Review Session 3

- Generating colored Gaussian variables
- Solving discrete-time Lyapunov equations

Generating colored Gaussian variables

- Suppose we want a random variable $y \sim \mathcal{N}(\bar{y}, \Sigma_y)$
- Have random number generator for $x \sim \mathcal{N}(0, I)$
- Perform linear transformation $y = Ax + b$, get $y \sim \mathcal{N}(b, AA^T)$

Generating Gaussian variables with specified covariance

- Can set $A = \Sigma_y^{1/2}$ (symmetric square root of Σ_y), so that

$$\begin{aligned} \Sigma_y &= \mathbf{E}(y-\bar{y})(y-\bar{y})^T \\ &= \mathbf{E}(\Sigma_y^{1/2}x + \bar{y} - \bar{y})(\Sigma_y^{1/2}x - \bar{y} - \bar{y})^T \\ &= \Sigma_y^{1/2}\,\mathbf{E}(xx^T)\Sigma_y^{1/2} \\ &= \Sigma_y \end{aligned}$$

- Can also set $A = \Sigma_y^{1/2}Q$, where Q is any orthogonal matrix

- Since $QQ^T = I$, we get
  $\Sigma_y = \Sigma_y^{1/2}Q\,\mathbf{E}(xx^T)Q^T\Sigma_y^{1/2} = \Sigma_y^{1/2}QQ^T\Sigma_y^{1/2} = \Sigma_y$

Using the Cholesky factorization

- Instead of computing the matrix square root, we can use the Cholesky factorization

- A little cheaper: about a tenth of the computational cost

- With $\Sigma_y = LL^T$, set $y = Lx + \bar{y}$, and

$$\mathbf{E}(y - \bar{y})(y - \bar{y})^T = L\,\mathbf{E}(xx^T)L^T = LL^T = \Sigma_y$$

- Can find an orthogonal Q that relates L and $\Sigma_y^{1/2}$, because

$$\Sigma_y = (\Sigma_y^{1/2}Q)(Q^T\Sigma_y^{1/2}) = LL^T$$

so if $Q = A^{-1/2}L$, then $L = \Sigma_y^{1/2}Q$

- Therefore, using L is equivalent to orthogonal transformation of x then multiplication by $A^{1/2}$

- Orthogonal transformation of random variable retains second-order statistics

- In Matlab, `sqrtm(Sigma_y)` or `chol(Sigma_y)`

Solving Lyapunov equations efficiently

- Used in (upcoming) lecture 13, *Linear quadratic Lyapunov theory*
- Continuous time, $A^T P + PA + Q = 0$
- Discrete time, $A^T PA - P + Q = 0$
- Solving for P with naïve method is $\mathcal{O}(n^6)$
- Using fast method, can solve in $\mathcal{O}(n^3)$
- Will show fast method for solving Sylvester equations (includes Lyapunov equations as special case)

Discrete-time Sylvester operator

- The *discrete-time Sylvester operator* $S : \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times n}$ is defined as

$$S(X) = AXB - X$$

where A, B, $X \in \mathbf{R}^{n\times n}$.

- We will show that the (n^2) eigenvalues of the Sylvester operator are

$$\lambda_i \mu_j - 1, \qquad i, j = 1, \ldots, n$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, and $\mu_1, \ldots, \mu_n$ are the eigenvalues of B.

Eigenvalues of the discrete-time Sylvester operator

- Let v_i be the right eigenvector of A associated with the eigenvalue λ_i. Then we know that

$$Av_i = \lambda_i v_i$$

- Let w_j be the left eigenvector of B associated with the eigenvalue μ_j. Then we know that

$$w_j^T B = \mu_j w_j^T$$

- Let $X = v_i w_j^T$. We will show that X is a (matrix) eigenvector of S associated with the eigenvalue $\lambda_i \mu_j - 1$.

Eigenvalues of the discrete-time Sylvester operator

$$
\begin{aligned}
AXB - X &= A\left(u_i w_j^T\right) B - u_i w_j^T \\
&= \left(A u_i\right)\left(w_j^T B\right) - u_i w_j^T \\
&= \left(\lambda_i u_i\right)\left(\mu_j w_j^T\right) - u_i w_j^T \\
&= \lambda_i \mu_j u_i w_j^T - u_i w_j^T \\
&= \left(\lambda_i \mu_j - 1\right) u_i w_j^T \\
&= \left(\lambda_i \mu_j - 1\right) X
\end{aligned}
$$

Then we have shown that

$$
S(X) = \left(\lambda_i \mu_j - 1\right) X
$$

which means that X is an eigenvector of S associated with the eigenvalue $(\lambda_i \mu_j - 1)$.

Discrete-time Sylvester operator (cnt'd)

- S is singular if and only if there exists a nonzero X with $S(X) = 0$.
- S is nonsingular if and only if, for all i, j, $\lambda_i \mu_j \neq 1$.
- If A and B are stable then S is nonsingular.

Discrete-time Sylvester equation

- The *discrete-time Sylvester equation* is

$$AXB - X = C$$

  where A, B, C, $X \in \mathbf{R}^{n\times n}$.

- The Sylvester equation has a unique solution for any C if and only if S is non-singular, which occurs if and only if $\lambda_i\mu_j \neq 1$ for all i, $j = 1, \ldots, n$.

- The Sylvester equation can be rewritten as a set of n^2 equations in n^2 variables $\mathcal{A}vec(X) = vec(C)$, which can be solved in $O(n^6)$ operations.

