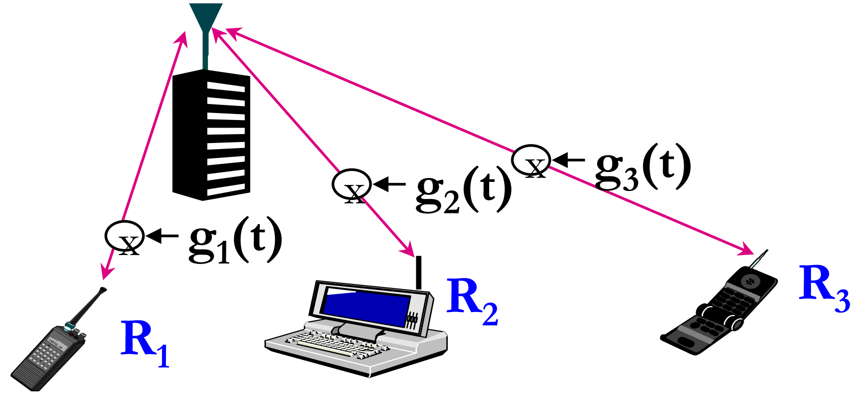
Lecture 3 Outline

- Announcements
  - Makeup lecture Feb 2, 5-6:15.
  - Presentation schedule will be sent out later today, presentations will start 1/30.
  - Next lecture: Random/Multiple Access, SS, MUD
- Capacity of Broadcast ISI Channels
- Capacity of MAC Channels
  - In AWGN
  - In Fading and ISI
- Duality between the MAC and the BC
- Capacity of MIMO Multiuser Channels

Review of Last Lecture

Broadcast:
One Transmitter
to Many Receivers.

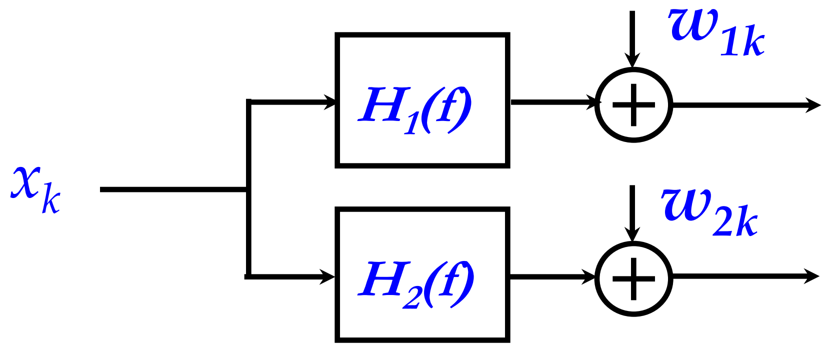


- Channel capacity region of broadcast channels
- Capacity in AWGN
  - Use superposition coding and optimal power allocation
- Capacity in fading
  - Ergodic capacity: optimally allocate resources over time
  - Outage capacity: maintain fixed rates in all states
  - Minimum rate capacity: fixed min. rate in all states, use excess rsources to optimize average rate above min.

Broadcast Channels with ISI

- ISI introduces memory into the channel
- The optimal coding strategy decomposes the channel into parallel broadcast channels
  - Superposition coding is applied to each subchannel.
- Power must be optimized across subchannels and between users in each subchannel.

Broadcast Channel Model

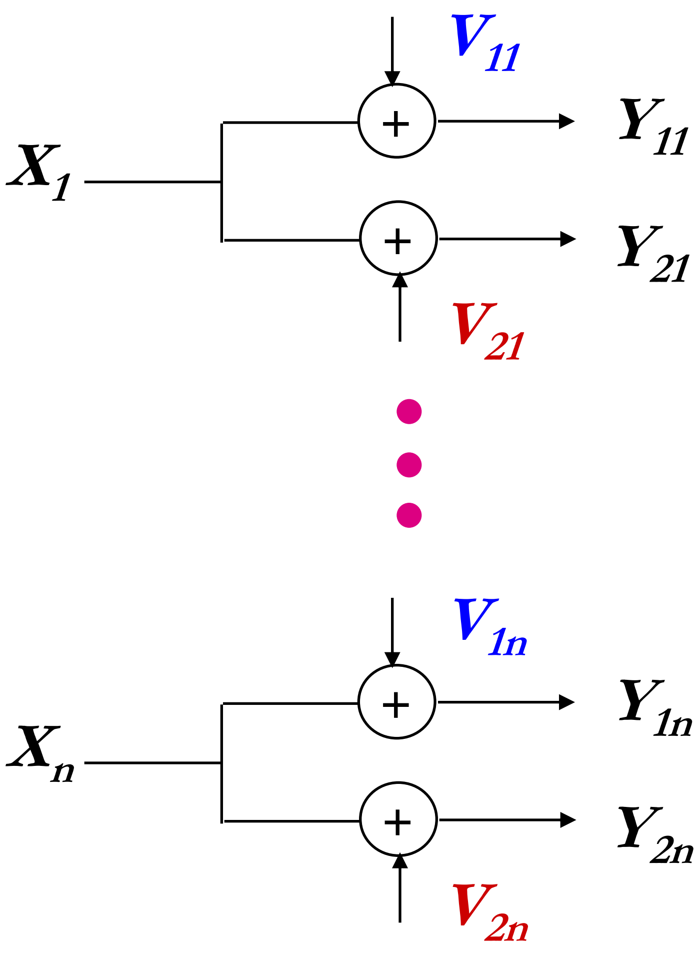


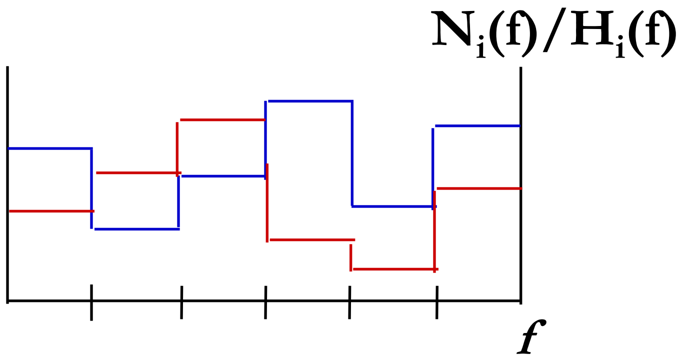
$$y_{1k} = \sum_{i=1}^{m} h_{1i} x_{k-i} + w_{1k}$$

$$y_{2k} = \sum_{i=1}^{m} h_{2i} x_{k-i} + w_{2k}$$

- Both H_1 and H_2 are finite IR filters of length m.
- The w_{1k} and w_{2k} are correlated noise samples.
- For $1<k\leq n$, we call this channel the n-block discrete Gaussian broadcast channel (n-DGBC).
- The channel capacity region is $C=(R_1,R_2)$.

Equivalent Parallel Channel Model





Channel Decomposition

- Via a DFT, the BC with ISI approximately decomposes into *n* parallel AWGN degraded broadcast channels.
  - As *n* goes to infinity, this parallel model becomes exact
- The capacity region of parallel degraded broadcast channels was obtained by El-Gamal (1980)
  - Optimal power allocation obtained by Hughes-Hartogs('75).
- The power constraint $\sum_{i=0}^{n-1} E[x_i^2] \le nP$ on the original channel is converted by Parseval's theorem to $\sum_{i=0}^{n-1} E[(X_i')^2] \le n^2 P$ on the equivalent channel.

Capacity Region of Parallel Set

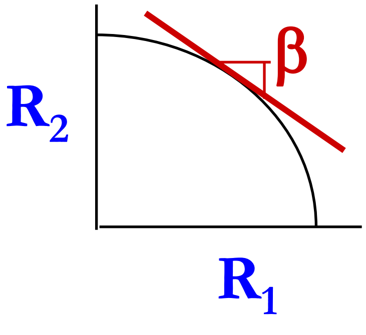
- Achievable Rates (no common information)

$$\left\{R_1 \le .5 \sum_{j:\sigma_{1j}<\sigma_{2j}} \log\left(1+\frac{\alpha_j P_j}{\sigma_{1j}}\right) + .5 \sum_{j:\sigma_{1j}\ge\sigma_{2j}} \log\left(1+\frac{\alpha_j P_j}{(1-\alpha_j)P_j+\sigma_{1j}}\right),\right.$$

$$R_2 \le .5 \sum_{j:\sigma_{1j}<\sigma_{2j}} \log\left(1+\frac{(1-\alpha_j)P_j}{\alpha_j P_j+\sigma_{2j}}\right) + .5 \sum_{j:\sigma_{1j}\ge\sigma_{2j}} \log\left(1+\frac{(1-\alpha_j)P_j}{\sigma_{2j}}\right),$$

$$\left. 0 \le \alpha_j \le 1, \sum P_j \le n^2 P \right\}$$

- Capacity Region
  - For $0<\beta\le\infty$ find $\{\alpha_j\}$, $\{P_j\}$ to maximize $R_1+\beta R_2+\lambda \sum P_j$.
  - Let $(R_1^*,R_2^*)_{n,\beta}$ denote the corresponding rate pair.
  - $C_n=\{(R_1^*,R_2^*)_{n,\beta} : 0<\beta \le\infty \}$, $C=\liminf_n \frac{1}{n} C_n$.



