Review Session 5

- a partial summary of the course
- no guarantees everything on the exam is covered here
- not designed to stand alone; use with the class notes

LQR

- balance good control and small input effort
- quadratic cost function

$$J(U) = \sum_{\tau=0}^{N-1} \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) + x_N^T Q_f x_N$$

- Q, Q_f and R are state cost, final state cost, input cost matrices

Solving LQR problems

- can solve as least-squares problem
- solve more efficiently with dynamic programming: use value function

$$V_t(z) = \min_{u_t,\ldots,u_{N-1}} \sum_{\tau=t}^{N-1} \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) + x_N^T Q_f x_N$$

subject to $x_t = z$, $x_{\tau+1} = Ax_\tau + Bu_\tau$, $\tau = t, \ldots, T$

- $V_t(z)$ is the minimum LQR cost-to-go from state z at time t
- can show by recursion that $V_t(z) = z^T P_t z$; $u_t^{\mathrm{lqr}} = K_t x_t$
- get Riccati recursion, runs backwards in time

Steady-state LQR

- usually P_t in value function converges rapidly as t decreases below N
- steady-state value P_{ss} satisfies

$$P_{\text{ss}} = Q + A^T P_{\text{ss}} A - A^T P_{\text{ss}} B (R + B^T P_{\text{ss}} B)^{-1} B^T P_{\text{ss}} A$$

- this is the discrete-time algebraic Riccati equation (ARE)
- for t not close to horizon N, LQR optimal input is approximately a linear, constant state feedback

LQR extensions

- time-varying systems
- time-varying cost matrices
- tracking problems (with state/input offsets)
- Gauss-Newton LQR for nonlinear dynamical systems
- can view LQR as solution of constrained minimization problem, via Lagrange multipliers

Infinite horizon LQR

- problem becomes: choose $u_0, u_1, \ldots$ to minimize

$$J = \sum_{\tau=0}^{\infty} \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right)$$

- infinite dimensional problem
- possibly no solution in general
- if (A, B) is controllable, then for any x^{init}, there's a length-n input sequence that steers x to zero and keeps it there

Hamilton-Jacobi equation

- define value function $V(z) = z^T P z$ as minimum LQR cost-to-go
- satisfies Hamilton-Jacobi equation

$$V(z) = \min_w \left(z^T Q z + w^T R w + V(Az + Bw) \right),$$

- after minimizing over w, HJ equation becomes

$$\begin{aligned} z^T P z &= z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P (Az + Bw^*) \\ &= z^T \left(Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A \right) z \end{aligned}$$

- holds for all z, so P satisfies the ARE (thus, constant state feedback)

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

Receding-horizon LQR control

- find sequence that minimizes first T-step-ahead LQR cost from current position then use just the first input

- in general, optimal T-step-ahead LQR control has constant state feedback

- state feedback gain converges to infinite horizon optimal as horizon becomes long (assuming controllability)

- closed loop system is stable if (Q, A) observable and (A, B) controllable

Continuous-time LQR

- choose $u : [0, T] \to \mathbf{R}^m$ to minimize

$$J = \int_0^T \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) \, d\tau + x(T)^T Q_f x(T)$$

- infinite dimensional problem
- can solve via dynamic programming, V_t again quadratic; P_t found from a differential equation, running backwards in time
- LQR optimal u easily expressed in terms of P_t
- can also handle time-varying/tracking problems

Continuous-time LQR in steady-state

- usually P_t converges rapidly as t decreases below T
- limit P_{ss} satisfies continuous-time ARE

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

- can solve using Riccati differential equation, or directly, via Hamiltonian
- for t not near T, LQR optimal input is approximately a linear constant state feedback
- (can also derive via discretization or Lagrange multipliers)

Linear quadratic stochastic control

- add IID process noise w_t: $x_{t+1} = Ax_t + Bu_t + w_t$
- objective becomes

$$J = \mathbf{E}\left(\sum_{t=0}^{N-1}\left(x_t^T Q x_t + u_t^T R u_t\right) + x_N^T Q_f x_N\right)$$

