EE363 homework 8 solutions

1. *Lyapunov condition for passivity.* The system described by $\dot{x} = f(x,u)$, $y = g(x)$, $x(0) = 0$, with $u(t),\ y(t) \in \mathbf{R}^m$, is said to be *passive* if

$$\int_0^t u(\tau)^T y(\tau)\, d\tau \geq 0$$

holds for all trajectories of the system, and for all t.

Here we interpret u and y as power-conjugate quantities (*i.e.*, quantities whose product gives power) such as voltage and current or force and velocity. The inequality above states that at all times, the total energy delivered to the system since $t = 0$ is nonnegative, *i.e.*, it is impossible to extract any energy from a passive system.

(a) Establish the following Lyapunov condition for passivity: If there exists a function V such that $V(z) \geq 0$ for all z, $V(0) = 0$, and $\dot{V}(z,w) \leq w^T g(z)$ for all w and z, then the system is passive.

(b) Now suppose the system is $\dot{x} = Ax + Bu$, $y = Cx$, and consider the quadratic Lyapunov function $V(z) = z^T P z$. Express the conditions found in part (a) as a matrix inequality involving A, B, C, and P.

Remark: you will not be surprised to learn that for a linear system, the condition you found in this problem is not only sufficient but also necessary for the system to be passive. This result is called the *Kalman-Yakubovich-Popov* (KYP) or *positive real* (PR) lemma.

(c) Now consider the specific case with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -5 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}.$$

Use an LMI solver to find a matrix P for which the Lyapunov function $V(z) = z^T P z$ establishes passivity of the system.

Solution:

(a) We need to show that

$$\int_0^t u(\tau)^T y(\tau) d\tau = \int_0^t u(\tau)^T g(x(\tau)) d\tau \geq 0,$$

given that there exists a function V such that for all w, z we have $V(z) \geq 0$, $V(0) = 0$, and $\dot{V}(z,w) \leq w^T g(z)$.

$$\int_0^t u(\tau)^T g(x(\tau)) d\tau \geq \int_0^t \dot{V}(x(\tau), u(\tau)) d\tau = V(x(t)) - V(0) \geq 0,$$

so the passivity condition is satisfied.

(b) Using a Lyapunov function of the form $V(z) = z^T P z$, the passivity condition of part (a) is satisfied if we find a matrix $P \geq 0$ such that $\dot{V}(z, w) = \dot{z}^T P z + z^T P \dot{z} \leq w^T C z$ holds for all w, z. These conditions can be restated as: There is a $P \geq 0$ such that, for all w and z,

$$\begin{aligned} &\dot{z}^T P z + z^T P \dot{z} \\ &= (Az + Bw)^T P z + z^T P (Az + Bw) \\ &= z^T (A^T P + PA) z + w^T B^T P z + z^T P B w \\ &\leq w^T C z, \end{aligned}$$

or simply that $z^T (A^T P + PA) z + w^T B^T P z + z^T P B w - w^T C z \leq 0$. But this expression is affine in w, so will be satisfied only if the linear portion, $w^T B^T P z + z^T P B w - w^T C z = 0$. This leaves the condition that $z^T (A^T P + PA) z \leq 0$, which will hold for all z if and only if $A^T P + PA \leq 0$. Hence, our conditions are that the system is passive if and only if $B^T P = (1/2)C$ and $A^T P + PA \leq 0$.

This is the well known *KYP lemma*. (The factor of 1/2 can be removed from the LMI above without affecting its feasibility, to obtain the KYP lemma in its more common form. This is because a system with a given C is passive if and only if it is also passive with $2C$.)

