# Lecture 9
# The Extended Kalman filter

- Nonlinear filtering
- Extended Kalman filter
- Linearization and random variables

# Nonlinear filtering

- nonlinear Markov model:

$$x_{t+1} = f(x_t, w_t), \qquad y_t = g(x_t, v_t)$$

  - $f$ is (possibly nonlinear) dynamics function
  - $g$ is (possibly nonlinear) measurement or output function
  - $w_0, w_1, \ldots, v_0, v_1, \ldots$ are independent
  - even if $w$, $v$ Gaussian, $x$ and $y$ need not be

- nonlinear filtering problem: find, $e.g.$,

$$\hat{x}_{t|t-1} = \mathbf{E}(x_t|y_0, \ldots, y_{t-1}), \qquad \hat{x}_{t|t} = \mathbf{E}(x_t|y_0, \ldots, y_t)$$

- general nonlinear filtering solution involves a PDE, and is not practical

# Extended Kalman filter

- extended Kalman filter (EKF) is *heuristic* for nonlinear filtering problem
- often works well (when tuned properly), but sometimes not
- widely used in practice
- based on
  - linearizing dynamics and output functions at current estimate
  - propagating an approximation of the conditional expectation and covariance

# Linearization and random variables

- consider $\phi : \mathbf{R}^n \to \mathbf{R}^m$
- suppose $\mathbf{E}\, x = \bar{x}$, $\mathbf{E}(x - \bar{x})(x - \bar{x})^T = \Sigma_x$, and $y = \phi(x)$
- if $\Sigma_x$ is small, $\phi$ is not too nonlinear,

$$y \approx \tilde{y} = \phi(\bar{x}) + D\phi(\bar{x})(x - \bar{x})$$

- gives *approximation* for mean and covariance of nonlinear function of random variable:

$$\bar{y} \approx \phi(\bar{x}), \qquad \Sigma_y \approx D\phi(\bar{x})\Sigma_x D\phi(\bar{x})^T$$

- if $\Sigma_x$ is not small compared to 'curvature' of $\phi$, these estimates are poor

- a good estimate can be found by Monte Carlo simulation:

$$\begin{aligned}\bar{y} &\approx \bar{y}^{\mathrm{mc}} = \frac{1}{N}\sum_{i=1}^{N} \phi(x^{(i)}) \\ \Sigma_y &\approx \frac{1}{N}\sum_{i=1}^{N} \left(\phi(x^{(i)}) - \bar{y}^{\mathrm{mc}}\right)\left(\phi(x^{(i)}) - \bar{y}^{\mathrm{mc}}\right)^T\end{aligned}$$

  where $x^{(1)}, \ldots, x^{(N)}$ are samples from the distribution of $x$, and $N$ is large

- another method: use Monte Carlo formulas, with a small number of nonrandom samples chosen as 'typical', $e.g.$, the $90\%$ confidence ellipsoid semi-axis endpoints

$$x^{(i)} = \bar{x} \pm \beta v_i, \qquad \Sigma_x = V \Lambda V^T$$

# Example

$x \sim \mathcal{N}(0,1)$, $y = \exp(x)$

(for this case we can compute mean and variance of $y$ exactly)

| | $\bar{y}$ | $\sigma_y$ |
|---|---:|---:|
| exact values | $e^{1/2} = 1.649$ | $\sqrt{e^2 - e} = 2.161$ |
| linearization | $1.000$ | $1.000$ |
| Monte Carlo ($N = 10$) | $1.385$ | $1.068$ |
| Monte Carlo ($N = 100$) | $1.430$ | $1.776$ |
| Sigma points ($x = \bar{x},\ \bar{x} \pm 1.5\sigma_x$) | $1.902$ | $2.268$ |