- choose input to minimize J, after knowing the current state, but before knowing the disturbance
- can solve via dynamic programming
- optimal policy is linear state feedback (same form as deterministic LQR)
- strangely, optimal policy is the same as LQR, doesn't depend on X, W

Invariant subspaces

- $\mathcal{V}$ is A-invariant if $A\mathcal{V} \subseteq \mathcal{V}$, *i.e.*, $v \in \mathcal{V} \implies Av \in \mathcal{V}$
- *e.g.*, controllable/unobservable subspaces for linear systems
- if $\mathcal{R}(M)$ is A-invariant, then there is a matrix X such that $AM = MX$
- converse is also true: if there is an X such that $AM = MX$, then $\mathcal{R}(M)$ is A-invariant

PBH controllability criterion

- (A, B) is controllable if and only if

$$\mathbf{Rank}\ [sI - A\ B] = n \text{ for all } s \in \mathbf{C}$$

or,

- (A, B) is uncontrollable if and only if there is a $w \neq 0$ with

$$w^T A = \lambda w^T, \qquad w^T B = 0$$

$i.e.$, a left eigenvector is orthogonal to columns of B

- mode associated with left eigenvector w is uncontrollable if $w^T B = 0$,

PBH observability criterion

- (C, A) is observable if and only if

$$\mathbf{Rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n \text{ for all } s \in \mathbf{C}$$

or,

- (C, A) is unobservable if and only if there is a $v \neq 0$ with

$$Av = \lambda v, \qquad Cv = 0$$

$i.e.$, a (right) eigenvector is in the nullspace of C

- mode associated with right eigenvector v is unobservable if $Cv = 0$

Estimation

- minimum mean-square estimator (MMSE) is, in general, $\mathbf{E}(x|y)$
- for jointly Gaussian x and y, MMSE estimator of x is affine function of y

$$\hat{x} = \phi_{\mathrm{mmse}}(y) = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

- when x, y aren't jointly Gaussian, best linear unbiased estimator is

$$\hat{x} = \phi_{\mathrm{blu}}(y) = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

- ϕ_{blu} is unbiased ($\mathbf{E}\,\hat{x} = \mathbf{E}\,x$), often works well, has MMSE among all affine estimators
- given A, Σ_x, Σ_v, can evaluate Σ_{est} before knowing measurements (can do experiment design)

Linear system with stochastic process

- covariance $\Sigma_x(t)$ satisfies a Lyapunov-like linear dynamical system

$$\Sigma_x(t+1) = A\Sigma_x(t)A^T + B\Sigma_u(t)B^T + A\Sigma_{xu}(t)B^T + B\Sigma_{ux}(t)A^T$$

- if $\Sigma_{xu}(t) = 0$ (x and u uncorrelated), we have the Lyapunov iteration

$$\Sigma_x(t+1) = A\Sigma_x(t)A^T + B\Sigma_u(t)B^T$$

- if (and only if) A is stable, converges to steady-state covariance which satisfies the Lyapunov equation

$$\Sigma_x = A\Sigma_x A^T + B\Sigma_u B^T$$

Kalman filter

- estimate current or next state, based on current and past outputs
- recursive, so computationally efficient (can express as Riccati recursion)
- measurement update

$$\begin{aligned}\hat{x}_{t|t} &= \hat{x}_{t|t-1} + \Sigma_{t|t-1} C^T \left(C \Sigma_{t|t-1} C^T + V \right)^{-1} (y_t - C\hat{x}_{t|t-1}) \\ \Sigma_{t|t} &= \Sigma_{t|t-1} - \Sigma_{t|t-1} C^T \left(C \Sigma_{t|t-1} C^T + V \right)^{-1} C \Sigma_{t|t-1}\end{aligned}$$

- time update

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t}, \qquad \Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W$$

- can compute $\Sigma_{t|t-1}$ before any observations are made
- steady-state error covariance satisfies ARE
  $\hat{\Sigma} = A\hat{\Sigma}A^T + W - A\hat{\Sigma}C^T(C\hat{\Sigma}C^T + V)^{-1}C\hat{\Sigma}A^T$