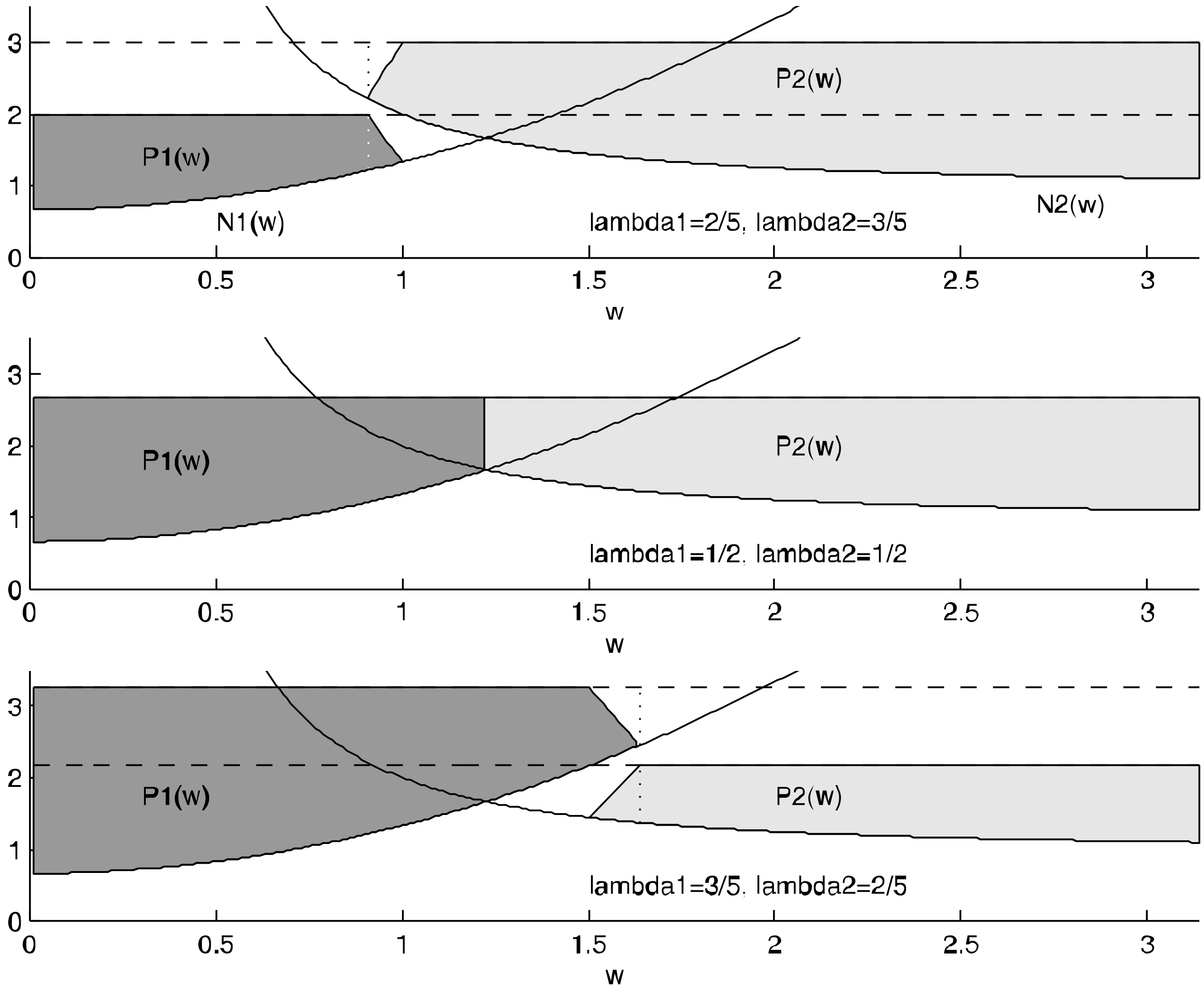
Limiting Capacity Region

$$\Big\{ \boldsymbol{R}_1 \le .5 \int_{f:\boldsymbol{H}_1(f)>\boldsymbol{H}_2(f)} \log\left(1+\frac{\alpha(f)\boldsymbol{P}(f)|\boldsymbol{H}_1(f)|^2}{.5\boldsymbol{N}_0}\right) + .5 \int_{f:\boldsymbol{H}_1(f)\le\boldsymbol{H}_2(f)} \log\left(1+\frac{\alpha_j \boldsymbol{P}_j}{(1-\alpha_j)\boldsymbol{P}_j+\sigma_{1j}}\right),$$

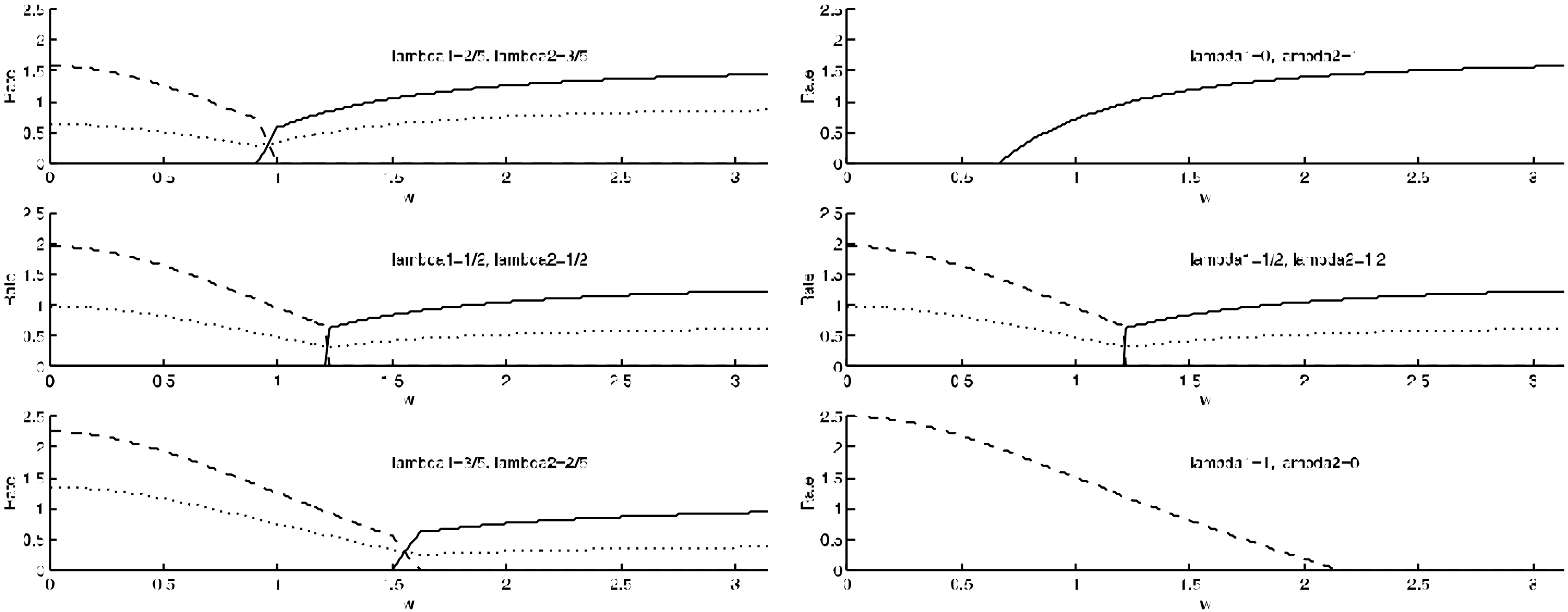
$$\boldsymbol{R}_2 \le .5 \int_{f:\boldsymbol{H}_1(f)>\boldsymbol{H}_2(f)} \log\left(1+\frac{(1-\alpha(f))\boldsymbol{P}(f)}{\alpha(f)\boldsymbol{P}(f)+.5\boldsymbol{N}_0/|\boldsymbol{H}_2(f)|^2}\right) + .5 \int_{f:\boldsymbol{H}_1(f)\le\boldsymbol{H}_2(f)} \log\left(1+\frac{(1-\alpha(f))\boldsymbol{P}(f)|\boldsymbol{H}_2(f)|^2}{.5\boldsymbol{N}_0}\right),$$

$$0 \le \alpha(f) \le 1, \qquad \int \boldsymbol{P}(f)\,df \le \boldsymbol{P} \Big\}$$

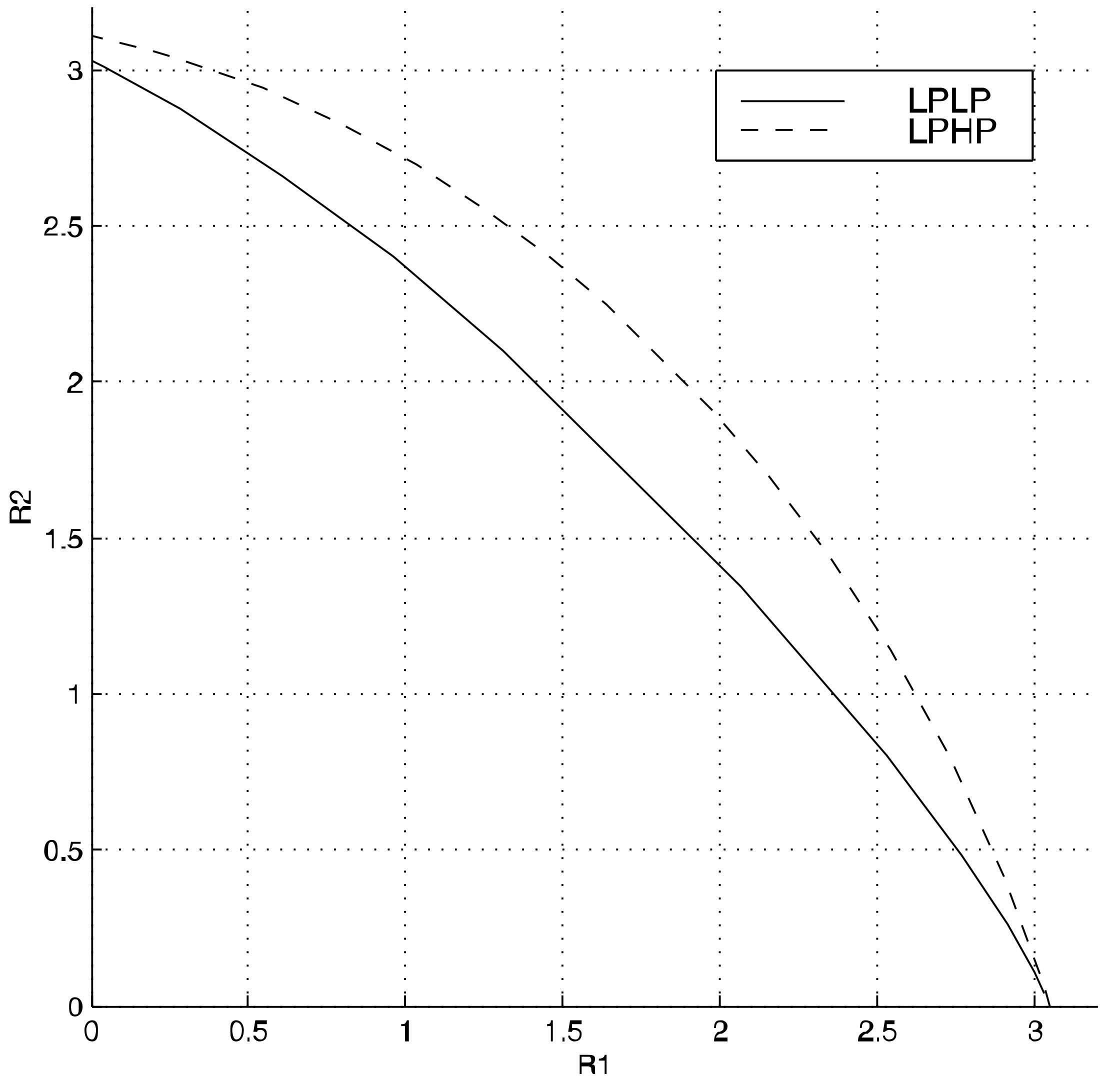
Optimal Power Allocation: Two Level Water Filling