# Extended Kalman filter

- *initialization:* $\hat{x}_{0|-1} = \bar{x}_0$, $\Sigma(0|-1) = \Sigma_0$
- *measurement update*
  - linearize output function at $x = \hat{x}_{t|t-1}$:

$$\begin{aligned} C &= \frac{\partial g}{\partial x}(\hat{x}_{t|t-1}, 0) \\ V &= \frac{\partial g}{\partial v}(\hat{x}_{t|t-1}, 0)\Sigma_v \frac{\partial g}{\partial v}(\hat{x}_{t|t-1}, 0)^T \end{aligned}$$

  - measurement update based on linearization

$$\begin{aligned} \hat{x}_{t|t} &= \hat{x}_{t|t-1} + \Sigma_{t|t-1} C^T \left( C \Sigma_{t|t-1} C^T + V \right)^{-1} \dots \\ &\quad \dots (y_t - g(\hat{x}_{t|t-1}, 0)) \\ \Sigma_{t|t} &= \Sigma_{t|t-1} - \Sigma_{t|t-1} C^T \left( C \Sigma_{t|t-1} C^T + V \right)^{-1} C \Sigma_{t|t-1} \end{aligned}$$

- *time update*
  - linearize dynamics function at $x = \hat{x}_{t|t}$:

$$
\begin{aligned}
A &= \frac{\partial f}{\partial x}(\hat{x}_{t|t}, 0) \\
W &= \frac{\partial f}{\partial w}(\hat{x}_{t|t}, 0)\Sigma_w \frac{\partial f}{\partial w}(\hat{x}_{t|t}, 0)^T
\end{aligned}
$$

  - time update based on linearization

$$
\hat{x}_{t+1|t} = f(\hat{x}_{t|t}, 0), \qquad \Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W
$$

- replacing linearization with Monte Carlo yields *particle filter*
- replacing linearization with sigma-point estimates yields *unscented Kalman filter* (UKF)

# Example

- $p_t$, $u_t \in \mathbf{R}^2$ are position and velocity of vehicle, with $(p_0, u_0) \sim \mathcal{N}(0, I)$
- vehicle dynamics:

$$p_{t+1} = p_t + 0.1 u_t, \qquad u_{t+1} = \begin{bmatrix} 0.85 & 0.15 \\ -0.1 & 0.85 \end{bmatrix} u_t + w_t$$

  $w_t$ are IID $\mathcal{N}(0, I)$

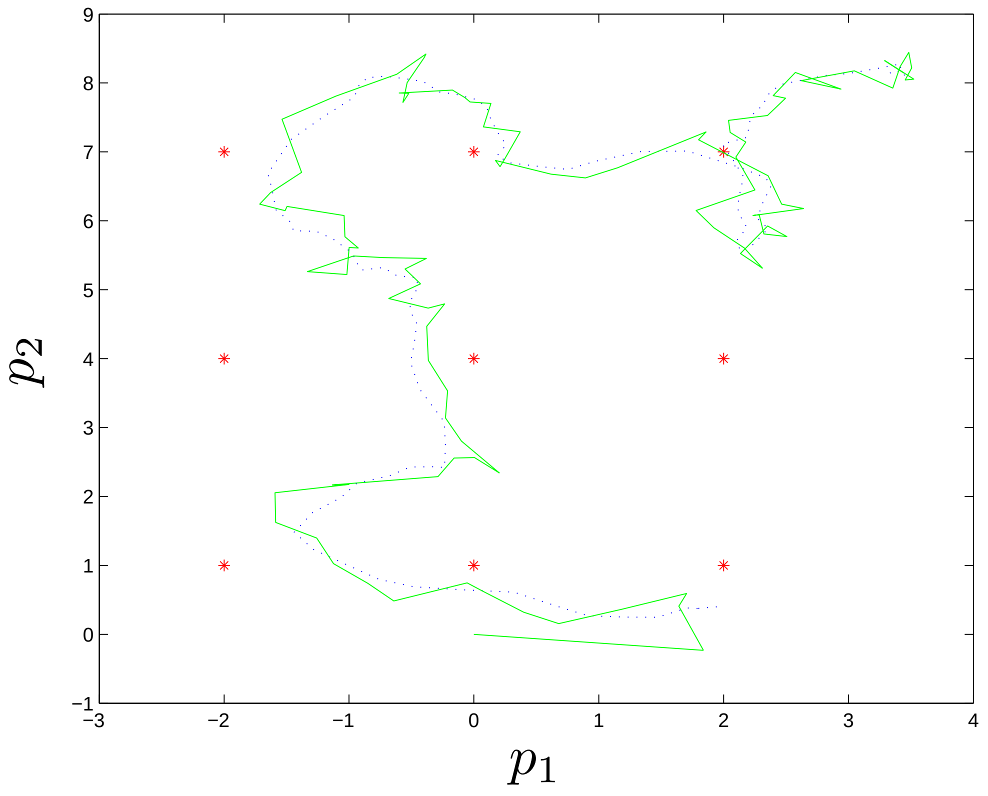
- measurements: noisy measurements of distance to 9 points $p_i \in \mathbf{R}^2$