Approximate nonlinear filtering

- in general, exact solution is impractical; requires propagating infinite dimensional conditional densities

- extended Kalman filter: use affine approximations of nonlinearities, Gaussian model

- other methods ($e.g.$, particle filters): based on Monte Carlo methods that sample the random variables

- usually heuristic, unless problems are very small

Conservation and dissipation

- a set $C \subseteq \mathbf{R}^n$ is invariant with respect to autonomous, time-invariant, nonlinear $\dot{x} = f(x)$ if for every trajectory x,

$$x(t) \in C \implies x(\tau) \in C \text{ for all } \tau \geq t$$

- every trajectory that enters or starts in C must stay there

- scalar valued function ϕ is a conserved quantity for $\dot{x} = f(x)$ if for every trajectory x, $\phi(x(t))$ is constant

- ϕ is a dissipated quantity for $\dot{x} = f(x)$ if for every trajectory x, $\phi(x(t))$ is (weakly) decreasing

Quadratic functions and linear dynamical systems

continuous time: linear system $\dot{x} = Ax$, quadratic form $\phi(z) = z^T P z$

- ϕ is conserved if and only if $A^T P + PA = 0$
- ϕ is dissipated if and only if $A^T P + PA \leq 0$

discrete time: linear system $x_{t+1} = Ax_t$, quadratic form $\phi(z) = z^T P z$

- ϕ is conserved if and only if $A^T PA - P = 0$
- ϕ is dissipated if and only if $A^T PA - P \leq 0$

Stability

consider nonlinear time-invariant system $\dot{x} = f(x)$

- x_e is an equilibrium point if $f(x_e) = 0$
- system is globally asymptotically stable (GAS) if for every trajectory x, $x(t) \to x_e$ as $t \to \infty$
- system is locally asymptotically stable (LAS) near or at x_e, if there is an $R > 0$ such that $\|x(0) - x_e\| \leq R \implies x(t) \to x_e$ as $t \to \infty$
- for linear systems (with $x_e = 0$), LAS $\Leftrightarrow$ GAS $\Leftrightarrow$ $\Re\lambda_i(A) < 0$

Energy and dissipation functions

consider nonlinear time-invariant system $\dot{x} = f(x)$, function $V : \mathbf{R}^n \to \mathbf{R}$

- define $\dot{V} : \mathbf{R}^n \to \mathbf{R}$ as $\dot{V}(z) = \nabla V(z)^T f(z)$

- $\dot{V}(z)$ gives $\frac{d}{dt}V(x(t))$ when $z = x(t)$, $\dot{x} = f(x)$

- can think of V as generalized energy function, $-\dot{V}$ as the associated generalized dissipation function

- V is positive definite if $V(z) \geq 0$ for all z, $V(z) = 0$ if and only if $z = 0$ and all sublevel sets of V are bounded ($V(z) \to \infty$ as $z \to \infty$)

Lyapunov theory

- used to make conclusions about of system trajectories, without finding the trajectories
- boundedness: if there is a (Lyapunov function) V with all sublevel sets bounded, and $\dot{V}(z) \leq 0$ for all z, then all trajectories are bounded
- global asymptotic stability: if there is a positive definite V with $\dot{V}(z) < 0$ for all $z \neq 0$ and $\dot{V}(0) = 0$, then every trajectory of $\dot{x} = f(x)$ converges to zero as $t \to \infty$
- exponential stability: if there is a positive definite V, and constant $\alpha > 0$ with $\dot{V}(z) \leq -\alpha V(z)$ for all z, then there is an M such that every trajectory satisfies $\|x(t)\| \leq M e^{-\alpha t/2} \|x(0)\|$

Lasalle's theorem

- can conclude GAS of a system with only $\dot{V} \leq 0$ and an observability-type condition

- if there is a positive definite V with $\dot{V}(z) \leq 0$, and the only solution of $\dot{w} = f(w)$, $\dot{V}(w) = 0$ is $w(t) = 0$ for all t, then the system is GAS

- requires time-invariance

Converse Lyapunov theorems

- if a linear system is GAS, there is a quadratic Lyapunov function that proves it