Capacity vs. Frequency



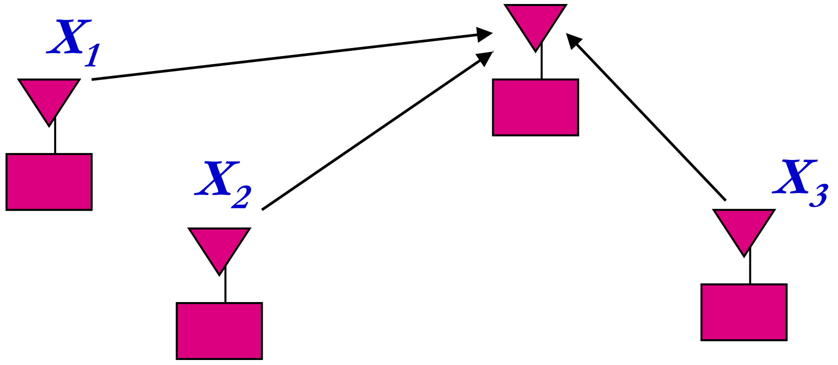
Capacity Region



Multiple Access Channel

- Multiple transmitters
  - Transmitter i sends signal X_i with power P_i
- Common receiver with AWGN of power N_0B
- Received signal:

$$Y = \sum_{i=1}^{M} X_i + N$$



MAC Capacity Region

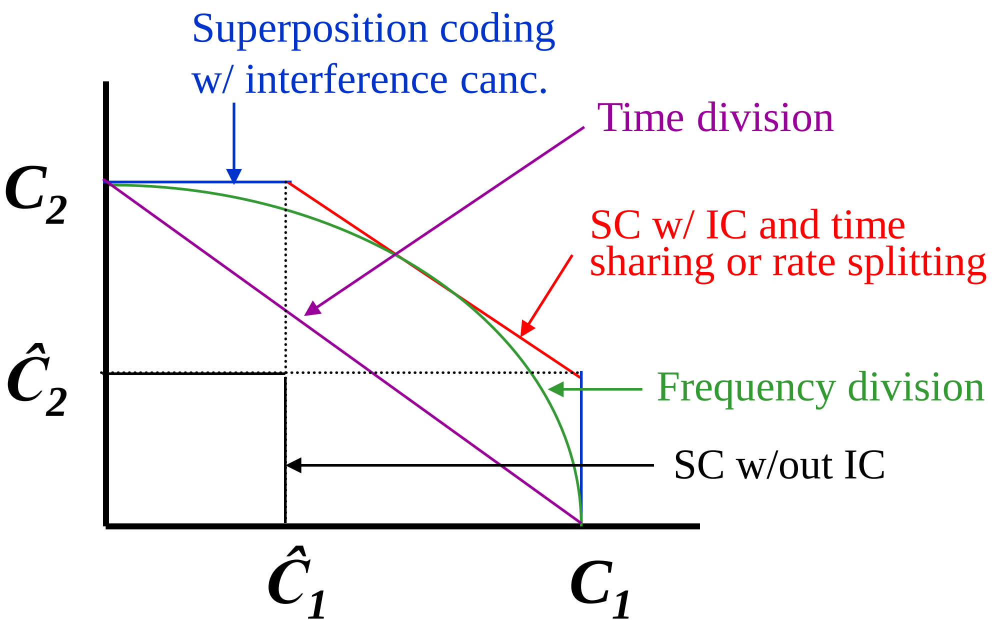
- **Closed convex hull of all ($R_1, \ldots, R_M$) s.t.**

$$\sum_{i \in S} R_i \le B \log\left[1 + \sum_{i \in S} P_i / N_0 B\right], \quad \forall S \subseteq \{1, \ldots, M\}$$

  - **For all subsets of users, rate sum equals that of 1 superuser with sum of powers from all users**

- **Power Allocation and Decoding Order**
  - **Each user has its own power (no power alloc.)**
  - **Decoding order depends on desired rate point**

Two-User Region



$$C_i = B\log\left[1+\frac{P_i}{N_0 B}\right], i = 1,2$$

$$\hat{C}_1 = B\log\left[1+\frac{P_1}{N_0 B + P_2}\right], \qquad \hat{C}_2 = B\log\left[1+\frac{P_2}{N_0 B + P_1}\right],$$

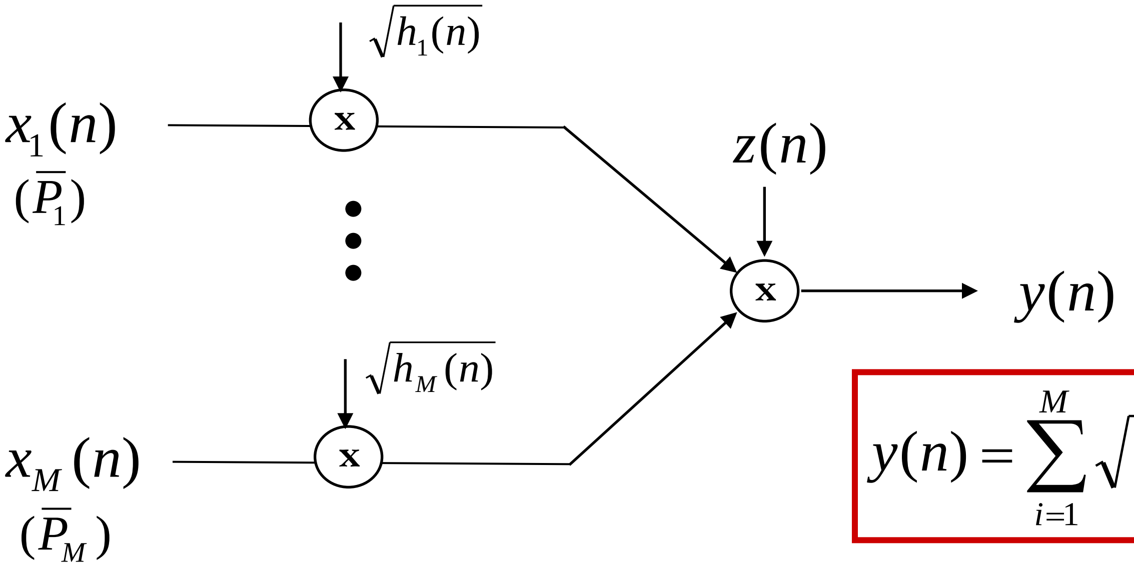
Fading and ISI

- MAC capacity under fading and ISI determined using similar techniques as for the BC
- In fading, can define ergodic, outage, and minimum rate capacity similar as in BC case
  - Ergodic capacity obtained based on AWGN MAC given fixed fading, averaged over fading statistics
  - Outage can be declared as common, or per user
- MAC capacity with ISI obtained by converting to equivalent parallel MAC channels over frequency

Characteristics

- Corner points achieved by 1 user operating at his maximum rate
  - Other users operate at rate which can be decoded perfectly and subtracted out (IC)
- Time sharing connects corner points
  - Can also achieve this line via rate splitting, where one user "splits" into virtual users
- FD has rate $R_i \leq B_i log[1+P_i/(N_0 B)]$
- TD is straight line connecting end points
  - With variable power, it is the same as FD
- CD without IC is box

Fading MAC Channels



$$y(n) = \sum_{i=1}^{M} \sqrt{h_i(n)}\, x_i(n) + z(n)$$

- **Noise is AWGN with variance σ^2.**
- **Joint fading state (known at TX and RX):**

$$\mathrm{h} = (h_1(n), \ldots, h_M(n))$$

Capacity Region*

- **Rate allocation R(h) $\in R^M$**
- **Power allocation P(h) $\in R^M$**
  - **Subject to power constraints: $E_h[P(h)] \leq P$**
- **Boundary points: R***
  - **$\exists\ \lambda, \mu \in R^M$ s.t. [R(h),P(h)] solves**

$$\max \mu \mathbf{R} - \lambda \mathbf{P} \quad s.t. \quad \sum_{i \in S} R_i \leq .5 \log\left[1 + \frac{\sum_{i \in S} h_i P_i}{\sigma^2}\right], \forall S \subseteq \{1, \ldots, M\}$$

with $E_h[R_i(h)] = R_i^*$

***Tse/Hanly, 1996**

Unique Decoding Order*

- For every boundary point R^*:
  - There is a unique decoding order that is the same for every fading state
  - Decoding order is reverse order of the priorities

$$\mu_1 \geq \ldots \geq \mu_M \Rightarrow \textbf{\textit{Decoding order}}: \ \boldsymbol{M}, \boldsymbol{M}-1, \ldots 1$$

- Implications:
  - Given decoding order, only need to optimally allocate power across fading states
  - Without unique decoding order, utility functions used to get optimal rate and power allocation

*S. Vishwanath

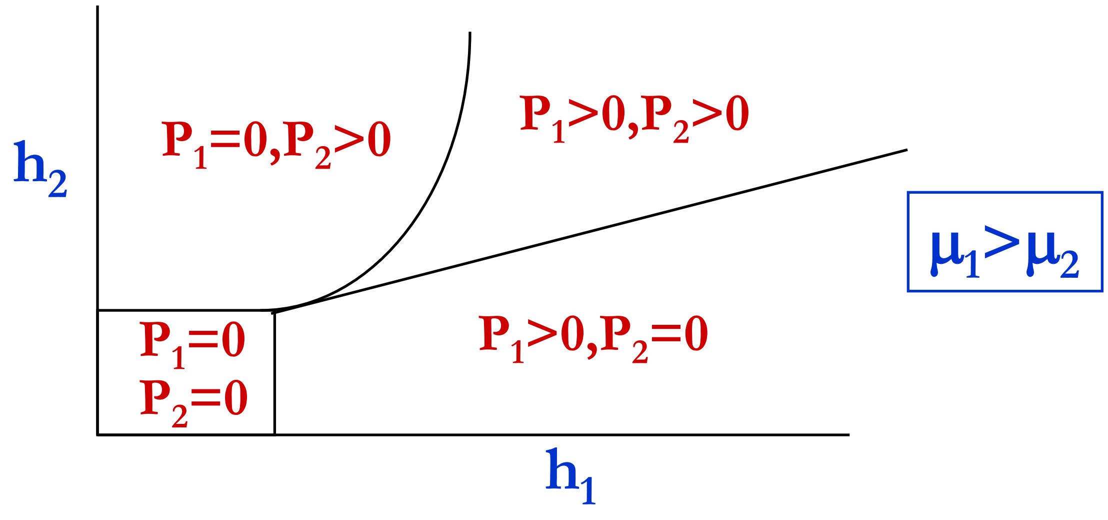
Characteristics of Optimum Power Allocation

