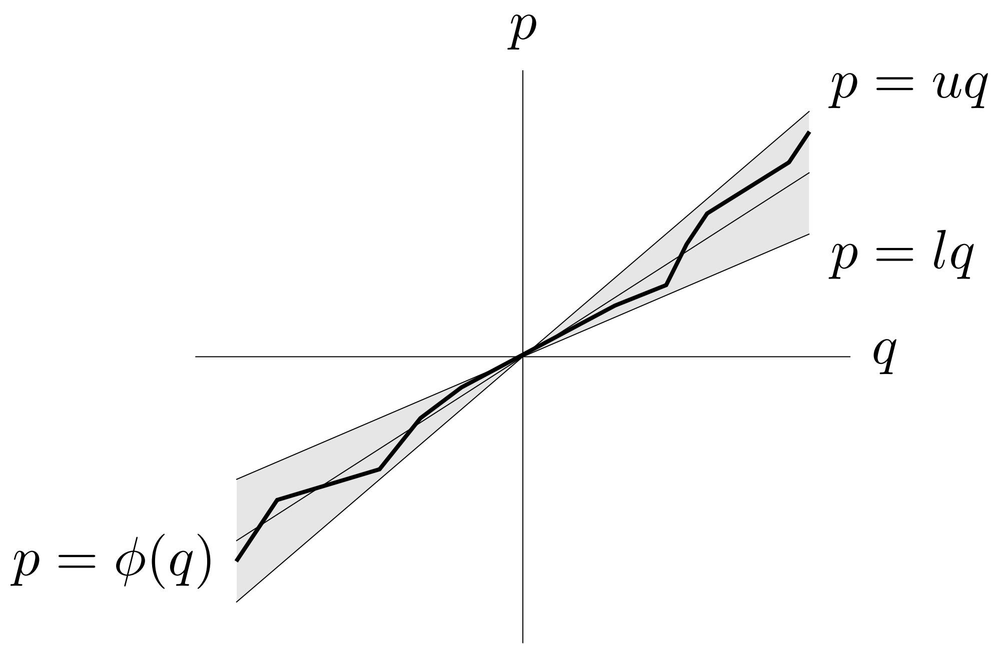
# Lecture 16
# Analysis of systems with sector nonlinearities

- Sector nonlinearities
- Lur'e system
- Analysis via quadratic Lyapunov functions
- Extension to multiple nonlinearities

# Sector nonlinearities

a function $\phi : \mathbf{R} \to \mathbf{R}$ is said to be in sector $[l, u]$ if for all $q \in \mathbf{R}$, $p = \phi(q)$ lies between $lq$ and $uq$




can be expressed as quadratic inequality

$$(p - uq)(p - lq) \leq 0 \text{ for all } q, \; p = \phi(q)$$

examples:

- sector $[-1, 1]$ means $|\phi(q)| \leq |q|$
- sector $[0, \infty]$ means $\phi(q)$ and $q$ always have same sign (graph in first & third quadrants)

some equivalent statements:

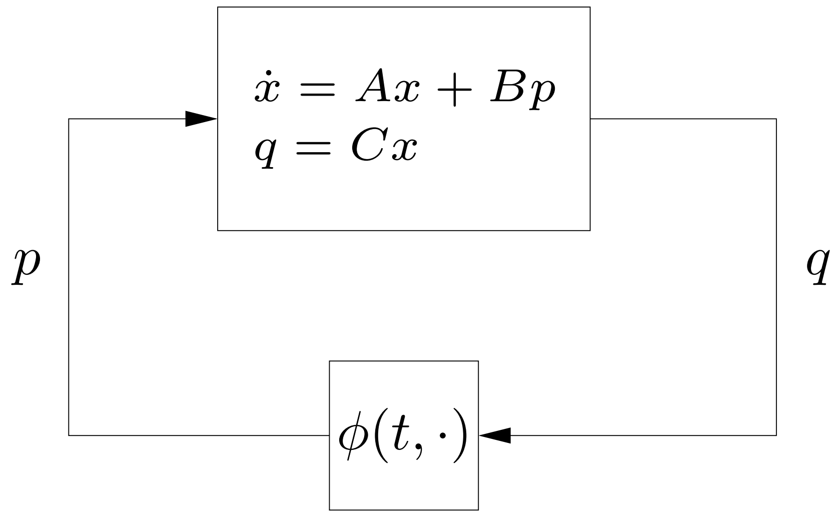
- $\phi$ is in sector $[l, u]$ iff for all $q$,

