EE363 homework 3

1. *Solution of a two-point boundary value problem.* We consider a linear dynamical system $\dot{x} = Ax$, with $x(t) \in \mathbf{R}^n$. There is an n-dimensional subspace of solutions of this equation, so to single out one of the trajectories we can impose, roughly speaking, n equations.

   In the most common situation, we specify $x(0) = x_0$, in which case the unique solution is $x(t) = e^{tA}x_0$. This is called an *initial value problem* since we specify the initial value of the state. In a *final value problem*, we specify the final state: $x(T) = x_f$. In this case the unique solution is $x(t) = e^{(t-T)A}x_f$.

   In a *two-point boundary value problem* we impose conditions on the initial and final states.

   (a) Find the solution to the two-point boundary value problem

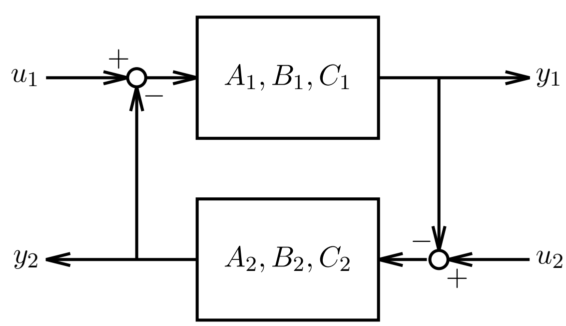
   $$\dot{x} = Ax, \qquad Fx(0) + Gx(T) = h.$$

   Here $F,\ G \in \mathbf{R}^{n\times n}$, and $h \in \mathbf{R}^n$. Express your answer in terms of A, F, G, and h. Your answer can contain a matrix exponential.

   What condition must hold to ensure that there is a unique solution to this equation?

   (b) Express the two-point boundary value problem that arises in the continuous time LQR problem (*i.e.*, with the Hamiltonian system) in the form given above, and then find the solution to this boundary value problem. (You may leave matrix exponentials in your solution.) How is the optimal input u obtained from this solution?

2. *Controllability of a feedback connection.* Consider the feedback connection of two linear dynamical systems:



   (a) Write state (linear dynamical system) equations with state $x = (x_1, x_2)$, input $u = (u_1, u_2)$, and output $y = (y_1, y_2)$.

(b) *Proof or counterexample:* the feedback system is observable (controllable) if and only if both subsystems are observable (controllable).

(c) Fix $u_2 = 0$ (*i.e.*, u_2 is not an input), and repeat part (b).

3. You know that $\mathcal{R}(\mathcal{C})$ is A-invariant, where $\mathcal{C} = [B\ AB\ \cdots\ A^{n-1}B]$ is the controllability matrix. Find a matrix X such that $A\mathcal{C} = \mathcal{C}X$.

4. *A matrix criterion for A-invariance of a nullspace.* We saw in lecture that $\mathcal{R}(M)$ is A-invariant if and only if there exists a matrix X such that $AM = MX$. In this problem you will derive a similiar condition for the nullspace of a matrix.

   (a) Show that $\mathcal{N}(N)$ is A-invariant if and only if $\mathcal{R}(N^T)$ is A^T-invariant.

   (b) Finish the statement "$\mathcal{N}(N)$ is A-invariant if and only if ...".

5. *Complex eigenvalues and invariant planes.* Let $A \in \mathbf{R}^{n\times n}$ satisfy $Av = (\alpha + j\beta)v$, where $v \in \mathbf{C}^n$ is nonzero, and $\beta \neq 0$. (In other words, v is an eigenvector of A corresponding to an eigenvalue that is not real.)

   (a) Show that $\mathrm{span}\{\Re v, \Im v\}$ is an A-invariant subspace.

   (b) With $M = [\Re v\ \Im v]$, find $X \in \mathbf{R}^{2\times 2}$ such that $AM = MX$.

6. *Stochastic LQR for a supply chain system with manufacturing delay.* We let $s_t \in \mathbf{R}$, $t = 1, 2, \ldots$, denote the stock level of some product available at the beginning of period t. This is measured with respect to some nominal value, so negative values just mean our stock is below the nominal value. We let o_t, $t = 1, 2, \ldots$, denote the amount of product ordered (say, from a factory) at the beginning of period t. The demand for the product in period t will be denoted d_t. The amount ordered and demand are also relative to some nominal values, so, *e.g.*, negative d_t just means that in period t there is less than the nominal demand for the product. We assume that the demands d_t are IID with zero mean and variance σ^2.

   When the order o_t is placed, you can assume that the current and previous stock levels $s_t, s_{t-1}, \ldots$ are known, as are the previous orders $o_{t-1}, o_{t-2}, \ldots$, and the previous demand levels $d_{t-1}, d_{t-2}, \ldots$. But the current period demand d_t is not known when the order o_t is placed.

   The stock level propagates as

   $$s_{t+1} = s_t + o_{t-D+1} - d_t,$$

   where D (a nonnegative integer) gives the delay between placing an order and receiving the product. For example, with $D = 3$, product ordered in period t arrives in stock in period $t + D$. In the equation above, we interpret o_t as zero for $t \leq 0$.

The objective J, which is to be minimized, is

$$J = \lim_{T\to\infty} \frac{1}{T} \mathbf{E} \left(\sum_{t=1}^{T} s_t^2 + \rho o_t^2 \right),$$

where $\rho > 0$. We can interpret J as the mean-square deviation of the stock level from the nominal value, plus ρ times the mean-square deviation of the orders from the nominal value.

(a) Explain how to find an optimal ordering policy. We do not need you to find an analytical solution: Your method can be computational, *e.g.*, involve solution of an ARE. Be sure to say what form the optimal ordering policy has. (For example, is it a linear function of s_t?)

Hint. Define a state that allows you to formulate the problem as a standard stochastic LQR problem.

(b) Find an optimal ordering policy for $\sigma = 1$, $\rho = 1$ and $D = 3$, and give the optimal mean-square deviation $J^\star$.

(c) Simulate the closed-loop system, using the optimal ordering policy, for 15000 steps, then discard the first 5000 steps, to get rid of the initial transient effect. Plot a histogram of the stage cost $s_t^2 + \rho o_t^2$, and find the (empirical) mean-square value, comparing to the (exact) value of $J^\star$. Plot a trace of s_t, o_t, and d_t over (say) 50 steps.