EE363 homework 6

1. *Constant norm and constant speed systems.* The linear dynamical system $\dot{x} = Ax$ is called *constant norm* if for every trajectory x, $\|x(t)\|$ is constant, *i.e.*, doesn't depend on t. The system is called *constant speed* if for every trajectory x, $\|\dot{x}(t)\|$ is constant, *i.e.*, doesn't depend on t.

   (a) Find the (general) conditions on A under which the system is constant norm.

   (b) Find the (general) conditions on A under which the system is constant speed.

   (c) Is every constant norm system a constant speed system?

   (d) Is every constant speed system a constant norm system?

2. *An iterative method for solving the ARE.* We consider the LQR problem with linear system $\dot{x} = Ax + Bu$, and state and input weight matrices Q and R, with $Q = Q^T > 0$ and $R = R^T > 0$. (The positive definiteness assumption on Q is made for convenience only; implies that (Q, A) is observable.) The optimal input has the form $u(t) = Kx(t)$, where $K = -R^{-1}B^TP$, where P is the unique positive definite solution of the ARE

$$A^TP + PA + Q - PBR^{-1}B^TP = 0.$$

   (a) Show that the ARE is equivalent to the two equations

$$(A + BK)^TP + P(A + BK) + (Q + K^TRK) = 0, \qquad K = -R^{-1}B^TP.$$

   The first equation is a Lyapunov equation (in P).

   (b) Let K_0 be any matrix for which $A + BK_0$ is stable. Let P_0 the solution of the Lyapunov equation above, with $K = K_0$. From P_0, we define $K_1 = -R^{-1}B^TP_0$. Now repeat, *i.e.*, let P_1 be solution of the Lyapunov equation above with $K = K_1$, and so on. Show that $A + BK_i$ are all stable, and P_i are all positive definite.

   Hint. Use induction. To show that $A + BK_i$ is stable, show that

$$(A+BK_i)^TP_{i-1}+P_{i-1}(A+BK_i)+\Big(Q + K_i^TRK_i + (K_i - K_{i-1})^TR(K_i - K_{i-1})\Big) = 0,$$

   and use a Lyapunov theorem.

   (c) Show that $P_{i+1} \leq P_i$. The sequence $P_1, P_2, \ldots$ is nonincreasing and bounded below by 0, so it converges to some limit P. Show that this limit is the solution of ARE.

   (d) Run the algorithm on a numerical example. You can choose A stable, for example as `A=randn(10); A=A-1.1*max(real(eig(A)))*eye(10);`, and use $K_0 = 0$. Plot $\|P_i - P\|$, where P is the solution of the ARE. If you've got it right, a small number of steps (say, 10) should more than suffice.

3. *A Lyapunov condition for attraction.* A set $C \subseteq \mathbf{R}^n$ is said to be *attractive* or an *attractor* for $\dot{x} = f(x)$, if every trajectory eventually ends up in (and stays in) C. More precisely, for any trajectory x, there is a time T (which can depend on the trajectory) such that $x(t) \in C$ for $t \geq T$.

   Note the subtle difference between an invariant set and an attractor. If a trajectory enters an invariant set, it will stay in the set thereafter. For an attractor set, *every* trajectory eventually enters (and then stays).

   Establish the following Lyapunov attractor theorem: Suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$, and constants $a > 0$ and b such that for all z,

$$V(z) \geq b \Longrightarrow \dot{V}(z) \leq -a.$$

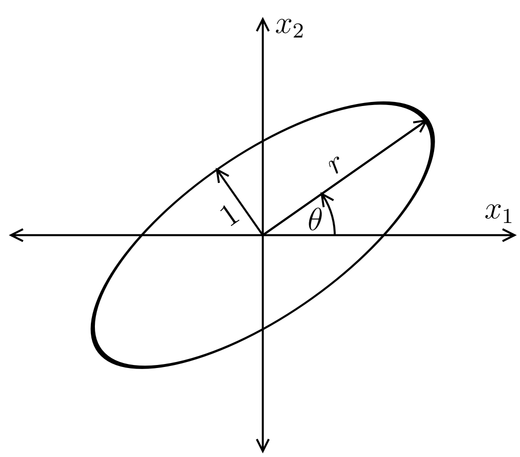
   Then the set $C = \{z \mid V(z) \leq b\}$ is an attractor.