(c) The following MATLAB code solves for P

```
clear all
close all

A = [ 0   1   0;
      0   0   1;
     -4  -5  -2];
B = [0 1 1]';
C = [2 4 1];

cvx_begin sdp
    variable P(3,3) symmetric;
    P >= zeros(3,3);
    A'*P+P*A <= 0;
    B'*P == (1/2)*C;
cvx_end
cvx_status
cvx_optval;

P
```

The output of the code is

```
P =
    2.0197    0.8472    0.1528
    0.8472    1.7439    0.2561
    0.1528    0.2561    0.2439
```

2. *Finding a discrete-time diagonal Lyapunov function.* Recall problem 4 from the last homework, which concerned the stability of a digital filter with saturation. In that problem you proved that the system $x_{t+1} = \mathbf{sat}(Ax_t)$ is globally asymptotically stable if there exists a nonsingular diagonal D such that $\|DAD^{-1}\| < 1$.

   (a) Show how to find a nonsingular diagonal D that satisfies $\|DAD^{-1}\| < 1$, or determine that no such D exists, using LMIs.

   Hints:

   - A matrix Z satisfies $\|Z\| < 1$ if and only if $Z^T Z < I$.
   - You might find it easier to search for $E = D^2$, which is positive and diagonal.

   (b) Use an LMI solver to find such a D for the specific case

$$A = \begin{bmatrix} -0.2 & 0.01 & -0.002 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Solution:

(a) We have to express the inequality $\|DAD^{-1}\| < 1$ as an LMI. This is done as follows

$$\begin{aligned} \|DAD^{-1}\| &< 1 \\ (DAD^{-1})^T(DAD^{-1}) &< I \\ D^{-1}A^T D^2 A D^{-1} &< I \\ A^T D^2 A - D^2 &< 0 \end{aligned}$$

By making the substitution $X = D^2$, we finally get that the original problem is equivalent to solving the following set of LMI's

$$A^T X A - X < 0, \qquad X > 0.$$

We can then retrieve D from $D = X^{1/2}$.

(b) It turns out that for this specific problem, $D = \mathbf{diag}(3.1522, 2.4624, 1.7630)$, gives $\|DAD^{-1}\| = 0.8$ and thus $x_{k+1} = \mathbf{sat}(Ax_k)$ is stable.

The following code segment solves the problem:

```
A = [-0.2 0.01 -0.002;
      1    0    0;
      0    1    0];

cvx_begin sdp
    variable X(3,3) diagonal
    A'*X*A - X <= -eye(3)
    X >= eye(3)
cvx_end

D = X^(1/2); norm(D*A*inv(D))
```

3. *Finding a stabilizing state feedback via LMIs.* We consider the time-varying LDS

$$\dot{x}(t) = A(t)x(t) + Bu(t),$$

with $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$, where $A(t) \in \{A_1, \ldots, A_M\}$. Thus, the dynamics matrix $A(t)$ can take any of M values, at any time. We seek a linear state feedback gain matrix $K \in \mathbf{R}^{m \times n}$ for which the closed-loop system

$$\dot{x} = (A(t) + BK)x(t),$$

is globally asymptotically stable. But even if you're given a specific state feedback gain matrix K, this is very hard to determine. So we'll require the existence of a quadratic Lyapunov function that establishes exponential stability of the closed-loop system, *i.e.*, a matrix $P = P^T > 0$ for which

$$\dot{V}(z,t) = z^T\left((A(t) + BK)^T P + P(A(t) + BK)\right) z \leq -\beta V(z)$$

for all z, and for any possible value of $A(t)$. (The parameter $\beta > 0$ is given, and sets a minimum decay rate for the closed-loop trajectories.)

So roughly speaking we seek

- a stabilizing state feedback gain, and
- a quadratic Lyapunov function that certifies the closed-loop performance.

In this problem, you will use LMIs to find both K and P, *simultaneously*.

(a) Pose the problem of finding P and K as an LMI problem.

Hint: Starting from the inequality above, you won't get an LMI in the variables P and K (although you'll have a set of matrix inequalities that are affine in K, for fixed P, and linear in P, for fixed K). Use the new variables $X = P^{-1}$ and $Y = KP^{-1}$. Be sure to explain why you can change variables.