- if a system is globally exponentially stable, there is a Lyapunov function that proves it

Linear quadratic Lyapunov theory

- Lyapunov equation: $A^TP + PA + Q = 0$

- for linear system $\dot{x} = Ax$, if $V(z) = z^TPz$, then
$\dot{V}(z) = (Az)^TPz + z^TP(Az) = -z^TQz$

- if z^TPz is the generalized energy, then z^TQz is the associated generalized dissipation

- boundedness: if $P > 0$, $Q \geq 0$, then all trajectories are bounded, and the ellipsoids $\{z \mid z^TPz \leq a\}$ are invariant

- stability: if $P > 0$, $Q > 0$, then the system is GAS

- an extension from Lasalle's theorem: if $P > 0$, $Q \geq 0$ and (Q, A) observable, then the system is GAS

- if $Q \geq 0$ and $P \not\geq 0$, then A is not stable

The Lyapunov operator

- the Lyapunov operator is given by

$$\mathcal{L}(P) = A^T P + PA$$

- if A is stable, Lyapunov operator is nonsingular

- if A has imaginary eigenvalue, then Lyapunov operator is singular

- thus, if A is stable, for any Q there is exactly one solution P of the Lyapunov equation $A^T P + PA + Q = 0$

- efficient ways to solve the Lyapunov equation (review session 3)

The Lyapunov integral

- if A is stable, explicit formula for solution of Lyapunov equation:

$$P = \int_0^\infty e^{tA^T} Q e^{tA}\, dt$$

- if A is stable, P is unique solution of Lyapunov equation, then

$$V(z) = z^T P z = \int_0^\infty x(t)^T Q x(t)\, dt$$

  (where $\dot{x} = Ax$ and $x(0) = z$)

- thus, $V(z)$ is cost-to-go from point z, and integral quadratic cost function with matrix Q

- can use to evaluate quadratic integrals

Further Lyapunov results

- all linear quadratic Lyapunov results have discrete-time counterparts
- discrete-time Lyapunov equation is

$$A^T PA - P + Q = 0$$

  (if $V(z) = z^T Pz$, then $\delta V(z) = -z^T Qz$)

- for a nonlinear system $\dot{x} = f(x)$ with x_e an equilibrium point, if the linearized system near x_e is stable, then the nonlinear system is locally asymptotically stable (and nearly vice versa)

LMIs

- the Lyapunov inequality $A^TP + PA + Q \le 0$ is an LMI in variable P

- P satisfies the Lyapunov LMI if and only if the quadratic form $V(z) = z^TPz$ satisfies $\dot{V}(z) \le -z^TQz$

- bounded-real LMI: if P satisfies

$$\begin{bmatrix} A^TP + PA + C^TC & PB \\ B^TP & -\gamma^2 I \end{bmatrix} \le 0, \qquad P \ge 0$$

  then the quadratic Lyapunov function $V(z) = z^TPz$ proves the RMS gain of the system is no more than γ

Using LMIs

- practical approach to strict matrix inequalities: if inequalities are homogeneous in x, replace $F_{\rm strict}(x) > 0$ with $F_{\rm strict}(x) \geq I$
- if inequalities aren't homogeneous, replace $F_{\rm strict}(x) > 0$ with $F_{\rm strict}(x) \geq \epsilon I$, with ϵ small and positive
- if we have $\dot{x}(t) = A(t)x(t)$, with $A(t) \in \{A_1, \ldots, A_K\}$, can use multiple simultaneous LMIs to find a simultaneous Lyapunov function that establishes a property for all trajectories
- can't be done analytically, but possible to do numerically
- more generally, can globally and efficiently solve SDPs:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0 \\ & Ax = b \end{array}$$

S-procedure

- for two quadratic forms, if and (with a constraint qualification) only if there is a $\tau \geq 0$ with $F_0 \geq \tau F_1$, then $z^T F_1 z \geq 0 \implies z^T F_0 z \geq 0$