$$\left|\phi(q) - \frac{u+l}{2}q\right| \leq \frac{u-l}{2}|q|$$

- $\phi$ is in sector $[l, u]$ iff for each $q$ there is $\theta(q) \in [l, u]$ with $\phi(q) = \theta(q)q$

# Nonlinear feedback representation

linear dynamical system with nonlinear feedback




closed-loop system: $\dot{x} = Ax + B\phi(t, Cx)$

- a common representation that separates linear and nonlinear time-varying parts
- often $p$, $q$ are scalar signals

# Lur'e system

a (single nonlinearity) *Lur'e system* has the form

$$\dot{x} = Ax + Bp, \qquad q = Cx, \qquad p = \phi(t, q)$$

where $\phi(t, \cdot) : \mathbf{R} \to \mathbf{R}$ is in sector $[l, u]$ for each $t$

here $A$, $B$, $C$, $l$, and $u$ are given; $\phi$ is otherwise not specified

- a common method for describing time-varying nonlinearity and/or uncertainty
- goal is to prove stability, or derive a bound, using only the sector information about $\phi$
- if we succeed, the result is strong, since it applies to a large family of nonlinear time-varying systems

# Stability analysis via quadratic Lyapunov functions

let's try to establish global asymptotic stability of Lur'e system, using quadratic Lyapunov function $V(z) = z^T P z$

we'll require $P > 0$ and $\dot{V}(z) \leq -\alpha V(z)$, where $\alpha > 0$ is given

second condition is:

$$\dot{V}(z) + \alpha V(z) = 2z^T P \left(Az + B\phi(t, Cz)\right) + \alpha z^T P z \leq 0$$

for all $z$ and all sector $[l, u]$ functions $\phi(t, \cdot)$

same as:

$$2z^T P \left(Az + Bp\right) + \alpha z^T P z \leq 0$$

for all $z$, and all $p$ satisfying $(p - uq)(p - lq) \leq 0$, where $q = Cz$

we can express this last condition as a quadratic inequality in $(z,p)$:

$$\left[\begin{array}{c} z \\ p \end{array}\right]^T \left[\begin{array}{cc} \sigma C^T C & -\nu C^T \\ -\nu C & 1 \end{array}\right] \left[\begin{array}{c} z \\ p \end{array}\right] \leq 0$$

where $\sigma = lu$, $\nu = (l+u)/2$

so $\dot{V} + \alpha V \leq 0$ is equivalent to:

$$\left[\begin{array}{c} z \\ p \end{array}\right]^T \left[\begin{array}{cc} A^T P + PA + \alpha P & PB \\ B^T P & 0 \end{array}\right] \left[\begin{array}{c} z \\ p \end{array}\right] \leq 0$$

whenever

$$\left[\begin{array}{c} z \\ p \end{array}\right]^T \left[\begin{array}{cc} \sigma C^T C & -\nu C^T \\ -\nu C & 1 \end{array}\right] \left[\begin{array}{c} z \\ p \end{array}\right] \leq 0$$

by (lossless) S-procedure this is equivalent to: there is a $\tau \geq 0$ with

$$\begin{bmatrix} A^TP + PA + \alpha P & PB \\ B^TP & 0 \end{bmatrix} \leq \tau \begin{bmatrix} \sigma C^TC & -\nu C^T \\ -\nu C & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} A^TP + PA + \alpha P - \tau\sigma C^TC & PB + \tau\nu C^T \\ B^TP + \tau\nu C & -\tau \end{bmatrix} \leq 0$$