Solving the discrete-time Sylvester equation

Suppose that A and B are diagonalizable. Then plugging in $A = T\Lambda T^{-1}$ and $B = SMS^{-1}$ where $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$ and $M = \mathbf{diag}(\mu_1, \ldots, \mu_n)$, we get

$$(T\Lambda T^{-1})X(SMS^{-1}) - X = C.$$

Multiplying on the left by T^{-1} and on the right by S, we obtain

$$\Lambda(T^{-1}XS)M - (T^{-1}XS) = T^{-1}CS.$$

Let $\tilde{X} = T^{-1}XS$ and $\tilde{C} = T^{-1}CS$. Then we have

$$\Lambda\tilde{X}M - \tilde{X} = \tilde{C}.$$

Solving the discrete-time Sylvester equation (cnt'd)

Denoting the (i, j)th entry of $\tilde{C}$ as $\tilde{c}_{ij}$ and of $\tilde{X}$ as $\tilde{x}_{ij}$, the equation reads as

$$\lambda_i \mu_j \tilde{x}_{ij} - \tilde{x}_{ij} = \tilde{c}_{ij},$$

which means that

$$\tilde{x}_{ij} = \frac{\tilde{c}_{ij}}{\lambda_i \mu_j - 1}.$$

Finally, once we compute $\tilde{X}$, we take $X = T\tilde{X}S^{-1}$.

Note that by exploiting the structure available in the Sylvester operator, we were able to solve the discrete-time Sylvester equation in $O(n^3)$ operations (since the eigenvectors and eigenvalues of a diagonalizable matrix can be found in $O(n^3)$ operations.)

Discrete-time Lyapunov operator

The *discrete-time Lyapunov operator* is a special case of the discrete-time Sylvester operator:

$$\mathcal{L}(P) = A^T PA - P.$$

- $\mathcal{L}$ is nonsingular if and only if

$$\lambda_i \lambda_j \neq 1 \qquad i, j = 1, \ldots, n,$$

  where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

- If A is stable then $\mathcal{L}$ is nonsingular.

Discrete-time Lyapunov equation

The *discrete-time Lyapunov equation* is

$$A^T P A - P + Q = 0.$$

- If A is stable then for any Q there is exactly one solution P to the discrete-time Lyapunov equation.

- If A is diagonalizable, i.e. if $A = T\Lambda T^{-1}$, then $P = T\tilde{P}T^{-1}$, where

  $$\tilde{p}_{ij} = \frac{-\tilde{q}_{ij}}{\lambda_i \lambda_j - 1},$$

  with $\tilde{Q} = T^{-1}QT$.

- The solution can be found efficiently in $O(n^3)$ operations.

Continuous-time Sylvester operator

The *continuous-time Sylvester operator* $S : \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times n}$ is defined as

$$S(X) = AX + XB$$

where A, B, $X \in \mathbf{R}^{n\times n}$. The (n^2) eigenvalues of the Sylvester operator are

$$\lambda_i + \mu_j, \qquad i, j = 1, \ldots, n$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, and $\mu_1, \ldots, \mu_n$ are the eigenvalues of B.

Continuous-time Sylvester operator (cnt'd)

- S is singular if and only if there exists a nonzero X with $S(X) = 0$.
- S is nonsingular if and only if, for all i, j, $\lambda_i \neq -\mu_j$, or equivalently, if and only if A and $-B$ share no eigenvalues.
- If A and B are stable, then S is nonsingular.

Continuous-time Sylvester equation

- The *continuous-time Sylvester equation* is

$$AX + XB = C$$

  where A, B, C, $X \in \mathbf{R}^{n \times n}$.

- The Sylvester equation has a unique solution for any C if and only if S is non-singular, which occurs if and only if for all i, j, $\lambda_i \neq -\mu_j$.

- The Sylvester equation can be rewritten as a set of n^2 equations in n^2 variables $\mathcal{A}vec(X) = vec(C)$, which can be solved in $O(n^6)$ operations.

- If we take advantage of the structure of the Sylvester operations, it can be solved in $O(n^3)$ operations.

Continuous-time Lyapunov operator

The *continuous-time Lyapunov operator* is a special case of the continuous-time Sylvester operator:

$$\mathcal{L}(P) = A^T P + PA.$$

$\mathcal{L}$ is nonsingular if and only if A and $-A$ have no common eigenvalues.

- If A is stable, then $\mathcal{L}$ is nonsingular.
- If A has an imaginary eigenvalue, then $\mathcal{L}$ is singular.

Continuous-time Lyapunov equation

The *continuous-time Lyapunov equation* is

$$A^T P + PA + Q = 0.$$

- If A is stable then for any Q there is exactly one solution P to the continuous-time Lyapunov equation.

- If A is diagonalizable, i.e. if $A = T\Lambda T^{-1}$, then $P = T\tilde{P}T^{-1}$, where

$$\tilde{p}_{ij} = \frac{-\tilde{q}_{ij}}{\lambda_i + \lambda_j},$$

  with $\tilde{Q} = T^{-1}QT$.

- The solution can be found efficiently in $O(n^3)$ operations.

Solving Sylvester equations in MATLAB

- `X = dlyap(A,B,-C)` solves the discrete-time Sylvester equation $AXB - X = C$.

- `P = dlyap(A',Q)` solves the discrete-time Lyapunov equation $A^TPA - P + Q = 0$.

- `X = lyap(A,B,-C)` solves the continuous-time Sylvester equation $AX + XB = C$.

- `X = lyap(A',Q)` solves the continuous-time Lyapunov equation $A^TP + PA + Q = 0$.