(b) Carry out your method for the specific problem instance

$$A_1 = \begin{bmatrix} -0.5 & 0.3 & 0.4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.7 & 0.1 & -0.2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.6 & -0.7 & 0.2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$B = (1, 0, 0)$, and $\beta = 1$. (Thus, we require a closed-loop decay at least as fast as $e^{-t/2}$.)

Solution:

(a) Let $A(t) = A_i$. For the closed loop system $\dot{x}(t) = (A_i + BK)x(t)$, we have

$$\dot{V}(x(t)) = 2x(t)^T P\dot{x}(t) = x(t)^T \left((A_i + BK)^T P + P(A_i + BK)\right) x(t).$$

Therefore the inequality $\dot{V}(x(t)) \leq -\beta V(x(t))$ can be expressed as the matrix inequality

$$(A_i + BK)^T P + P(A_i + BK) \leq -\beta P.$$

By pre and post-multiplying both sides of the above inequality with P^{-1} we get

$$P^{-1}(A_i + BK)^T + (A_i + BK)P^{-1} \leq -\beta P^{-1}.$$

Using the substitution $X = P^{-1}$ and $Y = KP^{-1}$, we can rewite this inequality as

$$XA_i^T + Y^T B^T + A_i X + BY + \beta X \leq 0.$$

This is an LMI with variables X and Y. We seek an X and a Y that satisfy M such LMI's (one for each A_i) together with the constraint that X must be positive definite (since P is), *i.e.*,

$$\begin{aligned} XA_i^T + Y^T B^T + A_i X + BY + X &\leq 0, \quad i = 1, \ldots, M \\ X &> 0 \end{aligned}$$

If we can find an X and a Y that satisfy these conditions, we can recover P and K from

$$P = X^{-1}, \qquad K = YX^{-1}.$$

(b) For the given system it turns out that for

$$P = \begin{bmatrix} 0.0643 & 0.1139 & 0.0726 \\ 0.1139 & 0.3381 & 0.2246 \\ 0.0726 & 0.2246 & 0.2000 \end{bmatrix}, \quad K = \begin{bmatrix} -5.2212 & -10.4396 & -6.6472 \end{bmatrix},$$

we have $\dot{V}(x(t)) \leq -V(x(t)), \ \forall t$. The following matlab code was used to solve this problem:

```
clear

A1 = [-0.5 0.3 0.4;
       1   0   0;
       0   1   0];

A2 = [-0.7 0.1 -0.2;
       1   0   0;
       0   1   0];

A3 = [0.6 -0.7 0.2;
       1   0   0;
       0   1   0];

B = [1;0;0];

cvx_begin sdp
    variable X(3,3) symmetric
    variable Y(1,3)
    X >= eye(3)
    X*A1'+Y'*B'+A1*X+B*Y+X <= 0
    X*A2'+Y'*B'+A2*X+B*Y+X <= 0
    X*A3'+Y'*B'+A3*X+B*Y+X <= 0
cvx_end

P = inv(X)
K = Y*inv(X)
```

4. *Stability of a switching system.* We consider the nonlinear dynamical system

$$\dot{x} = f(x),$$

where

$$f(x) = \begin{cases} A_1 x & x^T G x + b^T x + c > 0 \\ A_2 x & x^T G x + b^T x + c \leq 0. \end{cases}$$

You can assume that $c > 0$. Roughly speaking, the system switches between two linear dynamical systems, depending on the sign of a quadratic function of the state. The function f can be discontinuous, but don't let that worry you.

We seek a positive definite quadratic Lyapunov function $V(x) = x^T P x$ for which $\dot{V}(x) \leq -\beta V(x)$ for all x. (The parameter β is given.)

(a) Explain how to find such a P, or determine that no such P exists, by formulating the problem as an LMI.

