\| \cdot \|$

- lower bound property: $p^\star \geq b^T \nu$ if $\|A^T \nu\|_* \leq 1$

Two-way partitioning

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

- a nonconvex problem; feasible set contains 2^n discrete points
- interpretation: partition $\{1, \ldots, n\}$ in two sets encoded as $x_i = 1$ and $x_i = -1$
- W_{ij} is cost of assigning i, j to the same set; $-W_{ij}$ is cost of assigning to different sets
- dual function is

$$g(\nu) = \inf_x \left(x^T W x + \sum_i \nu_i (x_i^2 - 1) \right) = \inf_x x^T (W + \mathbf{diag}(\nu)) x - \mathbf{1}^T \nu = \begin{cases} -\mathbf{1}^T \nu & W + \mathbf{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- lower bound property: $p^\star \geq -\mathbf{1}^T \nu$ if $W + \mathbf{diag}(\nu) \succeq 0$

Lagrange dual and conjugate function

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax \preceq b, \quad Cx = d \end{array}$$

- dual function

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathbf{dom}\, f_0} \left(f_0(x) + (A^T\lambda + C^T\nu)^T x - b^T\lambda - d^T\nu \right) \\ &= -f_0^*(-A^T\lambda - C^T\nu) - b^T\lambda - d^T\nu \end{aligned}$$

  where $f^*(y) = \sup_{x \in \mathbf{dom}\, f}(y^T x - f(x))$ is conjugate of f_0
- simplifies derivation of dual if conjugate of f_0 is known
- **example: entropy maximization**

$$f_0(x) = \sum_{i=1}^n x_i \log x_i, \qquad f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

Outline

Lagrangian and dual function

Lagrange dual problem

KKT conditions

Sensitivity analysis

Problem reformulations

Theorems of alternatives

The Lagrange dual problem

(Lagrange) **dual problem**

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- finds best lower bound on $p^\star$, obtained from Lagrange dual function
- a convex optimization problem, even if original **primal** problem is not
- dual optimal value denoted $d^\star$
- λ, ν are dual feasible if $\lambda \succeq 0$, $(\lambda, \nu) \in \mathbf{dom}\, g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom}\, g$ explicit

Example: standard form LP

(see page 5.5)

- primal standard form LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

- dual problem is

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

  with $g(\lambda, \nu) = -b^T \nu$ if $A^T \nu - \lambda + c = 0$, $-\infty$ otherwise

- make implicit constraint explicit, and eliminate λ to obtain (transformed) dual problem

$$\begin{array}{ll} \text{maximize} & -b^T \nu \\ \text{subject to} & A^T \nu + c \succeq 0 \end{array}$$

Weak and strong duality

weak duality: $d^\star \leq p^\star$

- always holds (for convex and nonconvex problems)
- can be used to find nontrivial lower bounds for difficult problems, *e.g.*, solving the SDP

$$\begin{array}{ll} \text{maximize} & -\mathbf{1}^T \nu \\ \text{subject to} & W + \mathbf{diag}(\nu) \succeq 0 \end{array}$$

  gives a lower bound for the two-way partitioning problem on page 5.7

strong duality: $d^\star = p^\star$

- does not hold in general
- (usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called **constraint qualifications**

Slater's constraint qualification

strong duality holds for a convex problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

if it is **strictly feasible**, *i.e.*, there is an $x \in \textbf{int}\, \mathcal{D}$ with $f_i(x) < 0$, $i = 1, \ldots, m$, $Ax = b$

- also guarantees that the dual optimum is attained (if $p^\star > -\infty$)
- can be sharpened: *e.g.*,
  - can replace $\textbf{int}\, \mathcal{D}$ with $\textbf{relint}\, \mathcal{D}$ (interior relative to affine hull)
  - linear inequalities do not need to hold with strict inequality
- there are many other types of constraint qualifications

Inequality form LP

primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

dual function

$$g(\lambda) = \inf_x \left((c + A^T\lambda)^T x - b^T\lambda \right) = \begin{cases} -b^T\lambda & A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem

$$\begin{array}{ll} \text{maximize} & -b^T\lambda \\ \text{subject to} & A^T\lambda + c = 0, \quad \lambda \succeq 0 \end{array}$$

- from the sharpened Slater's condition: $p^\star = d^\star$ if the primal problem is feasible
- in fact, $p^\star = d^\star$ except when primal and dual are both infeasible

Quadratic program

primal problem (assume $P \in \mathbf{S}^n_{++}$)

$$\begin{array}{ll} \text{minimize} & x^T P x \\ \text{subject to} & Ax \preceq b \end{array}$$

dual function

$$g(\lambda) = \inf_x \left(x^T P x + \lambda^T (Ax - b) \right) = -\frac{1}{4} \lambda^T A P^{-1} A^T \lambda - b^T \lambda$$