an LMI in $P$ and $\tau$ ($2, 2$ block automatically gives $\tau \geq 0$)

by homogeneity, we can replace condition $P > 0$ with $P \geq I$

our final LMI is

$$\begin{bmatrix} A^TP + PA + \alpha P - \tau\sigma C^TC & PB + \tau\nu C^T \\ B^TP + \tau\nu C & -\tau \end{bmatrix} \leq 0, \qquad P \geq I$$

with variables $P$ and $\tau$

- hence, can efficiently determine if there exists a quadratic Lyapunov function that proves stability of Lur'e system

- this LMI can also be solved via an ARE-like equation, or by a graphical method that has been known since the 1960s

- this method is more sophisticated and powerful than the 1895 approach:
  - replace nonlinearity with $\phi(t, q) = \nu q$
  - choose $Q > 0$ ($e.g.$, $Q = I$) and solve Lyapunov equation

$$(A + \nu BC)^T P + P(A + \nu BC) + Q = 0$$

    for $P$
  - hope $P$ works for nonlinear system

# Multiple nonlinearities

we consider system

$$\dot{x} = Ax + Bp, \qquad q = Cx, \qquad p_i = \phi_i(t, q_i), \quad i = 1, \ldots, m$$

where $\phi_i(t, \cdot) : \mathbf{R} \to \mathbf{R}$ is sector $[l_i, u_i]$ for each $t$

we seek $V(z) = z^T P z$, with $P > 0$, so that $\dot{V} + \alpha V \leq 0$

last condition equivalent to:

$$\left[\begin{array}{c} z \\ p \end{array}\right]^T \left[\begin{array}{cc} A^T P + PA + \alpha P & PB \\ B^T P & 0 \end{array}\right] \left[\begin{array}{c} z \\ p \end{array}\right] \leq 0$$

whenever

$$(p_i - u_i q_i)(p_i - l_i q_i) \leq 0, \quad i = 1, \ldots, m$$

we can express this last condition as

$$\begin{bmatrix} z \\ p \end{bmatrix}^T \begin{bmatrix} \sigma c_i c_i^T & -\nu_i c_i e_i^T \\ -\nu_i e_i c_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} z \\ p \end{bmatrix} \leq 0, \quad i = 1, \ldots, m$$

where $c_i^T$ is the $i$th row of $C$, $e_i$ is the $i$th unit vector, $\sigma_i = l_i u_i$, and $\nu_i = (l_i + u_i)/2$

now we use (lossy) S-procedure to get a sufficient condition: there exists $\tau_1, \ldots, \tau_m \geq 0$ such that

$$\begin{bmatrix} A^T P + PA + \alpha P - \sum_{i=1}^m \tau_i \sigma_i c_i c_i^T & PB + \sum_{i=1}^m \tau_i \nu_i c_i e_i^T \\ B^T P + \sum_{i=1}^m \tau_i \nu_i e_i c_i^T & -\sum_{i=1}^m \tau_i e_i e_i^T \end{bmatrix} \leq 0$$

we can write this as:

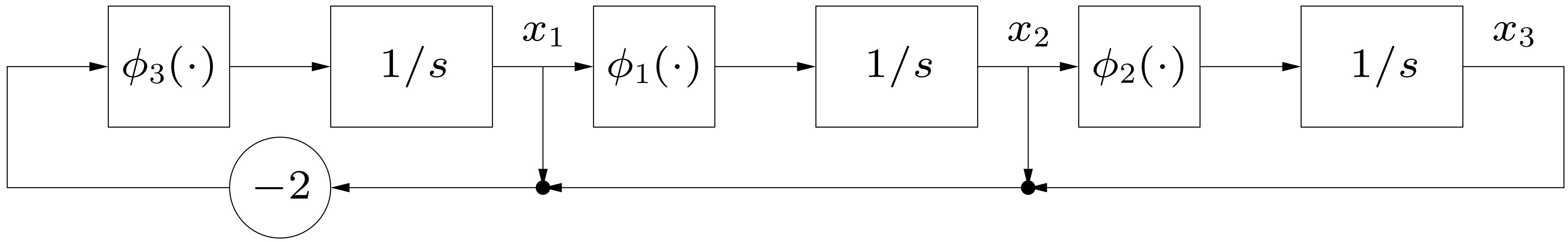
$$\begin{bmatrix} A^TP + PA + \alpha P - C^TDFC & PB + C^TDG \\ B^TP + DGC & -D \end{bmatrix} \leq 0$$

where

$$D = \mathbf{diag}(\tau_1, \ldots, \tau_m), \qquad F = \mathbf{diag}(\sigma_1, \ldots, \sigma_m), \qquad G = \mathbf{diag}(\nu_1, \ldots, \nu_m)$$

- this is an LMI in variables $P$ and $D$
- $2, 2$ block automatically gives us $\tau_i \geq 0$
- by homogeneity, we can add $P \geq I$ to ensure $P > 0$
- solving these LMIs allows us to (sometimes) find quadratic Lyapunov functions for Lur'e system with multiple nonlinearities
(which was impossible until recently)

# Example




we consider system

$$\dot{x}_2 = \phi_1(t, x_1), \qquad \dot{x}_3 = \phi_2(t, x_2), \qquad \dot{x}_1 = \phi_3(t, -2(x_1 + x_2 + x_3))$$

where $\phi_1(t, \cdot)$, $\phi_2(t, \cdot)$, $\phi_3(t, \cdot)$ are sector $[1 - \delta, 1 + \delta]$

- $\delta$ gives the percentage nonlinearity
- for $\delta = 0$, we have (stable) linear system $\dot{x} = \begin{bmatrix} -2 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$

let's put system in Lur'e form:

$$\dot{x} = Ax + Bp, \qquad q = Cx, \qquad p_i = \phi_i(q_i)$$

where

$$A = 0, \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & -2 \end{bmatrix}$$

the sector limits are $l_i = 1 - \delta$, $u_i = 1 + \delta$

define $\sigma = l_i u_i = 1 - \delta^2$, and note that $(l_i + u_i)/2 = 1$

we take $x(0) = (1, 0, 0)$, and seek to bound $J = \int_0^\infty \|x(t)\|^2 \, dt$

(for $\delta = 0$ we can calculate $J$ exactly by solving a Lyapunov equation)

we'll use quadratic Lyapunov function $V(z) = z^T P z$, with $P \geq 0$

Lyapunov conditions for bounding $J$: if $\dot{V}(z) \leq -z^T z$ whenever the sector conditions are satisfied, then $J \leq x(0)^T P x(0) = P_{11}$

use S-procedure as above to get sufficient condition:

$$\begin{bmatrix} A^T P + PA + I - \sigma C^T D C & PB + C^T D \\ B^T P + DC & -D \end{bmatrix} \leq 0$$