- A user's power in a given state depends only on:
  - His channel (h_{ik})
  - Channels of users decoded just before (h_{ik-1}) and just after (h_{ik+1})
  - Power increases with h_{ik} and decreases with h_{ik-1} and h_{ik+1}
  - Power allocation is a modified waterfilling, modified to interference from active users just before and just after

- User decoded first waterfills to SIR for all active users

Transmission Regions

- The region where no users transmit is a hypercube
  - Each user has a unique cutoff below which he does not transmit
- For highest priority user, always transmits above some h_1^*
- The lowest priority user, even with a great channel, doesn't transmit if some other user has a relatively good channel



Two User Example

- Power allocation for $\mu_1 > \mu_2$

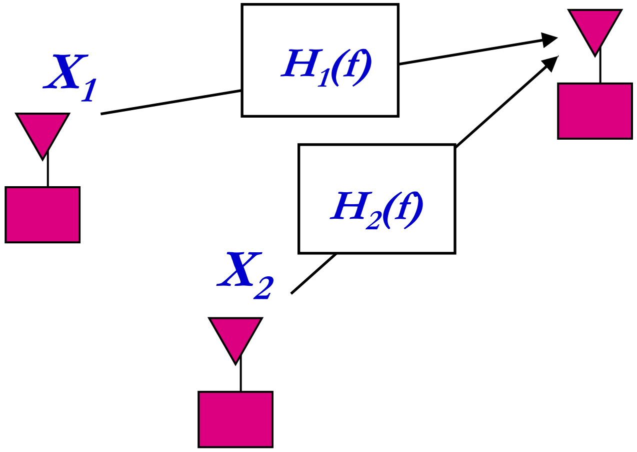
$$P_1(h) = \begin{cases} 0 & h_1 < \dfrac{\lambda_1}{\mu_1} \\ \dfrac{\mu_1}{\lambda_1} - \dfrac{1}{h_1} & h_1 > \dfrac{\lambda_1}{\mu_1}, P_2(h) = 0 \\ \dfrac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2 (h_1 / h_2)} - \dfrac{1}{h_1} & h_1 > \dfrac{\lambda_1}{\mu_1}, P_2(h) \neq 0 \end{cases}$$

$$P_2(h) = \begin{cases} 0 & \dfrac{h_2}{1 + h_1 P_1(h)} > \dfrac{\lambda_2}{\mu_2} \\ \dfrac{\mu_2}{\lambda_2} - \dfrac{1 + h_1 P_1(h)}{h_2} & \dfrac{h_2}{1 + h_1 P_1(h)} < \dfrac{\lambda_2}{\mu_2} \end{cases}$$

Ergodic Capacity Summary

- Rate region boundary achieved via optimal allocation of power and decoding order
- For any boundary point, decoding order is the same for all states
  - Only depends on user priorities
- Optimal power allocation obtained via Lagrangian optimization
  - Only depends on users decoded just before and after
  - Power allocation is a modified waterfilling
  - Transmission regions have cutoff and critical values

MAC Channel with ISI*

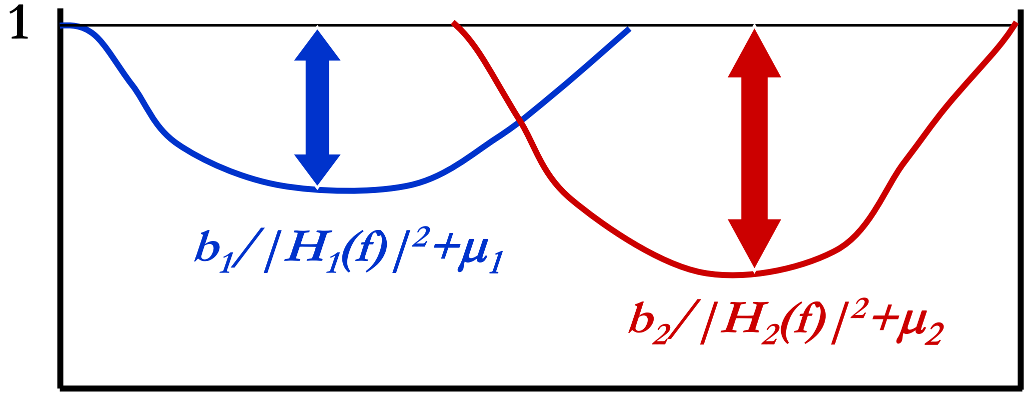


- Use DFT Decomposition
- Obtain parallel MAC channels
- Must determine each user's power allocation across subchannels and decoding order
- Capacity region no longer a pentagon

*Cheng and Verdu, IT'93

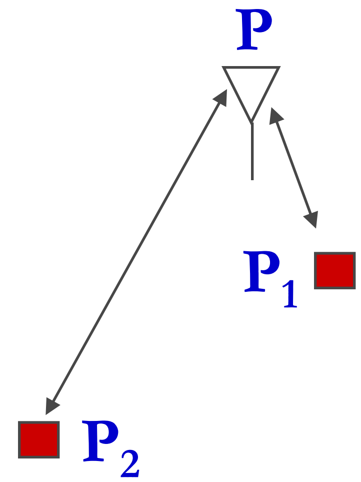
Optimal Power Allocation

- Capacity region boundary: maximize $\mu_1 R_1 + \mu_2 R_2$
- Decoding order based on priorities and channels
- Power allocation is a two-level water filling
  - Total power of both users is scaled water level
  - In non-overlapping region, best user gets all power (FD)
  - With overlap, power allocation and decoding order based on λs and user channels.



Comparison of MAC and BC

- Differences:
  - Shared vs. individual power constraints
  - Near-far effect in MAC



- Similarities:
  - Optimal BC "superposition" coding is also optimal for MAC (sum of Gaussian codewords)
  - Both decoders exploit successive decoding and interference cancellation

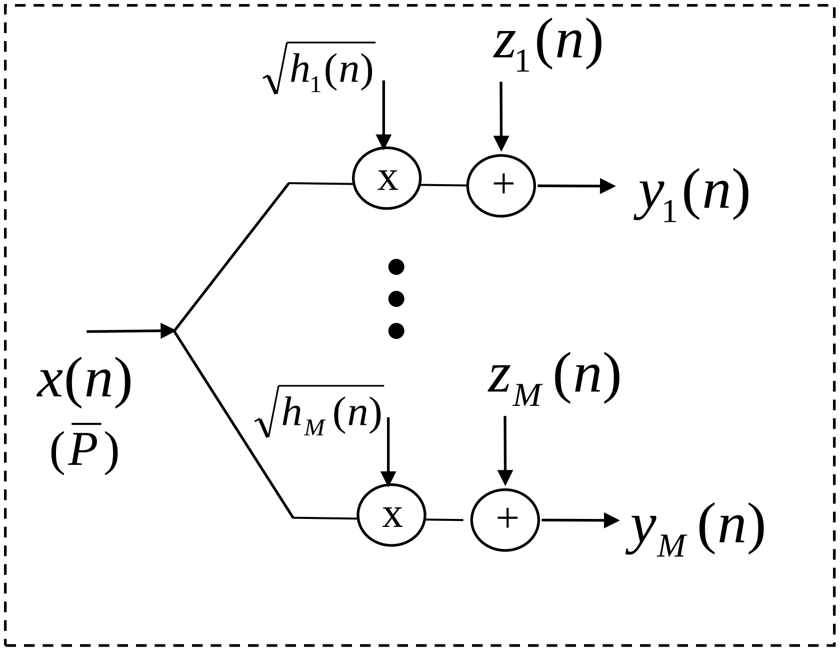
MAC-BC Capacity Regions

- MAC capacity region known for many cases
  - Convex optimization problem
- BC capacity region typically only known for (parallel) degraded channels
  - Formulas often not convex
- Can we find a connection between the BC and MAC capacity regions?

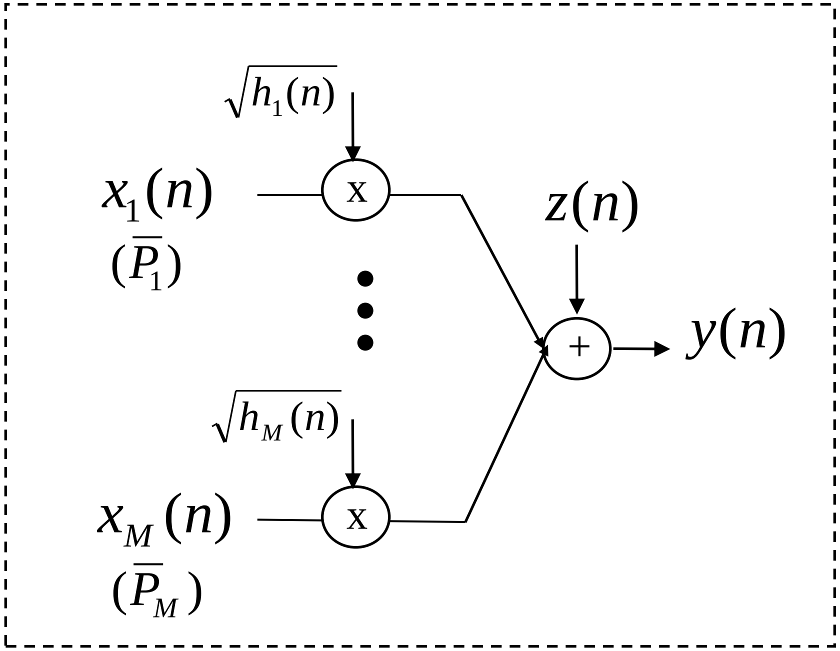
Duality

Dual Broadcast and MAC Channels

Gaussian BC and MAC with *same* channel gains and *same* noise power at each receiver