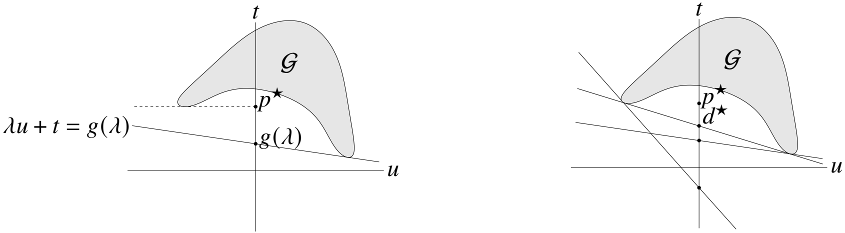
dual problem

$$\begin{array}{ll} \text{maximize} & -(1/4) \lambda^T A P^{-1} A^T \lambda - b^T \lambda \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- from the sharpened Slater's condition: $p^\star = d^\star$ if the primal problem is feasible
- in fact, $p^\star = d^\star$ always

Geometric interpretation

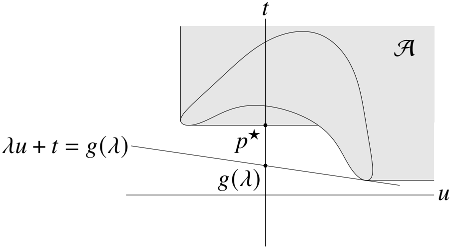
- for simplicity, consider problem with one constraint $f_1(x) \leq 0$
- $\mathcal{G} = \{(f_1(x), f_0(x)) \mid x \in \mathcal{D}\}$ is set of achievable (constraint, objective) values
- **interpretation of dual function:** $g(\lambda) = \inf_{(u,t)\in\mathcal{G}}(t + \lambda u)$



- $\lambda u + t = g(\lambda)$ is (non-vertical) supporting hyperplane to $\mathcal{G}$
- hyperplane intersects t-axis at $t = g(\lambda)$

Epigraph variation

- same with $\mathcal{G}$ replaced with $\mathcal{A} = \{(u,t) \mid f_1(x) \le u,\ f_0(x) \le t \text{ for some } x \in \mathcal{D}\}$



- strong duality holds if there is a non-vertical supporting hyperplane to $\mathcal{A}$ at $(0, p^\star)$
- for convex problem, $\mathcal{A}$ is convex, hence has supporting hyperplane at $(0, p^\star)$
- Slater's condition: if there exist $(\tilde{u}, \tilde{t}) \in \mathcal{A}$ with $\tilde{u} < 0$, then supporting hyperplane at $(0, p^\star)$ must be non-vertical

Outline

Complementary slackness

- assume strong duality holds, $x^\star$ is primal optimal, $(\lambda^\star, \nu^\star)$ is dual optimal

$$\begin{aligned} f_0(x^\star) = g(\lambda^\star, \nu^\star) &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^\star f_i(x) + \sum_{i=1}^p \nu_i^\star h_i(x) \right) \\ &\leq f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star f_i(x^\star) + \sum_{i=1}^p \nu_i^\star h_i(x^\star) \\ &\leq f_0(x^\star) \end{aligned}$$

- hence, the two inequalities hold with equality
- $x^\star$ minimizes $L(x, \lambda^\star, \nu^\star)$
- $\lambda_i^\star f_i(x^\star) = 0$ for $i = 1, \ldots, m$ (known as **complementary slackness**):

$$\lambda_i^\star > 0 \implies f_i(x^\star) = 0, \qquad f_i(x^\star) < 0 \implies \lambda_i^\star = 0$$

Karush-Kuhn-Tucker (KKT) conditions

the **KKT conditions** (for a problem with differentiable f_i, h_i) are

1. primal constraints: $f_i(x) \leq 0$, $i = 1, \ldots, m$, $h_i(x) = 0$, $i = 1, \ldots, p$
2. dual constraints: $\lambda \succeq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0$, $i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

if strong duality holds and x, λ, ν are optimal, they satisfy the KKT conditions

KKT conditions for convex problem

if $\tilde{x}$, $\tilde{\lambda}$, $\tilde{\nu}$ satisfy KKT for a convex problem, then they are optimal:

- from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

if Slater's condition is satisfied, then

x is optimal if and only if there exist λ, ν that satisfy KKT conditions

- recall that Slater implies strong duality, and dual optimum is attained
- generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

