# EE363 homework 1

1. *LQR for a triple accumulator.* We consider the system $x_{t+1} = Ax_t + Bu_t$, $y_t = Cx_t$, with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

This system has transfer function $H(z) = (z-1)^{-3}$, and is called a triple accumulator, since it consists of a cascade of three accumulators. (An accumulator is the discrete-time analog of an integrator: its output is the running sum of its input.) We'll use the LQR cost function

$$J = \sum_{t=0}^{N-1} u_t^2 + \sum_{t=0}^{N} y_t^2,$$

with $N = 50$.

(a) Find $P_t$ (numerically), and verify that the Riccati recursion converges to a steady-state value in fewer than about 10 steps. Find the optimal time-varying state feedback gain $K_t$, and plot its components $(K_t)_{11}$, $(K_t)_{12}$, and $(K_t)_{13}$, versus $t$.

(b) Find the initial condition $x_0$, with norm not exceeding one, that maximizes the optimal value of $J$. Plot the optimal $u$ and resulting $x$ for this initial condition.

2. *Linear quadratic state tracking.* We consider the system $x_{t+1} = Ax_t + Bu_t$. In the conventional LQR problem the goal is to make both the state and the input small. In this problem we study a generalization in which we want the state to follow a desired (possibly nonzero) trajectory as closely as possible. To do this we penalize the *deviations* of the state from the desired trajectory, *i.e.*, $x_t - x_t^d$, using the following cost function:

$$J = \sum_{\tau=0}^{N} (x_\tau - x_\tau^d)^T Q (x_\tau - x_\tau^d) + \sum_{\tau=0}^{N-1} u_\tau^T R u_\tau,$$

where we assume $Q = Q^T \geq 0$ and $R = R^T > 0$. (The desired trajectory $x_\tau^d$ is given.) Compared with the standard LQR objective, we have an extra linear term (in $x$) and a constant term.

In this problem you will use dynamic programming to show that the cost-to-go function $V_t(z)$ for this problem has the form

$$z^T P_t z + 2q_t^T z + r_t,$$

with $P_t = P_t^T \geq 0$. (*i.e.*, it has quadratic, linear, and constant terms.)

(a) Show that $V_N(z)$ has the given form.

(b) Assuming $V_{t+1}(z)$ has the given form, show that the optimal input at time $t$ can be written as

$$u_t^\star = K_t x_t + g_t,$$

where

$$K_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A, \quad g_t = -(R + B^T P_{t+1} B)^{-1} B^T q_{t+1}.$$

In other words, $u_t^\star$ is an affine (linear plus constant) function of the state $x_t$.

(c) Use backward induction to show that $V_0(z), \ldots, V_N(z)$ all have the given form. Verify that

$$\begin{aligned} P_t &= Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A, \\ q_t &= (A + BK_t)^T q_{t+1} - Q x_t^d, \\ r_t &= r_{t+1} + x_t^d Q x_t^d + q_{t+1}^T B g_t, \end{aligned}$$

for $t = 0, \ldots, N-1$.

3. *The Schur complement.* In this problem you will show that if we minimize a positive semidefinite quadratic form over *some* of its variables, the result is a positive semidefinite quadratic form in the *remaining* variables. Specifically, let

$$J(u, z) = \begin{bmatrix} u \\ z \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} u \\ z \end{bmatrix}$$

be a positive semidefinite quadratic form in $u$ and $z$. You may assume $Q_{11} > 0$ and $Q_{11}, Q_{22}$ are symmetric. Define $V(z) = \min_u J(u, z)$. Show that $V(z) = z^T P z$, where $P$ is symmetric positive semidefinite (find $P$ explicitly).

The matrix $P$ is called the *Schur complement* of the matrix $Q_{11}$ in the big matrix above. It comes up in many contexts.

4. *A useful determinant identity.* Suppose $X \in \mathbf{R}^{n \times m}$ and $Y \in \mathbf{R}^{m \times n}$.

(a) Show that $\det(I + XY) = \det(I + YX)$. *Hint:* Find a block lower triangular matrix $L$ for which

$$\begin{bmatrix} I & X \\ -Y & I \end{bmatrix} = L \begin{bmatrix} I & X \\ 0 & I \end{bmatrix},$$

and use this factorization to evaluate the determinant of this matrix. Then find a block upper triangular matrix $U$ for which

$$\begin{bmatrix} I & X \\ -Y & I \end{bmatrix} = U \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix},$$

and repeat.

(b) Show that the nonzero eigenvalues of $XY$ and $YX$ are exactly the same.

5. *When does a finite-horizon LQR problem have a time-invariant optimal state feedback gain?* Consider a discrete-time LQR problem with horizon $t = N$, with optimal input $u(t) = K_t x(t)$. Is there a choice of $Q_f$ (symmetric and positive semidefinite, of course) for which $K_t$ is constant, *i.e.*, $K_0 = \cdots = K_{N-1}$?