Broadcast Channel (BC)



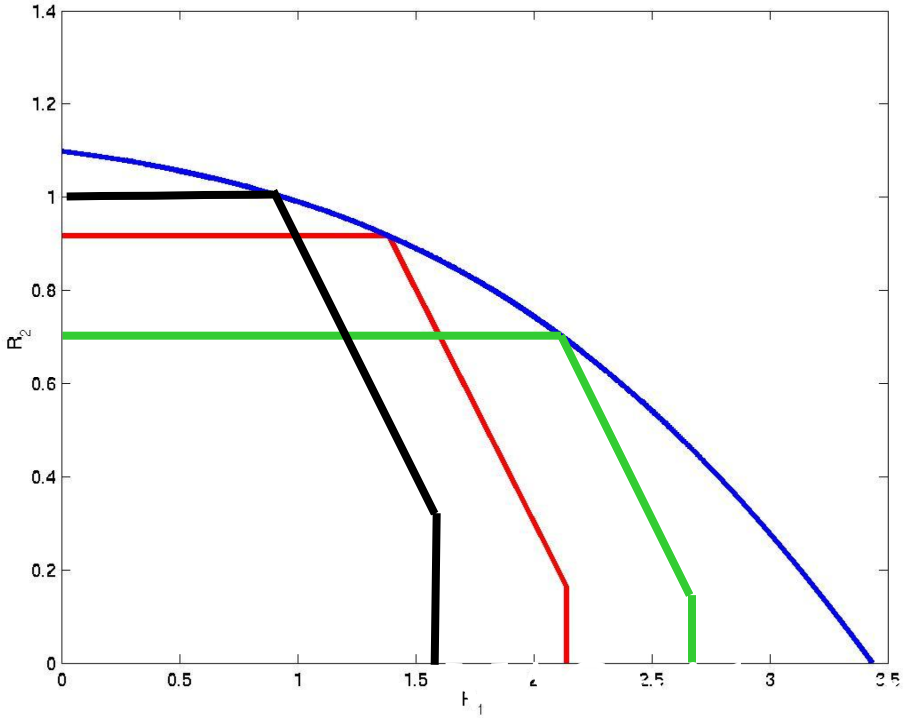
Multiple-Access Channel (MAC)

The BC from the MAC

$$C_{MAC}(P_1, P_2; h_1, h_2) \subseteq C_{BC}(P_1 + P_2; h_1, h_2)$$

$h_1 > h_2$

Blue = BC
Red = MAC



P_1=0.5, P_2=1.5

P_1=1, P_2=1

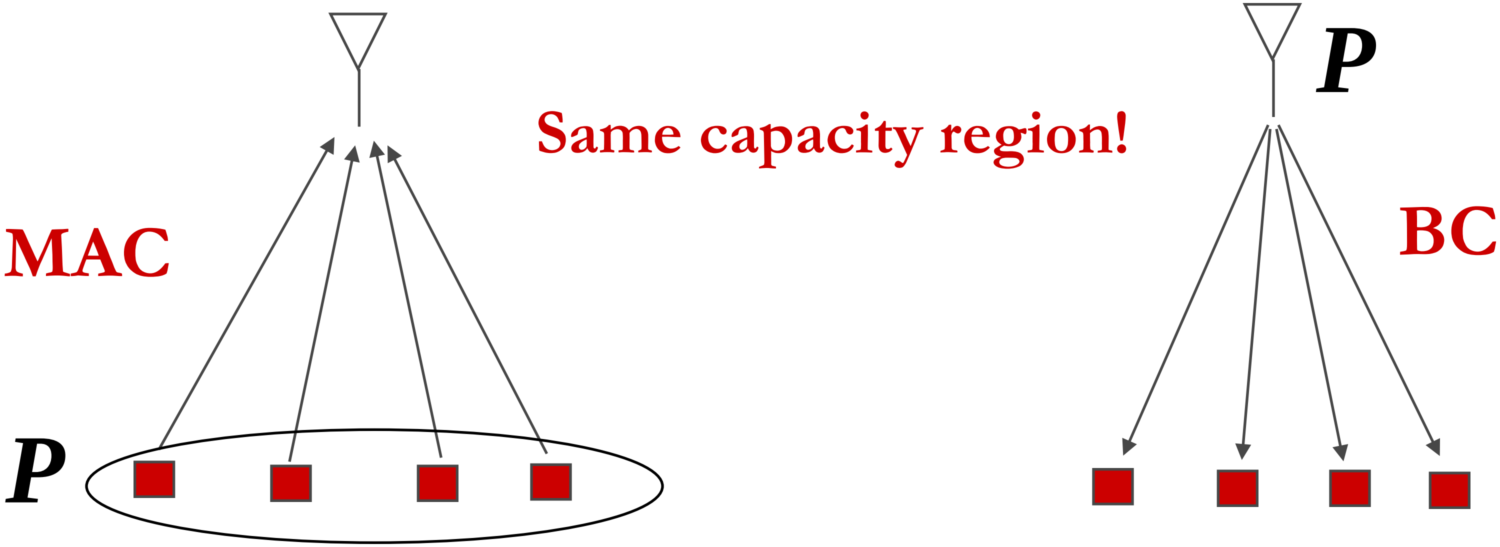
P_1=1.5, P_2=0.5

$$C_{BC}(P; h_1, h_2) = \bigcup_{0 \le P_1 \le P} C_{MAC}(P_1, P - P_1; h_1, h_2)$$

Sum-Power MAC

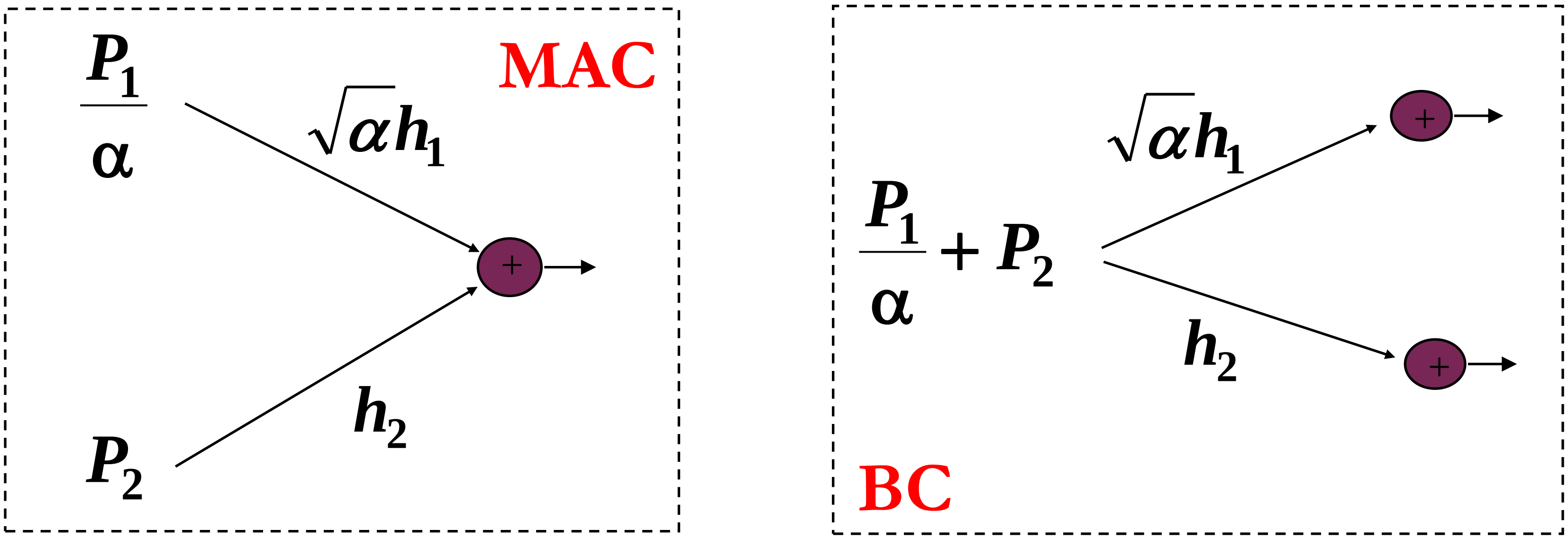
$$C_{BC}(P;h_1,h_2) = \bigcup_{0 \le P_1 \le P} C_{MAC}(P_1, P-P_1; h_1, h_2) \equiv C_{MAC}^{Sum}(P; h_1, h_2)$$

- MAC with <u>sum</u> power constraint
  - Power pooled between MAC transmitters
  - No transmitter coordination



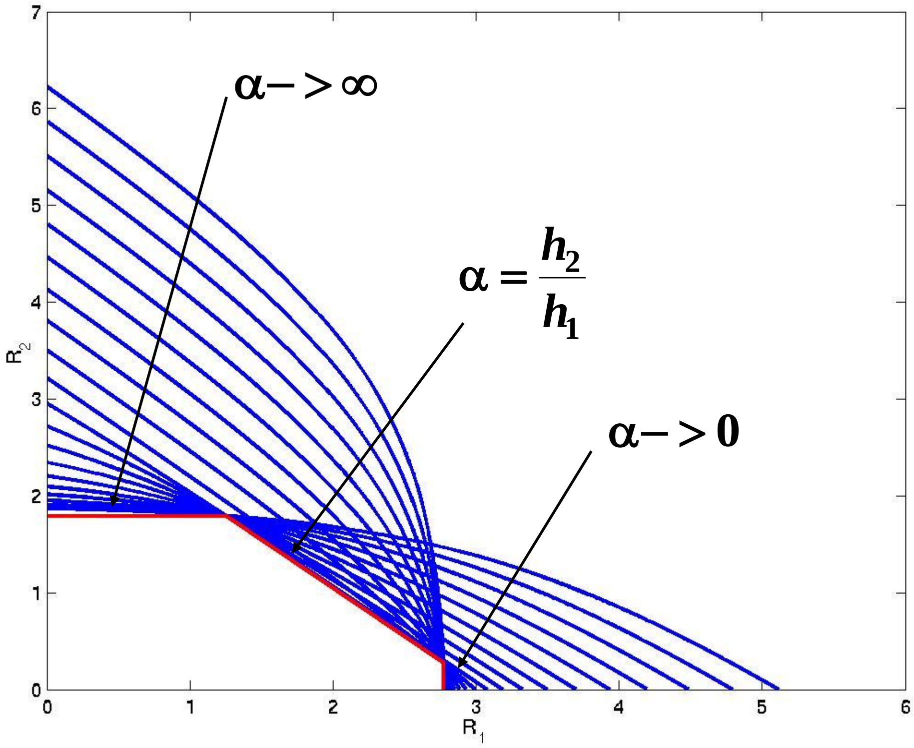
BC to MAC: Channel Scaling

- Scale channel gain by $\sqrt{\alpha}$, power by $1/\alpha$
- MAC capacity region <u>unaffected</u> by scaling
- Scaled MAC capacity region is a subset of the scaled BC capacity region for any α
- MAC region inside scaled BC region for any scaling



