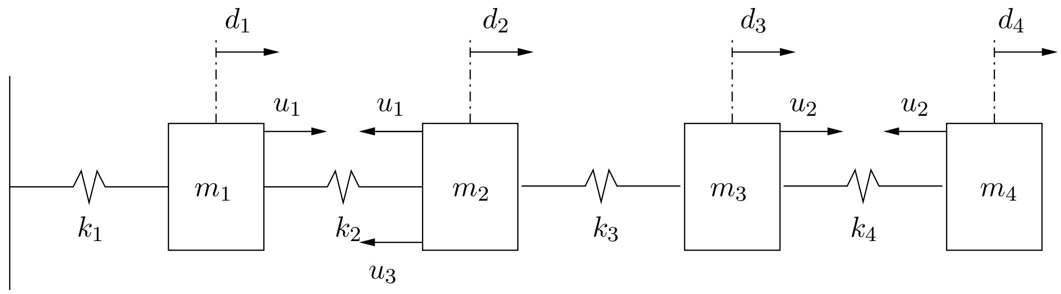
EE363 homework 2

1. *Derivative of matrix inverse.* Suppose that $X : \mathbf{R} \to \mathbf{R}^{n\times n}$, and that $X(t)$ is invertible. Show that
$$\frac{d}{dt}X(t)^{-1} = -X(t)^{-1}\left(\frac{d}{dt}X(t)\right)X(t)^{-1}.$$
Hint: differentiate $X(t)X(t)^{-1} = I$ with respect to t.

2. *Infinite horizon LQR for a periodic system.* Consider the system $x_{t+1} = A_t x_t + B_t u_t$, where
$$A_t = \begin{cases} A^{\mathrm{e}} & t \text{ even} \\ A^{\mathrm{o}} & t \text{ odd} \end{cases} \qquad B_t = \begin{cases} B^{\mathrm{e}} & t \text{ even} \\ B^{\mathrm{o}} & t \text{ odd} \end{cases}$$
In other words, A and B are periodic with period 2. We consider the infinite horizon LQR problem for this time-varying system, with cost
$$J = \sum_{\tau=0}^{\infty}\left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau\right).$$
In this problem you will use dynamic programming to find the optimal control for this system. You can assume that the value function is finite.
   (a) Conjecture a reasonable form for the value function. You do not have to show that your form is correct.
   (b) Derive the Hamilton-Jacobi equation, using your assumed form. *Hint:* you should get a pair of coupled nonlinear matrix equations.
   (c) Suggest a simple iterative method for solving the Hamilton-Jacobi equation. You do not have to prove that the iterative method converges, but do check your method on a few numerical examples.
   (d) Show that the Hamilton-Jacobi equation can be solved by solving a single (bigger) algebraic Riccati equation. How is the optimal u related to the solution of this equation?

   Remark: the results of this problem generalize to general periodic systems.

3. *LQR for a simple mechanical system.* Consider the mechanical system shown below:



Here $d_1, \ldots, d_4$ are displacements from an equilibrium position, and $u_1, \ldots, u_3$ are forces acting on the masses. Note that u_1 is a tension between the first and second masses, u_2 is a tension between the third and fourth masses, and u_3 is a force between the wall (at left) and the second mass. You can take the mass and stiffness constants to all be one: $m_1 = \cdots = m_4 = 1$, $k_1 = \cdots = k_4 = 1$.

(a) Describe the system as a linear dynamical system with state $(d, \dot{d}) \in \mathbf{R}^8$.

(b) Using the cost function

$$J = \int_0^\infty \left(\|d(t)\|^2 + \|u(t)\|^2 \right) \, dt,$$

find the optimal state feedback gain matrix K. You may find the Matlab function `lqr()` useful, but check sign conventions: it's not unusual for the optimal feedback gain to be defined as $u = -Kx$ instead of $u = Kx$ (which is what we use).

(c) Plot $d(t)$ versus t for the open loop ($u(t) = 0$) and closed loop ($u(t) = Kx(t)$) cases using an arbitrary initial condition (but not, of course, zero).

(d) Solve the ARE for this problem using the method based on the Hamiltonian, described in lecture 4. Verify that you get the same result as you did using the Matlab function `lqr()`.

4. *Hamiltonian matrices.* A matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

with $M_{ij} \in \mathbf{R}^{n \times n}$ is *Hamiltonian* if JM is symmetric, or equivalently

$$JMJ = M^T,$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

(a) Show that M is Hamiltonian if and only if $M_{22} = -M_{11}^T$ and M_{21} and M_{12} are each symmetric.

(b) Show that if v is an eigenvector of M, then Jv is an eigenvector of M^T.

(c) Show that if λ is an eigenvalue of M, then so is $-\lambda$.

(d) Show that $\det(sI - M) = \det(-sI - M)$, so that $\det(sI - M)$ is a polynomial in s^2.

5. *Value function for infinite-horizon LQR problem.* In this problem you will show that the minimum cost-to-go starting in state z is a quadratic form in z. Let $Q = Q^T \geq 0$, $R = R^T > 0$ and define

$$J(u, z) = \sum_{t=0}^{\infty} \left(x_t^T Q x_t + u_t^T R u_t \right)$$

where $x_{t+1} = Ax_t + Bu_t$, $x_0 = z$. Note that u is a sequence in $\mathbf{R}^m$, and $z \in \mathbf{R}^n$. Of course, for some u's and z's, $J(u,z) = \infty$. Define

$$V(z) = \min_u J(u,z)$$

Note that this is a minimum over all possible input sequences. This is just like the previous problem, except that here u has infinite dimension. Assume (A,B) is controllable, so $V(z) < \infty$ for all z.