(b) Use an LMI solver to find such a P for the specific case

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 33.1 & 30 \\ -36.6 & -32.9 \end{bmatrix}, \quad \beta = 0.1,$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -10 \\ -10 \end{bmatrix}, \quad c = 49.$$

Solution:

(a) We need to find $P > 0$ such that $\beta V(x) + \dot{V}(x) \leq 0$ both when $x^T Gx + b^T x + c \leq 0$ and $x^T Gx + b^T x + c > 0$. If $x^T Gx + b^T x + c > 0$ we have

$$\beta V(x) + \dot{V}(x) = \beta x^T Px + x^T (A_1^T P + PA_1)x,$$

and if $x^T Gx + b^T x + c \leq 0$

$$\beta V(x) + \dot{V}(x) = \beta x^T Px + x^T (A_2^T P + PA_2)x.$$

Therefore using the S-procedure for quadratic functions to impose that

$$x^T Gx + b^T x + c > 0 \Rightarrow \beta x^T Px + x^T (A_1^T P + PA_1)x \leq 0,$$

and

$$x^T Gx + b^T x + c \leq 0 \Rightarrow \beta x^T Px + x^T (A_2^T P + PA_2)x \leq 0,$$

we obtain

$$\begin{bmatrix} -\beta P - A_1 P - PA_1 & 0 \\ 0 & 0 \end{bmatrix} \geq \tau_1 \begin{bmatrix} G & b/2 \\ b^T/2 & c \end{bmatrix}$$

and

$$\begin{bmatrix} -\beta P - A_2 P - PA_2 & 0 \\ 0 & 0 \end{bmatrix} \geq \tau_2 \begin{bmatrix} -G & -b/2 \\ -b^T/2 & -c \end{bmatrix}$$

for $\tau_1 \geq 0$ and $\tau_2 \geq 0$. Equivalently we can write

$$\begin{bmatrix} -\beta P - A_1 P - PA_1 - \tau_1 G & -\tau_1 b/2 \\ -\tau_1 b^T/2 & -\tau_1 c \end{bmatrix} \geq 0$$

and

$$\begin{bmatrix} -\beta P - A_2 P - PA_2 + \tau_2 G & \tau_2 b/2 \\ \tau_2 b^T/2 & +\tau_2 c \end{bmatrix} \geq 0,$$

for $\tau_1 \geq 0$ and $\tau_2 \geq 0$. These are LMIs with variables P, q, τ_1, and τ_2. Since $c > 0$ τ_1 has to be equal to zero, the first LMI simplifies to $\tau_1 = 0$, and

$$\beta P + A_1 P + PA_1 \leq 0,$$

which means that the system $\dot{x} = A_1 x$ has to be stable.

Finally, the LMIs we need to solve are

$$\begin{aligned} P &> 0, \\ \tau &\geq 0, \\ \beta P + A_1 P + P A_1 &\leq 0, \\ \begin{bmatrix} -\beta P - A_2 P - P A_2 + \tau G & \tau b/2 \\ \tau b^T/2 & +\tau c \end{bmatrix} &\geq 0. \end{aligned}$$

(b) Since the equation $\dot{V}(x) \leq -\beta V(x)$ is homogeneous in P, for numerical reasons we can change the strict LMI $P > 0$ to the non-strict LMI $P \geq I$. The following MATLAB code solves these LMIs.

```
clear all
close all

beta=0.1;
A1=[-1 0; 0 -2];
A2=[ 33.1   30
  -36.6  -32.9];

G=[1, 0; 0 ,1];
b=[-10;-10];
c=49;

cvx_begin sdp
    variable P(2,2) symmetric;
    variable tau;

    P>=eye(2);
    tau>=0;
    beta*P+A1'*P+P*A1<=0;
    [ -beta*P-A2'*P-P*A2+ tau*G, tau*b/2;  tau*b'/2, tau*c]>=0;
cvx_end
cvx_status
cvx_optval;

P
tau
```

The output of the code is

```
P =
   11.5501   11.8338
   11.8338   14.2737
```

```
tau =
   48.9608
```

5. *Perron-Frobenius theorem for nonnegative but not regular matrices.* Suppose $A \in \mathbf{R}^{n\times n}$ and is nonnegative, with Perron-Frobenius eigenvalue $\lambda_{\rm pf}$. Show by example that the following can occur:

   (a) The multiplicity of $\lambda_{\rm pf}$ can exceed one.