The BC from the MAC

Blue = Scaled BC
Red = MAC

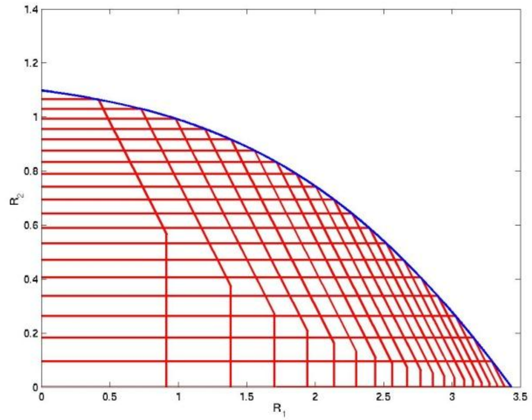


$$C_{MAC}(P_1, P_2; h_1, h_2) = \bigcap_{\alpha>0} C_{BC}\left(\frac{P_1}{\alpha} + P_2; \sqrt{\alpha} h_1, h_2\right)$$

Duality: Constant AWGN Channels

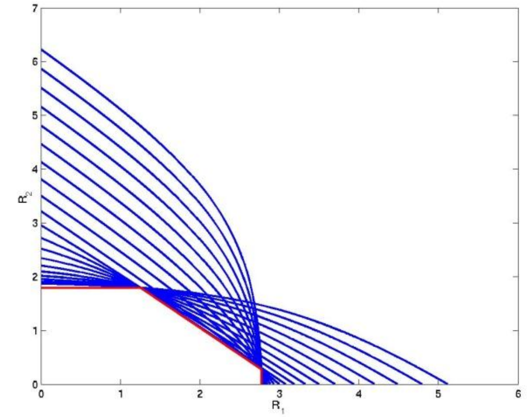
- BC in terms of MAC

$$C_{BC}(P;h_1,h_2) = \bigcup_{0 \le P_1 \le P} C_{MAC}(P_1, P-P_1; h_1, h_2)$$



- MAC in terms of BC

$$C_{MAC}(P_1,P_2;h_1,h_2) = \bigcap_{\alpha > 0} C_{BC}\left(\frac{P_1}{\alpha} + P_2; \alpha h_1, h_2\right)$$



What is the relationship between the optimal transmission strategies?

Transmission Strategy Transformations

- Equate rates, solve for powers

$$R_1^M = \log(1 + \frac{h_1^2 P_1^M}{h_2 P_2^M + \sigma^2}) = \log(1 + \frac{h_1^2 P_1^B}{\sigma^2}) = R_1^B$$

$$R_2^M = \log(1 + \frac{h_2^2 P_2^M}{\sigma^2}) = \log(1 + \frac{h_2^2 P_2^B}{h_2^2 P_1^B + \sigma^2}) = R_2^B$$

- <u>Opposite</u> decoding order
  - Stronger user (User 1) decoded last in BC
  - Weaker user (User 2) decoded last in MAC

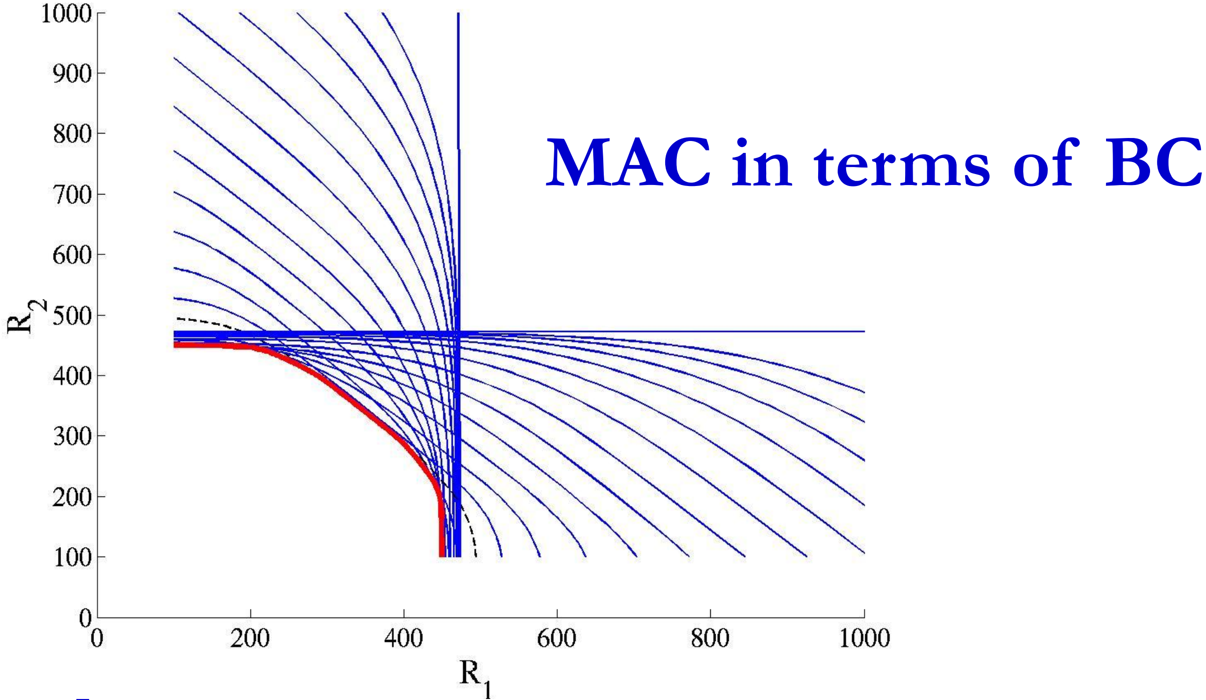
Duality Applies to Different Fading Channel Capacities

- Ergodic (Shannon) capacity: maximum rate averaged over all fading states.
- Zero-outage capacity: maximum rate that can be maintained in all fading states.
- Outage capacity: maximum rate that can be maintained in all nonoutage fading states.
- Minimum rate capacity: Minimum rate maintained in all states, maximize average rate in excess of minimum

Explicit transformations between transmission strategies

Duality: Minimum Rate Capacity

Blue = Scaled BC
Red = MAC

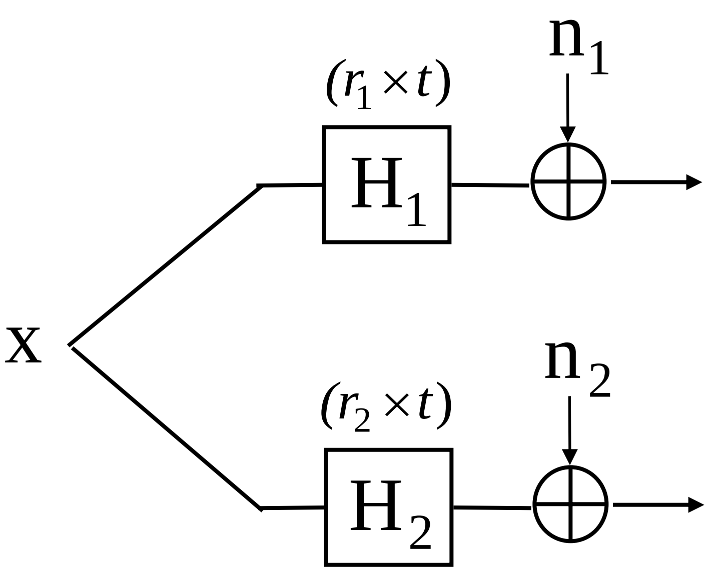


- BC region known
- MAC region can only be obtained by duality

What other capacity regions can be obtained by duality?

Broadcast MIMO Channels

Broadcast MIMO Channel



$y_1 = H_1 x + n_1$

$y_2 = H_2 x + n_2$

$t \geq 1$ TX antennas
$r_1 \geq 1$, $r_2 \geq 1$ RX antennas

Perfect CSI at TX and RX

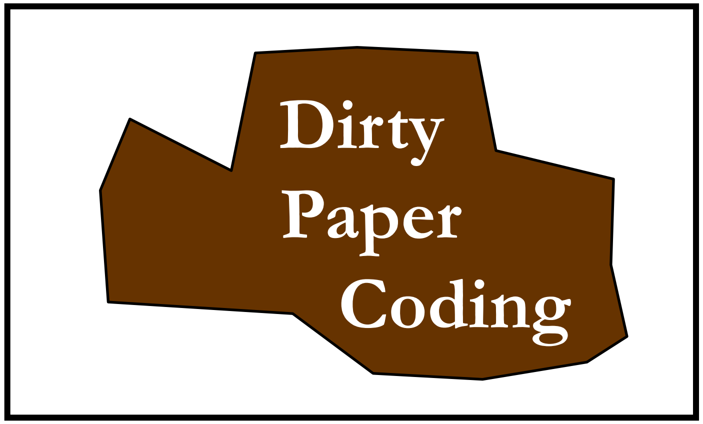
$$n_1 \sim N(0, I_{r_1}) \quad n_2 \sim N(0, I_{r_2})$$

Non-degraded broadcast channel

Dirty Paper Coding (Costa'83)

- Basic premise
  - If the interference is known, channel capacity same as if there is no interference
  - Accomplished by cleverly distributing the writing (codewords) and coloring their ink
  - Decoder must know how to read these codewords