Outline

Lagrangian and dual function

Lagrange dual problem

KKT conditions

Sensitivity analysis

Problem reformulations

Theorems of alternatives

Perturbation and sensitivity analysis

(unperturbed) optimization problem and its dual

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array} \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

perturbed problem and its dual

$$\begin{array}{ll} \text{minimoize} & f_0(x) \\ \text{subject to} & f_i(x) \le u_i, \quad i = 1, \ldots, m \\ & h_i(x) = v_i, \quad i = 1, \ldots, p \end{array} \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \nu) - u^T \lambda - v^T \nu \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- x is primal variable; u, v are parameters
- $p^\star(u, v)$ is optimal value as a function of u, v
- $p^\star(0, 0)$ is optimal value of unperturbed problem

Global sensitivity via duality

- assume strong duality holds for unperturbed problem, with $\lambda^\star$, $\nu^\star$ dual optimal
- apply weak duality to perturbed problem:

$$p^\star(u,v) \geq g(\lambda^\star, \nu^\star) - u^T\lambda^\star - v^T\nu^\star = p^\star(0,0) - u^T\lambda^\star - v^T\nu^\star$$

- **implications**
  - if $\lambda_i^\star$ large: $p^\star$ increases greatly if we tighten constraint i ($u_i < 0$)
  - if $\lambda_i^\star$ small: $p^\star$ does not decrease much if we loosen constraint i ($u_i > 0$)
  - if $\nu_i^\star$ large and positive: $p^\star$ increases greatly if we take $v_i < 0$
  - if $\nu_i^\star$ large and negative: $p^\star$ increases greatly if we take $v_i > 0$
  - if $\nu_i^\star$ small and positive: $p^\star$ does not decrease much if we take $v_i > 0$
  - if $\nu_i^\star$ small and negative: $p^\star$ does not decrease much if we take $v_i < 0$

Local sensitivity via duality

if (in addition) $p^\star(u,v)$ is differentiable at $(0,0)$, then

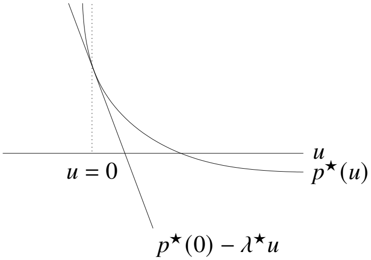
$$\lambda_i^\star = -\frac{\partial p^\star(0,0)}{\partial u_i}, \qquad \nu_i^\star = -\frac{\partial p^\star(0,0)}{\partial v_i}$$

proof (for $\lambda_i^\star$): from global sensitivity result,

$$\frac{\partial p^\star(0,0)}{\partial u_i} = \lim_{t \searrow 0} \frac{p^\star(te_i,0) - p^\star(0,0)}{t} \geq -\lambda_i^\star \qquad \frac{\partial p^\star(0,0)}{\partial u_i} = \lim_{t \nearrow 0} \frac{p^\star(te_i,0) - p^\star(0,0)}{t} \leq -\lambda_i^\star$$

hence, equality

$p^\star(u)$ for a problem with one (inequality) constraint:



Outline

Lagrangian and dual function

Lagrange dual problem

KKT conditions

Sensitivity analysis

Problem reformulations

Theorems of alternatives

Duality and problem reformulations

- equivalent formulations of a problem can lead to very different duals
- reformulating primal problem can be useful when dual is difficult to derive, or uninteresting

common reformulations

- introduce new variables and equality constraints
- make explicit constraints implicit or vice-versa
- transform objective or constraint functions, *e.g.*, replace $f_0(x)$ by $\phi(f_0(x))$ with ϕ convex, increasing

Introducing new variables and equality constraints

- unconstrained problem: minimize $f_0(Ax+b)$
- dual function is constant: $g = \inf_x L(x) = \inf_x f_0(Ax+b) = p^\star$
- we have strong duality, but dual is quite useless

- introduce new variable y and equality constraints $y = Ax+b$

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & Ax+b-y=0 \end{array}$$

- dual of reformulated problem is

$$\begin{array}{ll} \text{maximize} & b^T\nu - f_0^*(\nu) \\ \text{subject to} & A^T\nu = 0 \end{array}$$

- a nontrivial, useful dual (assuming the conjugate f_0^* is easy to express)

Example: Norm approximation

- minimize $\|Ax - b\|$
- reformulate as minimize $\|y\|$ subject to $y = Ax - b$
- recall conjugate of general norm:

$$\|z\|^* = \begin{cases} 0 & \|z\|_* \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

- dual of (reformulated) norm approximation problem:

$$\begin{array}{ll} \text{maximize} & b^T \nu \\ \text{subject to} & A^T \nu = 0, \quad \|\nu\|_* \leq 1 \end{array}$$