$$(y_t)_i = \|p_t - p_i\| + (v_t)_i, \quad i = 1, \ldots, 9,$$

  $(v_t)_i$ are IID $\mathcal{N}(0, 0.3^2)$

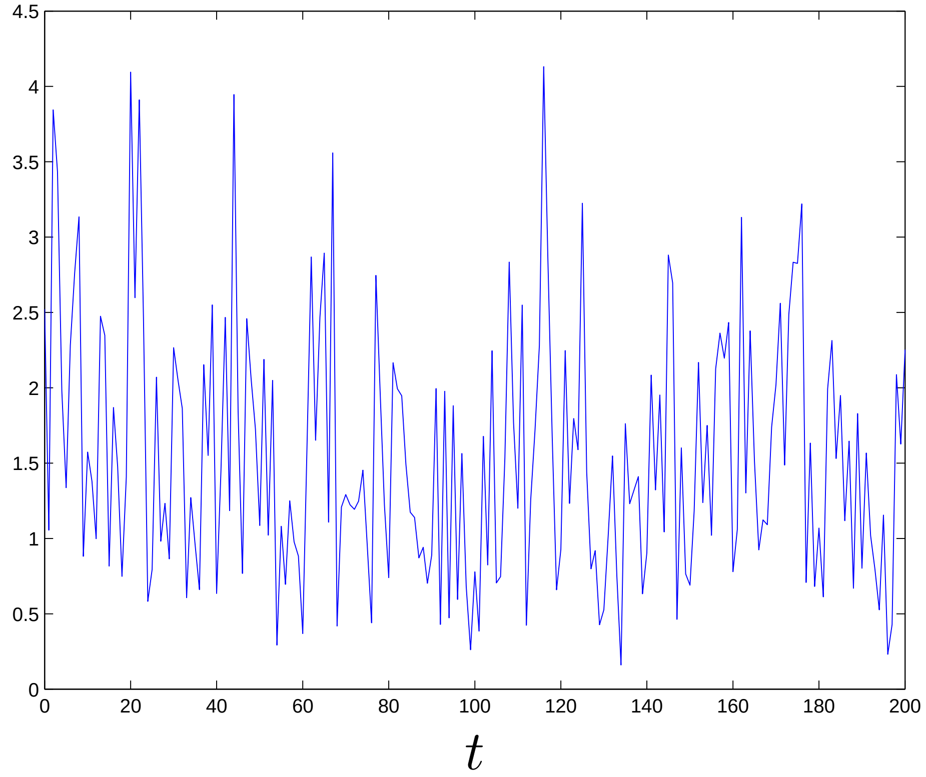
# EKF results

- EKF initialized with $\hat{x}_{0|-1} = 0$, $\Sigma(0|-1) = I$, where $x = (p, u)$
- $p_i$ shown as stars; $p_t$ as dotted curve; $\hat{p}_{t|t}$ as solid curve



# Current position estimation error

$\|\hat{p}_{t|t} - p_t\|$ versus $t$



# Current position estimation predicted error

$(\Sigma(t|t)_{11} + \Sigma(t|t)_{22})^{1/2}$ versus $t$