which is an LMI in variables $P$ and $D = \mathbf{diag}(\tau_1, \tau_2, \tau_3)$

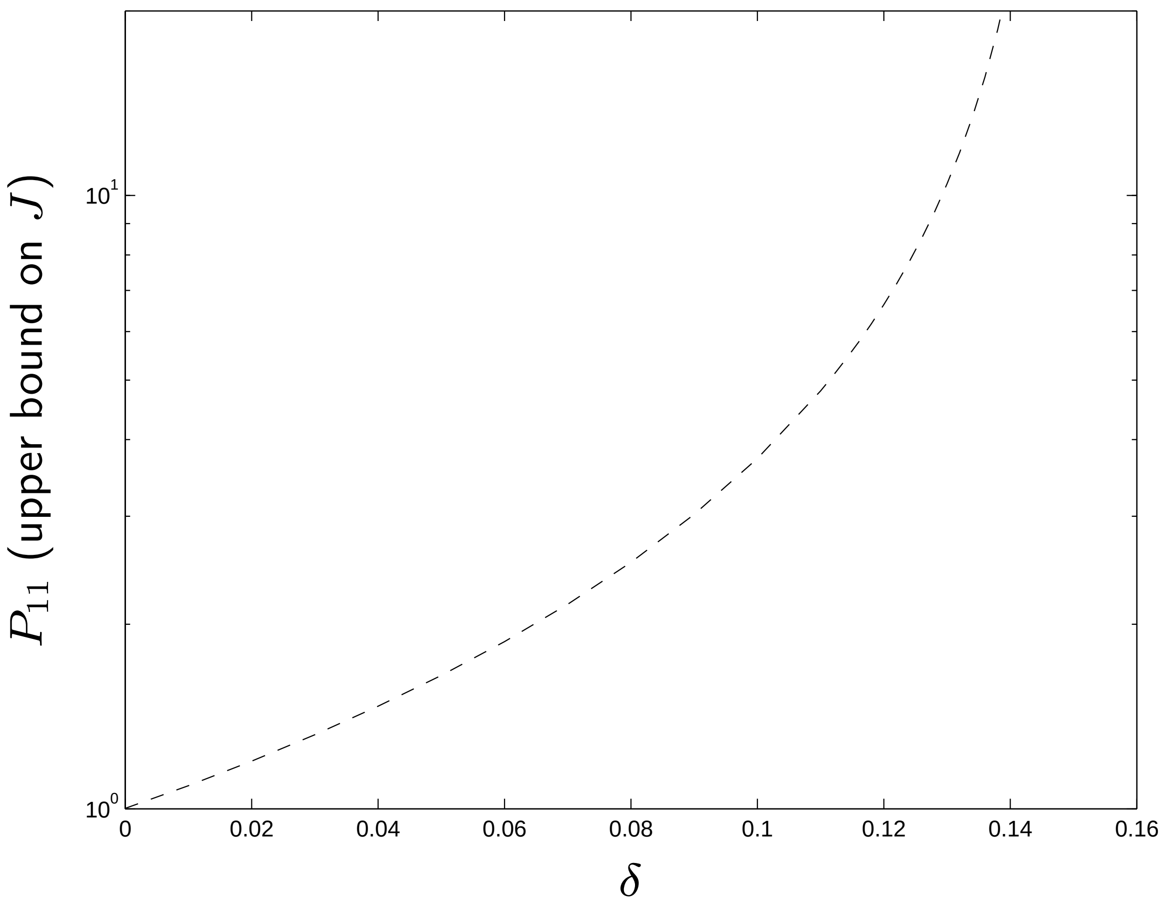
note that LMI gives $\tau_i \geq 0$ automatically to get best bound on $J$ for given $\delta$, we solve SDP

$$\begin{array}{ll} \text{minimize} & P_{11} \\ \text{subject to} & \left[\begin{array}{cc} A^TP + PA + I - \sigma C^T DC & PB + C^T D \\ B^T P + DC & -D \end{array}\right] \leq 0 \\ & P \geq 0 \end{array}$$

with variables $P$ and $D$ (which is diagonal)

optimal value gives best bound on $J$ that can be obtained from a quadratic Lyapunov function, using S-procedure