Outline

Lagrangian and dual function

Lagrange dual problem

KKT conditions

Sensitivity analysis

Problem reformulations

Theorems of alternatives

Theorems of alternatives

- consider two systems of inequality and equality constraints
- called **weak alternatives** if no more than one system is feasible
- called **strong alternatives** if exactly one of them is feasible
- examples: for any $a \in \mathbf{R}$, with variable $x \in \mathbf{R}$,
  - $x > a$ and $x \leq a - 1$ are weak alternatives
  - $x > a$ and $x \leq a$ are strong alternatives

- a **theorem of alternatives** states that two inequality systems are (weak or strong) alternatives
- can be considered the extension of duality to feasibility problems

Feasibility problems

- consider system of (not necessarily convex) inequalities and equalities

$$f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad h_i(x) = 0, \quad i = 1, \ldots, p$$

- express as **feasibility problem**

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- if system if feasible, $p^\star = 0$; if not, $p^\star = \infty$

Duality for feasibility problems

- dual function of feasibility problem is $g(\lambda, \nu) = \inf_x \left(\sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right)$
- for $\lambda \succeq 0$, we have $g(\lambda, \nu) \leq p^\star$
- it follows that feasibility of the inequality system

$$\lambda \succeq 0, \qquad g(\lambda, \nu) > 0$$

  implies the original system is infeasible
- so this is a weak alternative to original system
- it is strong if f_i convex, h_i affine, and a constraint qualification holds
- g is positive homogeneous so we can write alternative system as

$$\lambda \succeq 0, \qquad g(\lambda, \nu) \geq 1$$

Example: Nonnegative solution of linear equations

- consider system
$$Ax = b, \qquad x \succeq 0$$
- dual function is $g(\lambda, \nu) = \begin{cases} -\nu^T b & A^T \nu = \lambda \\ -\infty & \text{otherwise} \end{cases}$
- can express strong alternative of $Ax = b$, $x \succeq 0$ as
$$A^T \nu \succeq 0, \qquad \nu^T b \leq -1$$

(we can replace $\nu^T b \leq -1$ with $\nu^T b = -1$)

Farkas' lemma

- Farkas' lemma:

$$Ax \preceq 0, \quad c^T x < 0 \quad \text{and} \quad A^T y + c = 0, \quad y \succeq 0$$

  are strong alternatives

- proof: use (strong) duality for (feasible) LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq 0 \end{array}$$

Investment arbitrage

- we invest x_j in each of n assets $1, \ldots, n$ with prices $p_1, \ldots, p_n$
- our initial cost is $p^T x$
- at the end of the investment period there are only m possible outcomes $i = 1, \ldots, m$
- V_{ij} is the **payoff** or final value of asset j in outcome i
- first investment is risk-free (cash): $p_1 = 1$ and $V_{i1} = 1$ for all i

- **arbitrage** means there is x with $p^T x < 0$, $Vx \geq 0$
- arbitrage means we receive money up front, and our investment cannot lose
- standard assumption in economics: the prices are such that **there is no arbitrage**

Absence of arbitrage

- by Farkas' lemma, there is no arbitrage $\Longleftrightarrow$ there exists $y \in \mathbf{R}_+^m$ with $V^T y = p$
- since first column of V is $\mathbf{1}$, we have $\mathbf{1}^T y = 1$
- y is interpreted as a **risk-neutral probability** on the outcomes $1, \ldots, m$
- $V^T y$ are the expected values of the payoffs under the risk-neutral probability
- interpretation of $V^T y = p$:

  asset prices equal their expected payoff under the risk-neutral probability

- **arbitrage theorem**: there is no arbitrage $\Leftrightarrow$ there exists a risk-neutral probability distribution under which each asset price is its expected payoff

Example

$$V = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 1.0 & 0.8 & 0.0 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 1.3 & 4.0 \end{bmatrix}, \qquad p = \begin{bmatrix} 1.0 \\ 0.9 \\ 0.3 \end{bmatrix}, \qquad \tilde{p} = \begin{bmatrix} 1.0 \\ 0.8 \\ 0.7 \end{bmatrix}$$

- with prices p, there is an arbitrage

$$x = \begin{bmatrix} 6.2 \\ -7.7 \\ 1.5 \end{bmatrix}, \qquad p^T x = -0.2, \qquad \mathbf{1}^T x = 0, \qquad Vx = \begin{bmatrix} 2.35 \\ 0.04 \\ 0.00 \\ 2.19 \end{bmatrix}$$

- with prices $\tilde{p}$, there is no arbitrage, with risk-neutral probability

$$y = \begin{bmatrix} 0.36 \\ 0.27 \\ 0.26 \\ 0.11 \end{bmatrix} \qquad V^T y = \begin{bmatrix} 1.0 \\ 0.8 \\ 0.7 \end{bmatrix}$$