- can also replace $\geq$ with $>$

- for multiple quadratic forms, if there are $\tau_1, \ldots, \tau_k \geq 0$ with

$$F_0 \geq \tau_1 F_1 + \cdots + \tau_k F_k$$

then, for all z,

$$z^T F_1 z \geq 0, \ldots, z^T F_k z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0$$

- can solve using LMIs

Systems with sector nonlinearities

- a function ϕ : **R** $\rightarrow$ **R** is said to be in sector $[l, u]$ if for all $q \in$ **R**, $p = \phi(q)$ lies between lq and uq

- a (single nonlinearity) Lur'e system has the form

$$\dot{x} = Ax + Bp, \qquad q = Cx, \qquad p = \phi(t, q)$$

  where $\phi(t, \cdot)$: **R** $\rightarrow$ **R** is in sector $[l, u]$ for each t

- goal: prove stability or bound using only the sector information

GAS of Lur'e system

- can express GAS of Lur'e system using quadratic Lyapunov function $V(z) = z^T P z$ as requiring $\dot{V} + \alpha V \leq 0$, equivalent to

$$\begin{bmatrix} z \\ p \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \alpha P & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} z \\ p \end{bmatrix} \leq 0$$

  whenever

$$\begin{bmatrix} z \\ p \end{bmatrix}^T \begin{bmatrix} \sigma C^T C & -\nu C^T \\ -\nu C & 1 \end{bmatrix} \begin{bmatrix} z \\ p \end{bmatrix} \leq 0$$

- can convert this to the LMI (with variables P and τ)

$$\begin{bmatrix} A^T P + PA + \alpha P - \tau\sigma C^T C & PB + \tau\nu C^T \\ B^T P + \tau\nu C & -\tau \end{bmatrix} \leq 0, \qquad P \geq I$$

- can sometimes extend to case with multiple nonlinearities

Perron-Frobenius theory

- a nonegative matrix A is regular if for some $k \geq 1$, $A^k > 0$ (path of length k from every node to every other node)
- if A is regular, then there is a real, positive, strictly dominant, simple Perron-Frobenius eigenvalue λ_{pf}, with positive left and right eigenvectors
- if we only have $A \geq 0$, then there is an eigenvalue λ_{pf} of A that is real, nonnegative and (non-strictly) dominant, and has (possibly not unique) nonnegative left and right eigenvectors
- For a Markov chain with transition matrix P, if P is regular, the distribution always converges to the unique invariant distribution $\pi > 0$, associated with a simple, dominant eigenvalue of 1
- rate of convergence depends on second largest eigenvalue magnitude

Max-min/min-max ratio characterization

- Perron-Frobenius eigenvalue is optimal value of two optimization problems

$$\begin{array}{ll} \text{maximize} & \min_i \frac{(Ax)_i}{x_i} \\ \text{subject to} & x > 0 \end{array}$$

and

$$\begin{array}{ll} \text{minimize} & \max_i \frac{(Ax)_i}{x_i} \\ \text{subject to} & x > 0 \end{array}$$

- the optimal x is the Perron-Frobenius eigenvector

Linear Lyapunov functions

- suppose $c > 0$, and consider the linear Lyapunov function $V(z) = c^T z$
- if $V(Az) \leq \delta V(z)$ for some $\delta < 1$ and all $z \geq 0$, then V proves (nonnegative) trajectories converge to zero
- a nonnegative regular system is stable if and only if there is a linear Lyapunov function that proves it

Continuous time results

- $\mathbf{R}_+^n$ is invariant under $\dot{x} = Ax$ if and only if $A_{ij} \geq 0$ for $i \neq j$
- such matrices are called Metzler matrices
- A has a real, dominant eigenvalue $\lambda_{\mathrm{metzler}}$ that is real and has associated nonnegative left and right eigenvectors
- analogs exist with other discrete-time results

Exam advice

- five questions
- determine the topic(s) each question covers
- guess the form the problem statement should take
- manipulate ('hammer') the question into that standard form
- explain things as simply as possible; if your solution is extremely complicated, you're probably missing something
- we're not especially concerned about boundary conditions or edge cases, but mention any assumptions you make