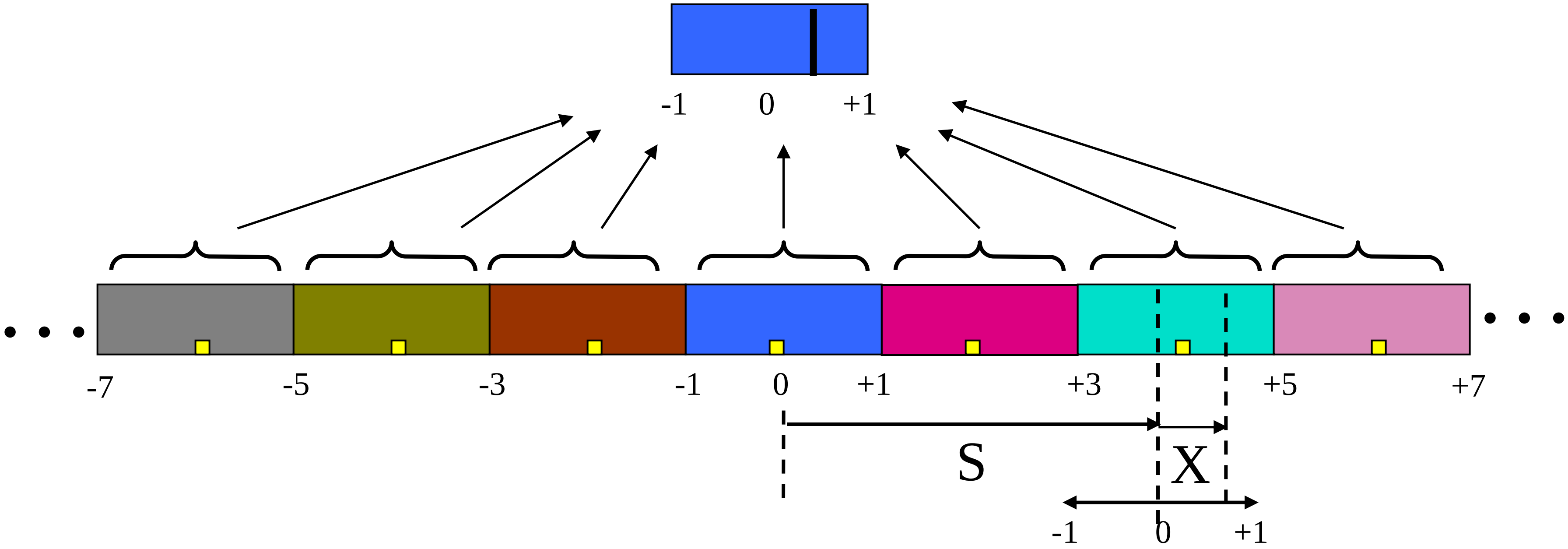
Clean Channel



Dirty Channel

Modulo Encoding/Decoding

- Received signal Y=X+S, $-1 \leq X \leq 1$
  - S known to transmitter, not receiver
- Modulo operation removes the interference effects
  - Set X so that $\lfloor Y \rfloor_{[-1,1]}$=desired message (e.g. 0.5)
  - Receiver demodulates modulo [-1,1]



Capacity Results

- Non-degraded broadcast channel
  - Receivers not necessarily "better" or "worse" due to multiple transmit/receive antennas
  - Capacity region for general case unknown
- Pioneering work by Caire/Shamai (Allerton'00):
  - Two TX antennas/two RXs (1 antenna each)
  - Dirty paper coding/lattice precoding (achievable rate)
    - Computationally very complex
  - MIMO version of the Sato upper bound
  - Upper bound is achievable: capacity known!

Dirty-Paper Coding (DPC) for MIMO BC

- Coding scheme:
  - Choose a codeword for user 1
  - Treat this codeword as interference to user 2
  - Pick signal for User 2 using "pre-coding"
- Receiver 2 experiences no interference:

$$\mathrm{R}_2 = \log(\det(\mathrm{I} + H_2 \Sigma_2 H_2^T))$$

- Signal for Receiver 2 interferes with Receiver 1:

$$\mathrm{R}_1 = \log\left(\frac{\det(\mathrm{I} + H_1(\Sigma_1 + \Sigma_2)H_1^T)}{\det(\mathrm{I} + H_1 \Sigma_2 H_1^T)}\right)$$

- Encoding order can be switched
- DPC optimization highly complex

Does DPC achieve capacity?

- DPC yields MIMO BC achievable region.
  - We call this the dirty-paper region
- Is this region the capacity region?
- We use duality, dirty paper coding, and Sato's upper bound to address this question
- First we need MIMO MAC Capacity

MIMO MAC Capacity

- **MIMO MAC follows from MAC capacity formula**

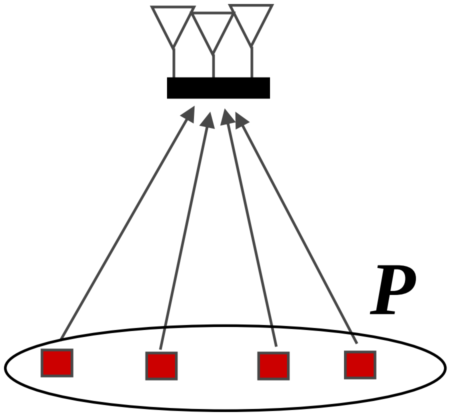
$$C_{MAC}(P_1,\ldots,P_k) = \bigcup \left\{ (R_1,\ldots,R_k) : \sum_{k\in S} R_k \le \log_2 \det\left[I + \sum_{k\in S} H_k Q_k H_k^H \right], \ \forall S \subseteq \{1,\ldots,K\} \right\}$$

- **Basic idea same as single user case**
  - **Pick some subset of users**
  - **The sum of those user rates equals the capacity as if the users pooled their power**

- **Power Allocation and Decoding Order**
  - **Each user has its own power (no power alloc.)**
  - **Decoding order depends on desired rate point**

MIMO MAC with sum power

- **MAC with sum power:**
  - **Transmitters code independently**
  - **Share power**



$$C_{MAC}^{Sum}(P) = \bigcup_{0 \le P_1 \le P} C_{MAC}(P_1, P - P_1)$$

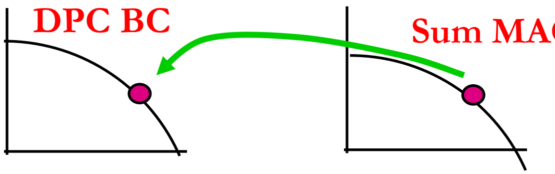
- **Theorem: Dirty-paper BC region equals the dual sum-power MAC region**

$$C_{BC}^{DPC}(P) = C_{MAC}^{Sum}(P)$$

Transformations: MAC to BC

- Show any rate achievable in sum-power MAC also achievable with DPC for BC:

$$C_{BC}^{DPC}(P) \supseteq C_{MAC}^{Sum}(P)$$

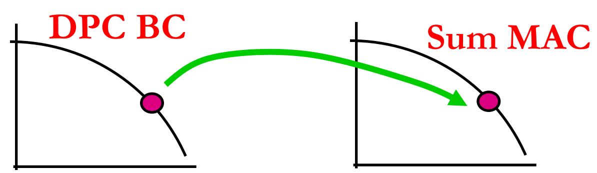


  - A sum-power MAC strategy for point $(R_1,\ldots R_N)$ has a given input covariance matrix and encoding order
  - We find the corresponding PSD covariance matrix and encoding order to achieve $(R_1,\ldots,R_N)$ with DPC on BC
    - The rank-preserving transform "flips the effective channel" and reverses the order
    - Side result: beamforming is optimal for BC with 1 Rx antenna at each mobile

Transformations: BC to MAC

- **Show any rate achievable with DPC in BC also achievable in sum-power MAC:**

$$C_{BC}^{DPC}(P) \subseteq C_{MAC}^{Sum}(P)$$



  - **We find transformation between optimal DPC strategy and optimal sum-power MAC strategy**
    - **"Flip the effective channel" and reverse order**

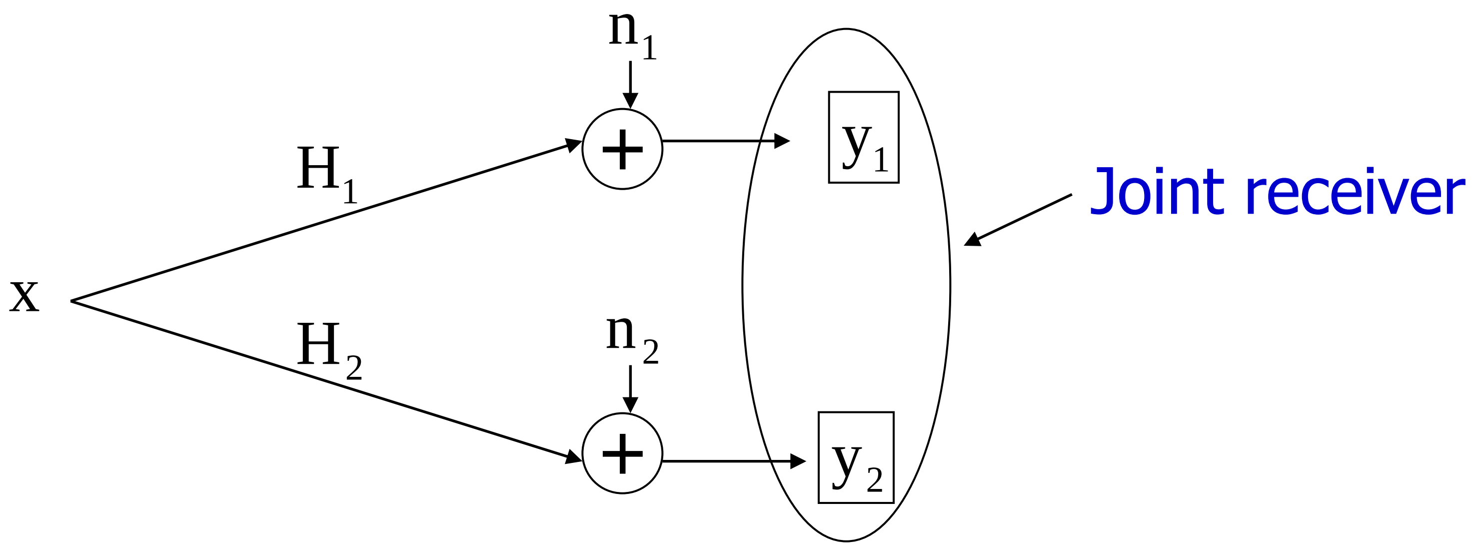
Computing the Capacity Region

$$C_{BC}^{DPC}(P) = C_{MAC}^{Sum}(P)$$

- Hard to compute DPC region (Caire/Shamai'00)
- "Easy" to compute the MIMO MAC capacity region
  - Obtain DPC region by solving for sum-power MAC and applying the theorem
  - Fast iterative algorithms have been developed
  - Greatly simplifies calculation of the DPC region and the associated transmit strategy

