EE360: Multiuser Wireless Systems and Networks

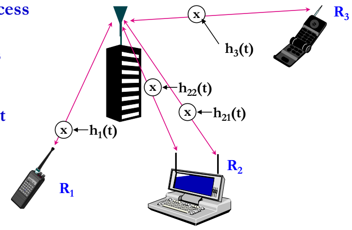
Lecture 2 Outline

- Announcements
  - HW 0 due today
  - Makeup lecture for first class (sorry for confusion):
    - Thurs eve or Friday lunch (w/ food)? Feb 2/3 or Feb 9/10?
- Bandwidth Sharing in Multiuser Channels
  - FD, TD, CD, SD, Hybrid
- Overview of Multiuser Channel Capacity
- Capacity of Broadcast Channels
  - AWGN, Fading, and ISI
- Capacity of MAC Channels
- MIMO Channels

Review of Last Lecture: Uplink and Downlink

Uplink (Multiple Access Channel or MAC):
Many Transmitters to One Receiver.

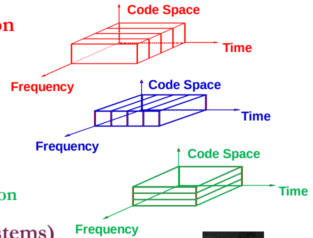
Downlink (Broadcast Channel or BC):
One Transmitter to Many Receivers.



Uplink and Downlink typically duplexed in time or frequency

Bandwidth Sharing

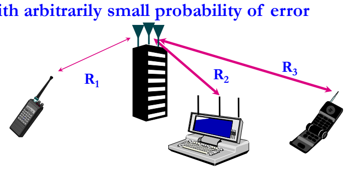
- Frequency Division
- Time Division
- Code Division
  - Multiuser Detection
- Space (MIMO Systems)
- Hybrid Schemes

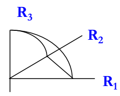


What is optimal? Look to Shannon.

Multiuser Shannon Capacity
Fundamental Limit on Data Rates

Capacity: The set of simultaneously achievable rates $\{R_1,\ldots,R_n\}$ with arbitrarily small probability of error





- Main drivers of channel capacity
  - Bandwidth and received SINR
  - Channel model (fading, ISI)
  - Channel knowledge and how it is used
  - Number of antennas at TX and RX
- Duality connects capacity regions of uplink and downlink

Broadcast Channel Capacity Region in AWGN

- Model
  - One transmitter, two receivers with spectral noise density n_1, n_2: $n_1<n_2$.
  - Transmitter has average power Pand total bandwidth B.
- Single User Capacity:
  - Maximum achievable rate with asymptotically small P_e

$$C_i = B\log\left[1+\frac{P}{n_i B}\right]$$

- Set of achievable rates includes $(C_1,0)$ and $(0,C_2)$, obtained by allocating all resources to one user.

Rate Region: Time Division

- Time Division (Constant Power)
  - Fraction of time τ allocated to each user is varied

$$\{U(\boldsymbol{R}_1=\tau\boldsymbol{C}_1, \boldsymbol{R}_2=(1-\tau)\boldsymbol{C}_2);0\le\tau\le 1\}$$

- Time Division (Variable Power)
  - Fraction of time τ and power σ_i allocated to each user is varied

$$\left\{\begin{matrix}\mathbf{U}\left(R_1=\tau B\log\left[1+\frac{\sigma_1}{n_1 B}\right], R_2=(1-\tau)B\log\left[1+\frac{\sigma_2}{n_2 B}\right]\right); \\ \tau\sigma_1+(1-\tau)\sigma_2=P, \qquad 0\le\tau\le 1.\end{matrix}\right\}$$

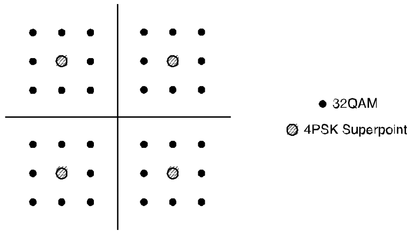
Rate Region: Frequency Division

- **Frequency Division**
  - **Bandwidth B_i and power S_i allocated to each user is varied.**

$$\left\{ \mathbf{U}\left(R_1 = B_1 \log\left[1+\frac{P_1}{n_1 B_1}\right], R_2 = B_2 \log\left[1+\frac{P_2}{n_2 B_2}\right]\right); \atop P_1 + P_2 = P, B_1 + B_2 = B \right\}$$

Equivalent to TD for $B_i = \tau_i B$ and $P_i = \tau_i \sigma_i$.