# Upper bound on $J$




- bound is tight for $\delta = 0$; for $\delta \geq 0.15$, LMI is infeasible

# Approximate worst-case simulation

- heuristic method for finding 'bad' $\phi_i$'s, *i.e.*, ones that lead to large $J$
- find $V$ from worst-case analysis as above
- at time $t$, choose $p_i$'s to maximize $\dot{V}(x(t))$ subject to sector constraints $|p_i - q_i| \leq \delta |q_i|$
- using $\dot{V}(x(t)) = 2x^T P(Ax + Bp)$, we get

$$p = q + \delta\, \mathbf{diag}(\mathbf{sign}(B^T P x))|q|$$

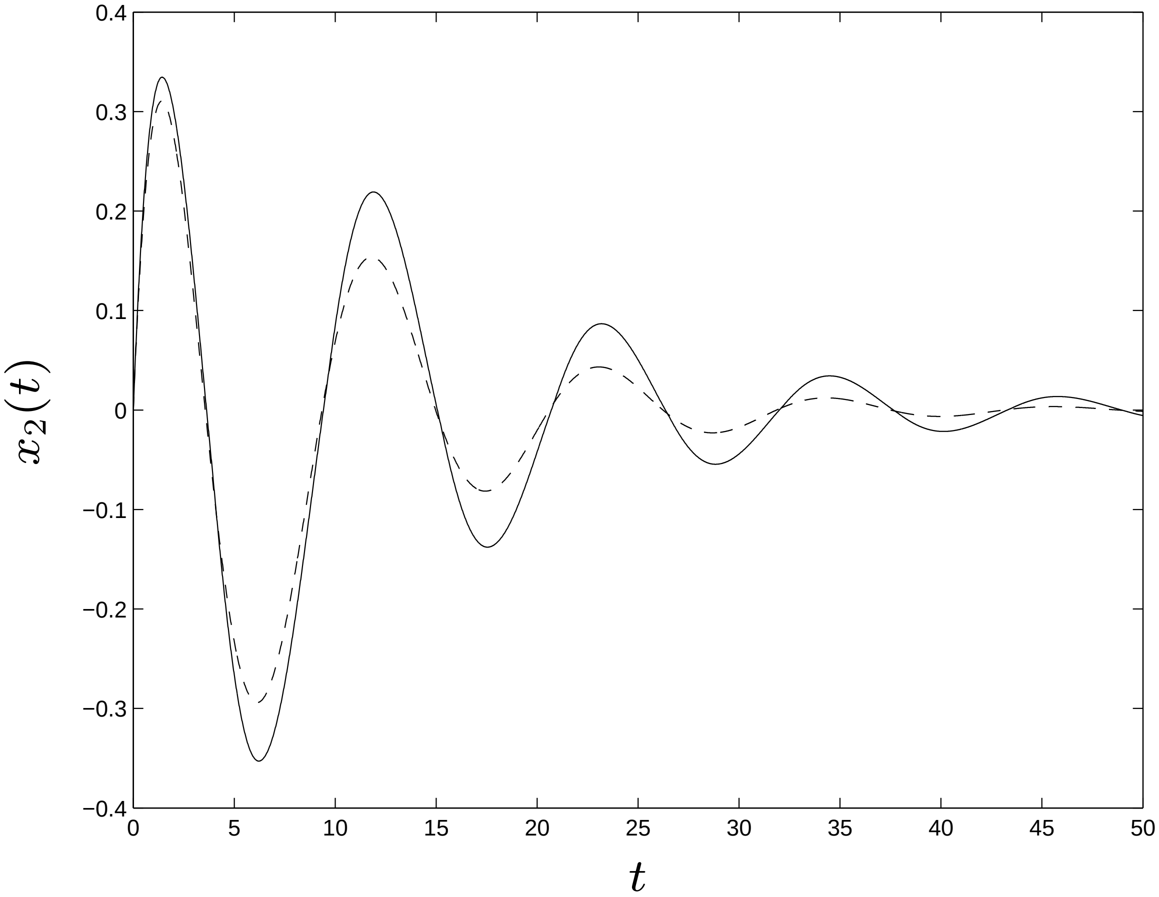
- simulate

$$\dot{x} = Ax + Bp, \qquad p = q + \delta\, \mathbf{diag}(\mathbf{sign}(B^T P x))|q|$$