Sato Upper Bound on the BC Capacity Region

- Based on receiver cooperation



- BC sum rate capacity ≤ Cooperative capacity

$$C_{BC}^{\text{sum rate}}(P,H) \le \max_{\Sigma_x} \frac{1}{2}\log|I + H\Sigma_x H^T|$$

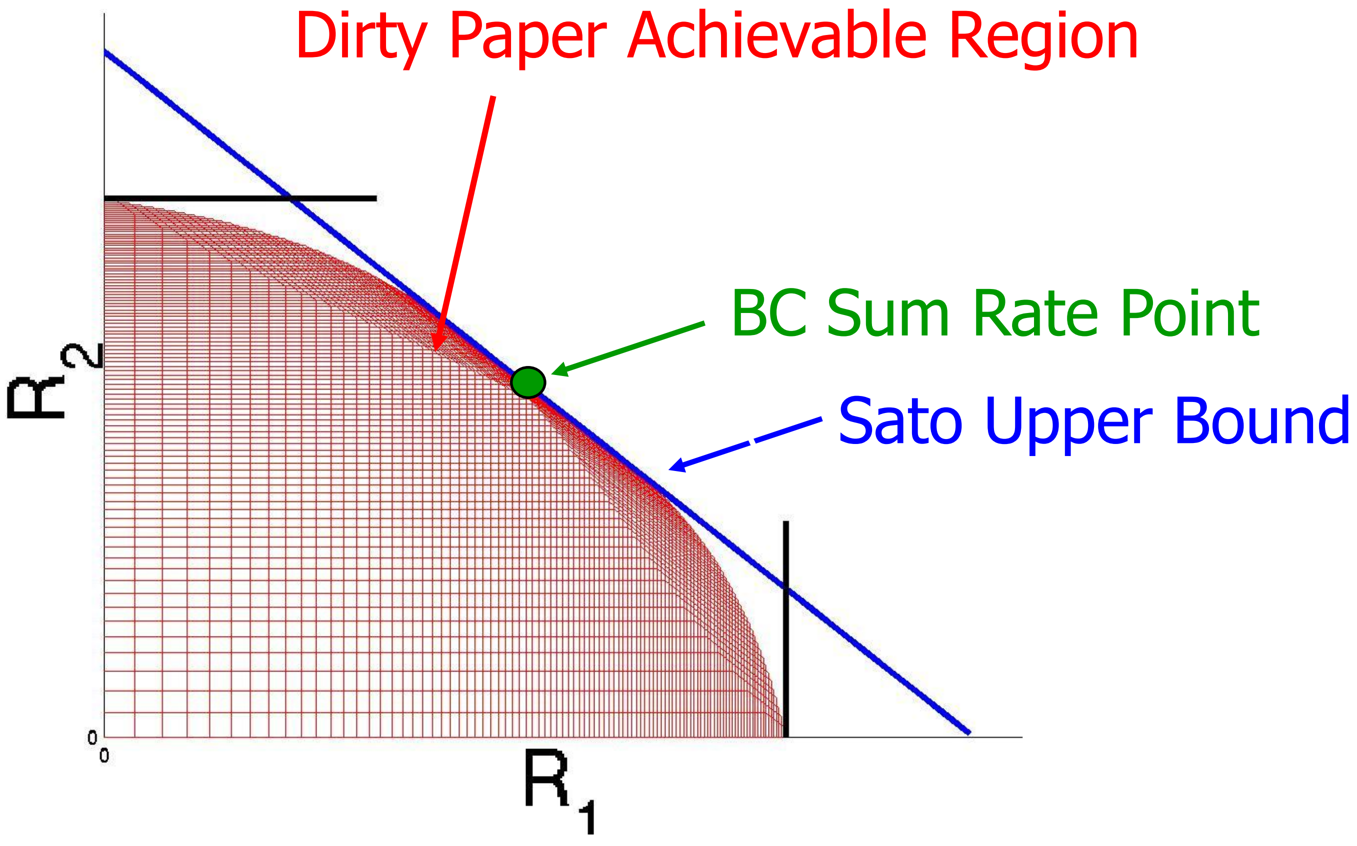
The Sato Bound for MIMO BC

- Introduce noise correlation between receivers
- BC capacity region unaffected
  - Only depends on noise marginals
- Tight Bound (Caire/Shamai'00)
  - Cooperative capacity with <u>worst-case</u> noise correlation

$$C_{BC}^{\text{sum rate}}(P,H) \leq \inf_{\Sigma_z} \max_{\Sigma_x} \frac{1}{2} \log |I + \Sigma_z^{-1/2} H \Sigma_x H^T \Sigma_z^{-1/2}|$$

- Explicit formula for worst-case noise covariance
- By Lagrangian duality, cooperative BC region equals the sum-rate capacity region of MIMO MAC

MIMO BC Capacity Bounds



Does the DPC region equal the capacity region?

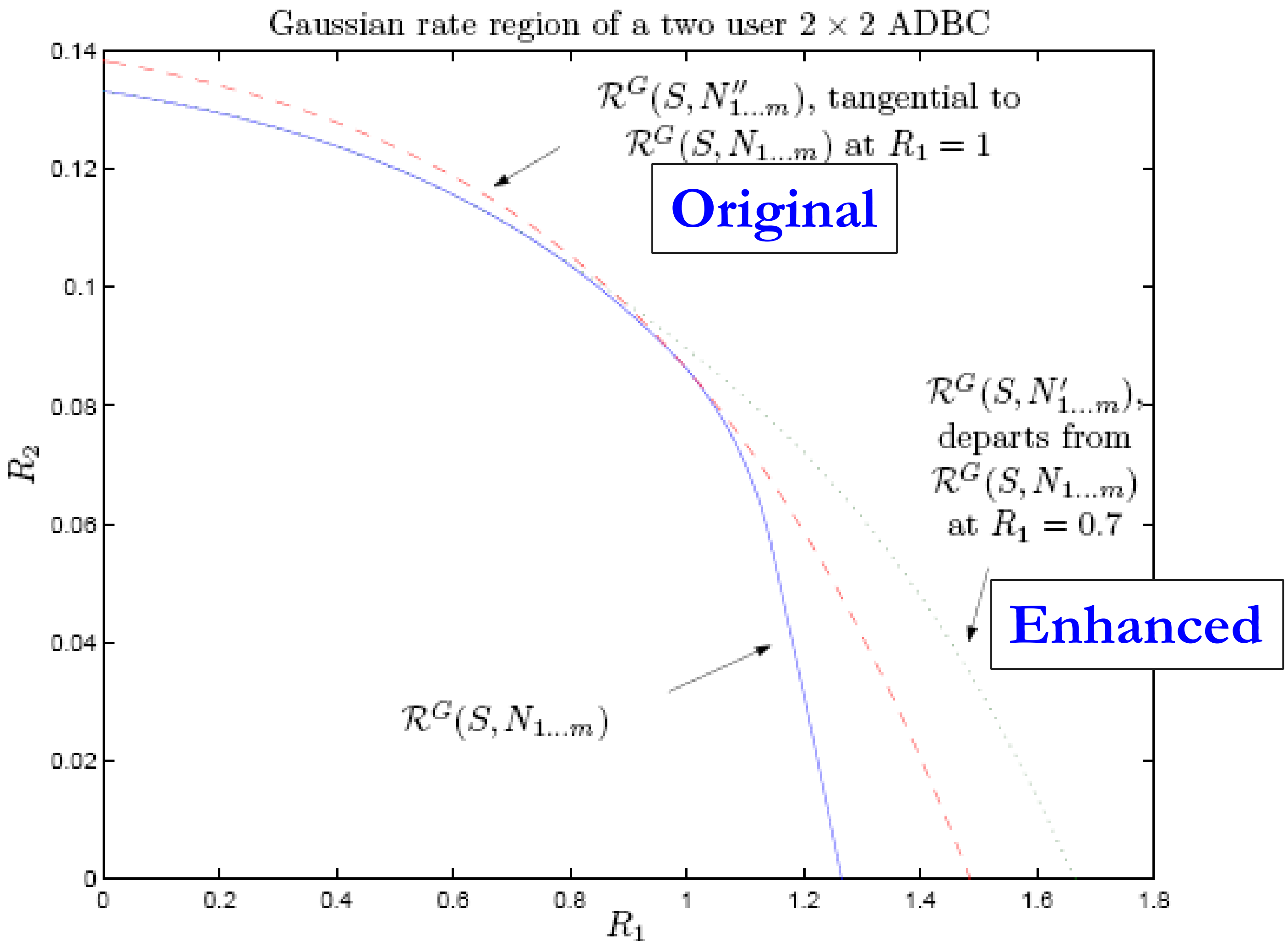
Full Capacity Region

- DPC gives us an achievable region
- Sato bound only touches at sum-rate point
- Bergman's entropy power inequality is not a tight upper bound for nondegraded broadcast channel
- A tighter bound was needed to prove DPC optimal
  - It had been shown that if Gaussian codes optimal, DPC was optimal, but proving Gaussian optimality was open.
- Breakthrough by Weingarten, Steinberg and Shamai
  - Introduce notion of enhanced channel, applied Bergman's converse to it to prove DPC optimal for MIMO BC.

Enhanced Channel Idea

- The aligned and degraded BC (AMBC)
  - Unity matrix channel, noise innovations process
  - Limit of AMBC capacity equals that of MIMO BC
  - Eigenvalues of some noise covariances go to infinity
  - Total power mapped to covariance matrix constraint
- Capacity region of AMBC achieved by Gaussian superposition coding and successive decoding
  - Uses entropy power inequality on enhanced channel
  - Enhanced channel has less noise variance than original
  - Can show that a power allocation exists whereby the enhanced channel rate is inside original capacity region
- By appropriate power alignment, capacities equal

Illustration



Main Points

- Shannon capacity gives fundamental data rate limits for multiuser wireless channels
- Fading multiuser channels optimize at each channel instance for maximum average rate
- Outage capacity has higher (fixed) rates than with no outage.
- OFDM is near optimal for broadcast channels with ISI
- Duality connects BC and MAC channels
  - Used to obtain capacity of one from the other
- Capacity of broadcast MIMO channel obtained using duality and the notion of an enhanced channel