4. *Finding an invariant ellipsoid for a linear system.* Consider the linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   Recall that an ellipsoid $\mathcal{E}$ is said to be invariant for this system if all trajectories that start in $\mathcal{E}$ stay in $\mathcal{E}$, *i.e.*, $x(0) \in \mathcal{E}$ implies that $x(t) \in \mathcal{E}$ for all $t \geq 0$.

   You will find an invariant ellipsoid for this system. We will describe the ellipsoid by the length $r \geq 1$ of its major semi-axis (the length of the minor semi-axis is set to one) and the angle θ of the major semi-axis with respect to the x_1-axis, as shown in the figure below.



   (a) First present a general description of how you will go about finding r and θ, briefly justifying each step.

   (b) Carry out the individual steps in your description from part (a) to find specific values of r and θ.

5. *Global asymptotic stability for a system with small nonlinearity.* We consider the system $\dot{x} = Ax + q(x)$, where $x(t) \in \mathbf{R}^n$ and $q : \mathbf{R}^n \to \mathbf{R}^n$. We assume that q satisfies $\|q(z)\| \leq \alpha \|z\|$ for all z, but otherwise is unknown. We assume that A is stable, *i.e.*, all its eigenvalues have negative real part.

   Intuition suggests that for α small, the system is globally asymptotically stable. Show that this intuition is correct, by finding a positive number $\bar{\alpha}$ such that for $\alpha \leq \bar{\alpha}$, you can guarantee global asymptotic stability of the system. The number $\bar{\alpha}$ must be explicit, *i.e.*, it should be easily calculated using standard matrix operations (such as computing eigenvalues and singular values, solving Lyapunov or Riccati equations, etc.), and the problem data.

6. *Stability analysis of system with intermitent failures.* We consider the system $\dot{x} = A(t)x$, where $A(t) = A_{\text{nom}} \in \mathbf{R}^{n \times n}$ when system is working (*i.e.*, in *nominal mode*) and $A(t) = A_{\text{fail}} \in \mathbf{R}^{n \times n}$ when system is not working (*i.e.*, in *failure mode*). The nominal system is stable, *i.e.*, $\dot{z} = A_{\text{nom}} z$ is stable.

   We suppose that the system fails in various failure episodes, which are the intervals of time during which $A(t) = A_{\text{fail}}$. We assume that no failure episode can last longer than T_1 seconds. (A failure can, however, last shorter than T_1 seconds.) We also assume that once a failure has finished, no new failure occurs for at least T_2 seconds. In other words, T_2 gives the minimum time between two successive failure episodes, *i.e.*, a minimum time between failures. (Again, it is possible that the time between two successive failures exceeds T_2.) To simplify things, you can assume that the first failure does not occur until at least time T_2, *i.e.*, the system starts off at $t = 0$ in a working period, that is at least T_2 seconds long. We let $\alpha = T_1/(T_1 + T_2)$, which is an upper bound on the fraction of time the system can be in failure mode.

   Intuition suggests that if the failures occur for only a small fraction of time, then the system should still be globally asymptotically stable, *i.e.*, if α is small enough, the system is globally asymptotically stable. The goal in this problem is to verify and quantify this statement.

   Find a positive number $\bar{\alpha}$ such that for $\alpha \leq \bar{\alpha}$, you can guarantee global asymptotic stability of the system. The number $\bar{\alpha}$ must be explicit, *i.e.*, it should be easily calculated using standard matrix operations (such as computing eigenvalues and singular values, solving Lyapunov or Riccati equations, etc.), starting from the problem data A_{nom}, A_{fail}, T_1, T_2, and α.

   Remark: to save you some trouble, we should point out a common misperception. Many people assume that the 'worst' sequence of failure events is to have the system fail for the maximum possible time, *i.e.*, T_1, then work normally for the smallest possible time, *i.e.*, T_2, and then repeat this pattern. *This assumption is false.*