(a) Show that for all $\lambda \in \mathbf{R}$, $J(\lambda u, \lambda z) = \lambda^2 J(u,z)$, and conclude that

$$V(\lambda z) = \lambda^2 V(z). \tag{1}$$

(b) Let u and $\tilde{u}$ be two input sequences, and let z and $\tilde{z}$ be two initial states. Show that

$$J(u+\tilde{u}, z+\tilde{z}) + J(u-\tilde{u}, z-\tilde{z}) = 2J(u,z) + 2J(\tilde{u},\tilde{z})$$

Minimize the RHS with respect to u and $\tilde{u}$, and conclude

$$V(z+\tilde{z}) + V(z-\tilde{z}) \leq 2V(z) + 2V(\tilde{z})$$

(c) Apply the above inequality with $\frac{1}{2}(z+\tilde{z})$ substituted for z and $\frac{1}{2}(z-\tilde{z})$ substituted for $\tilde{z}$ to get:

$$V(z+\tilde{z}) + V(z-\tilde{z}) = 2V(z) + 2V(\tilde{z}) \tag{2}$$

(d) The two properties (1) and (2) of V are enough to guarantee that V is a quadratic form. Here is one way to see it (you supply all details): take gradients in (2) with respect to z and $\tilde{z}$ and add to get

$$\nabla V(z+\tilde{z}) = \nabla V(z) + \nabla V(\tilde{z}) \tag{3}$$

From (1) show that

$$\nabla V(\lambda z) = \lambda \nabla V(z) \tag{4}$$

(3) and (4) mean that $\nabla V(z)$, which is a vector, is linear in z, and hence has a matrix representation:

$$\nabla V(z) = Mz \tag{5}$$

where $M \in \mathbf{R}^{n\times n}$.

(e) Show that $V(z) = z^T P z$, where $P = \frac{1}{4}(M + M^T)$. Show that $P = P^T \geq 0$. Thus we are done. *Hint*

$$V(z) = V(0) + \int_0^1 \nabla V(\theta z)^T z \, d\theta.$$

6. *Closed-loop stability for receding horizon LQR.* We consider the system $x_{t+1} = Ax_t + Bu_t$, $y_t = Cx_t$, with

$$A = \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ -0.6 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & -0.6 \\ 0 & 0 & 0.4 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the receding horizon LQR control problem with horizon T, using state cost matrix $Q = C^T C$ and control cost $R = 1$. For each T, the receding horizon control has linear state feedback form $u_t = K_T x_t$. The associated closed-loop system has the form $x_{t+1} = (A + BK_T)x_t$. What is the smallest horizon T for which the closed-loop system is stable?

Interpretation. As you increase T, receding horizon control becomes less myopic and greedy; it takes into account the effects of current actions on the long-term system behavior. If the horizon is too short, the actions taken can result in an unstable closed-loop system.

7. *LQR with exponential weighting.* A common variation on the LQR problem includes explicit time-varying weighting factors on the state and input costs,

$$J = \sum_{\tau=0}^{N-1} \gamma^\tau \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) + \gamma^N x_N^T Q_f x_N,$$

where $x_{t+1} = Ax_t + Bu_t$, x_0 is given, and, as usual, we assume $Q = Q^T \geq 0$, $Q_f = Q_f^T \geq 0$ and $R = R^T > 0$ are constant. The parameter γ, called the exponential weighting factor, is positive. For $\gamma = 1$, this reduces to the standard LQR cost function. For $\gamma < 1$, the penalty for future state and input deviations is smaller than in the present; in this case we call γ the *discount factor* or *forgetting factor*. When $\gamma > 1$, future costs are accentuated compared to present costs. This gives added incentive for the input to steer the state towards zero quickly.

   (a) Note that we can find the input sequence $u_0^*, \ldots, u_{N-1}^*$ that minimizes J using standard LQR methods, by considering the state and input costs as time varying, with $Q_t = \gamma^t Q$, $R_t = \gamma^t R$, and final cost given by $\gamma^N Q_f$. Thus, we know at least one way to solve the exponentially weighted LQR problem. Use this method to find the recursive equations that give u^*.

   (b) Exponential weights can also be incorporated directly into a dynamic programming formulation. We define

$$W_t(z) = \min \sum_{\tau=t}^{N-1} \gamma^{\tau-t} \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) + \gamma^{N-t} x_N^T Q_f x_N,$$

where $x_t = z$, $x_{\tau+1} = Ax_\tau + Bu_\tau$, and the minimum is over $u_t, \ldots, u_{N-1}$. This is the minimum cost-to-go, if we started in state z at time t, with the time weighting also starting at t. Argue that we have

$$W_N(z) = x_N^T Q_f x_N,$$

$$W_t(z) = \min_w \left(z^T Q z + w^T R w + \gamma W_{t+1}(Az + Bw) \right),$$

and that the minimizing w is in fact u_t^*. In other words, work out a backwards recursion for W_t, and give an expression for $u^*(t)$ in terms of W_t. Show that this method yields the same u^* as the first method.

(c) Yet another method can be used to find u^*. Define a new system as

$$y_{t+1} = \gamma^{1/2} A y_t + \gamma^{1/2} B z_t, \qquad y_0 = x_0.$$

Argue that we have $y_t = \gamma^{t/2} x_t$, provided $z_t = \gamma^{t/2} u_t$, for $t = 0, \ldots, N-1$. With this choice of z, the exponentially weighted LQR cost J for the original system is given by

$$J = \sum_{\tau=0}^{N-1} \left(y_\tau^T Q y_\tau + z_\tau^T R z_\tau \right) + y_N^T Q_f y_N,$$

i.e., the unweighted LQR cost for the modified system. We can use the standard formulas to obtain the optimal input for the modified system z^*, and from this, we can get u^*. Do this, and verify that once again, you get the same u^*.