   (b) The eigenvalue $\lambda_{\rm pf}$ is associated with a Jordan block of size larger than 1×1.

   (c) There are multiple PF eigenvectors, *i.e.*, there are nonzero nonnegative vectors v and $\tilde{v}$, not multiples of each others, such that $Av = \lambda_{\rm pf} v$ and $A\tilde{v} = \lambda_{\rm pf}\tilde{v}$.

   (None of these can occur if A is regular, *i.e.*, $A^k > 0$ for some k.)

   Solution:

   (a) Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (b) Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

   (c) Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

6. *A bound on the Perron-Frobenius eigenvalue.* Let $A \geq 0$, with PF eigenvalue $\lambda_{\rm pf}$. Show that

$$\min_i \sum_j A_{ij} \leq \lambda_{\rm pf} \leq \max_i \sum_j A_{ij},$$

   i.e., $\lambda_{\rm pf}$ lies between the minimum and maximum of all row sums of A. Show that the same holds for the column sums.

   Solution: We'll assume that A is regular. Since

$$\lambda_{\rm pf} = \min_{x>0} \max_i \frac{(Ax)_i}{x_i} = \max_{x>0} \min_i \frac{(Ax)_i}{x_i},$$

   we have

$$\min_i \frac{(Ax)_i}{x_i} \leq \lambda_{\rm pf} \leq \max_i \frac{(Ax)_i}{x_i}$$

   for $\forall x > 0$. Taking $x = \mathbf{1}$ gives the result.

   For the column sums, simply consider A^T and notice that the eigenvalues of A and A^T are the same.

7. *Some relations between a matrix and its absolute value.* In this problem, $A \in \mathbf{R}^{n\times n}$, and $|A|$ denotes the matrix with entries $|A|_{ij} = |A_{ij}|$.

   For each of the following statements, give a proof of the statement or provide a specific counterexample.

(a) If all eigenvalues of $|A|$ have magnitude less than one, then all eigenvalues of A have magnitude less than one.

(b) If all eigenvalues of A have magnitude less than one, then all eigenvalues of $|A|$ have magnitude less than one.

(c) If $\||A|\| < 1$, then $\|A\| < 1$.

(d) If $\|A\| < 1$, then $\||A|\| < 1$.

Solution.

(a) The statement "If all eigenvalues of $|A|$ have magnitude less than one, then all eigenvalues of A have magnitude less than one" is true. To prove it, suppose all eigenvalues of $|A|$ have magnitude less than one. This means that the discrete time system with dynamics matrix $|A|$ is stable, *i.e.*, $|A|^t \to 0$ as $t \to \infty$. Now we observe that for any matrices F and G, we have

$$|FG| \leq |F||G|$$

(these are componentwise inequalities, of course). In particular, we have $|A^2| \leq |A|^2$, and by applying the result many times, we get $|A^t| \leq |A|^t$. Since we know that $|A^t| \to 0$, it follows that $|A^t| \to 0$ which means that A has all eigenvalues with magnitude less than one.

(b) It is false that if all eigenvalues of A have magnitude less than one, then all eigenvalues of $|A|$ have magnitude less than one. To see this, consider

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

which has eigenvalues $1 \pm j$, which have magnitude $\sqrt{2}$. Its absolute value,

$$|A| = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

has eigenvalues 2 and 0. So the matrix $(2/3)A$ has all its eigenvalues with magnitude less than one, but the matrix $(2/3)|A|$ has an eigenvalue with magnitude greater than one.