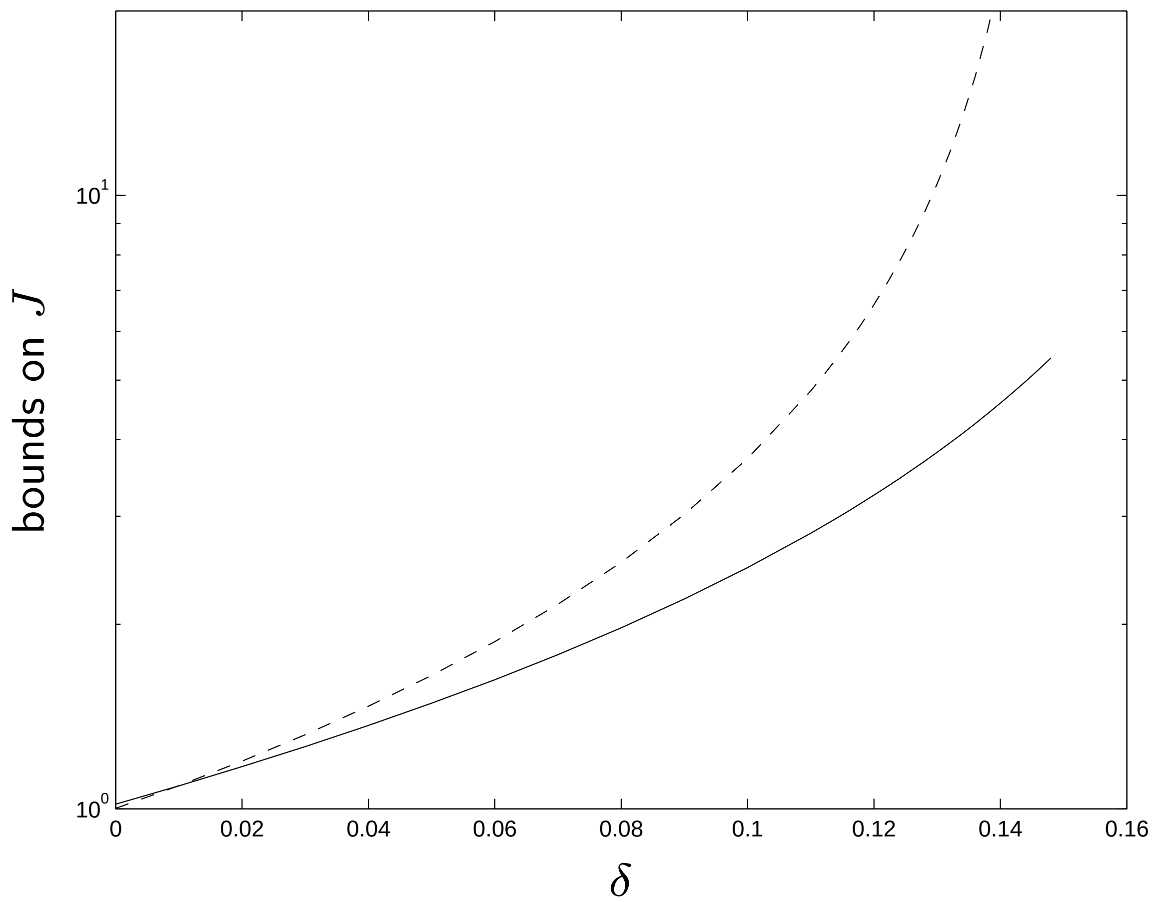
starting from $x(0) = (1, 0, 0)$

# Approximate worst-case simulation

AWC simulation with $\delta = 0.05$: $J_{\mathrm{awc}} = 1.49$; $J_{\mathrm{ub}} = 1.65$
for comparison, linear case ($\delta = 0$): $J_{\mathrm{lin}} = 1.00$



# Upper and lower bounds on worst-case $J$




- lower curve gives $J$ obtained from approximate worst-case simulation