(c) Let's prove that if $\||A|\| < 1$, then $\|A\| < 1$. To prove this, let v be a right singular vector of A corresponding to its largest singular value, so $\|v\| = 1$ and $\|Av\| = \|A\|$. Using the fact that $\||x|\| = \|x\|$, we have

$$\|A\| = \|Av\| = \|\,|Av|\,\| \leq \|\,|A|\,|v|\,\| \leq \|\,|A|\,\|\,\|\,|v|\,\| \leq \|\,|A|\,\|.$$

(Whew!) This means that $\|A\| \leq \||A|\|$, which proves the given statement.

(d) The statement "If $\|A\| < 1$, then $\||A|\| < 1$" is false. To see this, consider (as above)

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

which has norm (maximum singular value) $\sqrt{2}$. Its absolute value,

$$|A| = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

has norm 2. So the matrix $(2/3)A$ gives a counterexample: it has norm less than one, but its absolute value has norm greater than one.

8. *A weighted maximum Lyapunov function.* Suppose A is nonnegative, regular, and stable, and let v be the PF eigenvector of A, and $\lambda_{\rm pf}$ the PF eigenvalue. Consider the Lyapunov function

$$V(z) = \max_i |z_i/v_i|$$

and the system $x_{t+1} = Ax_t$. Show that along trajectories of this system, V decreases at each step by at least the factor $\lambda_{\rm pf}$.

Solution: We need to show that for any x, $V(Ax) \le \lambda_{\rm pf} V(x)$, *i.e.*,

$$\max_j \left|\frac{(Ax)_j}{v_j}\right| \le \lambda_{\rm pf} \max_i \left|\frac{x_i}{v_i}\right|.$$

Notice that $|x| \le (\max_i |\frac{x_i}{v_i}|)v$, and since A is nonnegative, we have $A|x| \le A((\max_i |\frac{x_i}{v_i}|)v) = \lambda_{\rm pf}(\max_i |\frac{x_i}{v_i}|)v$, *i.e.*. for any j, $\frac{(A|x|)_j}{v_j} \le \lambda_{\rm pf} \max_i |\frac{x_i}{v_i}|$.

It is also true that $|Ax| \le A|x|$. Hence

$$\left|\frac{(Ax)_j}{v_j}\right| \le \frac{(A|x|)_j}{v_j} \le \lambda_{\rm pf} \max_i \left|\frac{x_i}{v_i}\right|,$$

which implies that $\max_j \left|\frac{(Ax)_j}{v_j}\right| \le \lambda_{\rm pf} \max_i |\frac{x_i}{v_i}|$.

9. *Iterative power control with receiver noise.* We consider the power control problem described in the lecture, with one modification: we include a receiver noise term, so the signal to interference plus noise ratio (SINR) is

$$\frac{G_{ii}P_i}{N_i + \sum_{k\neq i} G_{ik}P_k}.$$

Note that by increasing the powers of all transmitters, we can make the effects of the noise on the SINR small, so we can achieve a minimum SINR as close as we like (but not equal) to the optimal SIR when there is no noise.

Now suppose the following iterative power control scheme is used to set the powers: at each step of the iteration, the power P_i is adjusted to that the SINR of receiver i would equal γ, provided the other powers are not changed.

Show that this scheme works, provided $\gamma < 1/\lambda_{\rm pf}$, where $\lambda_{\rm pf}$ is the PF eigenvalue of $\tilde{G}$. ('Works' means the powers converge to a power allocation for which each SINR is equal to γ). You can assume that $G_{ij} > 0$, and that $N_i > 0$.

Solution: We have

$$P(t+1) = \gamma \tilde{G} P(t) + \gamma \tilde{N}$$

where $\tilde{N}_i = N_i/G_{ii}$. Since $\tilde{N} \geq 0$, following the same arguments in the lecture, we know this linear system is stable iff $\gamma < 1/\lambda_{\rm pf}$. The same argument as in the lecture shows that the equilibrium SIRs are all equal to γ.