Superposition Coding



Best user decodes fine points
Worse user decodes coarse points

Code Division

- **Superposition Coding**
  - **Coding strategy allows better user to cancel out interference from worse user.**

$$\left\{ \mathbf{U}\left(R_1 = B \log\left[1+\frac{P_1}{n_1 B}\right], R_2 = B \log\left[1+\frac{P_2}{n_2 B + S_1}\right]\right); P_1 + P_2 = P \right\}$$

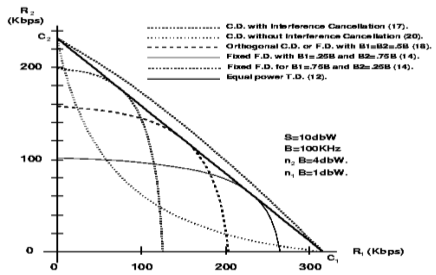
- **DS spread spectrum with spreading gain G and cross correlation $\rho_{12} = \rho_{21} = G$:**

$$\left\{ \mathbf{U}\left(R_1 = \frac{B}{G} \log\left[1+\frac{P_1}{n_1 B/G}\right], R_2 = \frac{B}{G} \log\left[1+\frac{P_2}{n_2 B/G + S_1/G}\right]\right); P_1 + P_2 = P \right\}$$

  - **By concavity of the log function, G=1 maximizes the rate region.**

- **DS without interference cancellation**

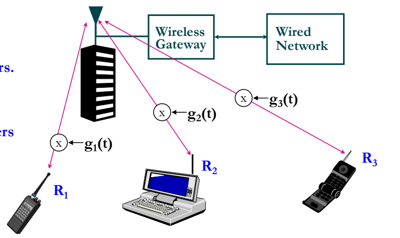
$$\left\{ \mathbf{U}\left(R_1 = \frac{B}{G} \log\left[1+\frac{P_1}{n_1 B/G + P_2/G}\right], R_2 = \frac{B}{G} \log\left[1+\frac{P_2}{n_2 B/G + P_1/G}\right]\right); P_1 + P_2 = P \right\}$$



Broadcast and MAC Fading Channels

Broadcast:
One Transmitter
to Many Receivers.

Multiple Access:
Many Transmitters
to One Receiver.



Goal: Maximize the rate region $\{R_1,\ldots,R_n\}$, subject to some minimum rate constraints, by dynamic allocation of power, rate, and coding/decoding.

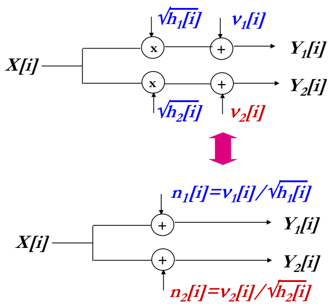
Assume transmit power constraint and perfect TX and RX CSI

Fading Capacity Definitions

- **Ergodic (Shannon) capacity: maximum long-term rates averaged over the fading process.**
  - **Shannon capacity applied directly to fading channels.**
  - **Delay depends on channel variations.**
  - **Transmission rate varies with channel quality.**
- **Zero-outage (delay-limited*) capacity: maximum rate that can be maintained in all fading states.**
  - **Delay independent of channel variations.**
  - **Constant transmission rate – much power needed for deep fading.**
- **Outage capacity: maximum rate that can be maintained in all nonoutage fading states.**
  - **Constant transmission rate during nonoutage**
  - **Outage avoids power penalty in deep fades**

***Hanly/Tse, IT, 11/98**

Two-User Fading Broadcast Channel



At each time i:
$n=\{n_1[i],n_2[i]\}$

Ergodic Capacity Region*

- Capacity region: $C_{ergodic}(\bar{P}) = \cup_{\mathcal{P}\in\mathcal{F}} C(\mathcal{P})$, where

$$C(\mathcal{P}) = \left\{R_j \le E_n\left[B\log\left(1+\frac{P_j(n)}{n_jB+\sum_{i=1}^{M}P_i(n)1[n_j>n_i]}\right)\right],\quad 1\le j\le M\right\}$$

  - The power constraint implies $E_n\sum_{j=1}^{M}P_j(n)=\bar{P}$
- Superposition coding and successive decoding achieve capacity
  - Best user in each state decoded last
  - Power and rate adapted using multiuser water-filling: power allocated based on noise levels and user priorities

*Li/Goldsmith, IT, 3/01

Zero-Outage Capacity Region*

- The set of rate vectors that can be maintained for all channel states under power constraint $\bar{P}$

$$C_{zero}(\bar{P}) = \cup_{\mathcal{P}\in\mathcal{F}} \cap_{n\in N} C(\mathcal{P})$$

$$C(\mathcal{P}) = \left\{R_j \le B\log\left(1+\frac{P_j(n)}{n_jB+\sum_{i=1}^{M}P_i(n)1[n_j>n_i]}\right),\quad 1\le j\le M\right\}$$

- Capacity region defined implicitly relative to power:
  - For a given rate vector R and fading state n we find the minimum power $P^{min}(R,n)$ that supports R.
  - $R\in C_{zero}(\bar{P})$ if $E_n[P^{min}(R,n)] \le \bar{P}$

*Li and Goldsmith, IT, 3/01

Outage Capacity Region

- Two different assumptions about outage:
  - All users turned off simultaneously (common outage Pr)
  - Users turned off independently (outage probability vector <u>Pr</u>)
- Outage capacity region implicitly defined from the minimum outage probability associated with a given rate
- Common outage: given (R,n), use threshold policy
  - If $P^{min}(R,n)>s^*$ declare an outage, otherwise assign this power to state n.
  - Power constraint dictates s^* : $\bar{P} = E_{n:P^{\min}(R,n)\le s^*}\left[P^{\min}(R,n)\right]$
  - Outage probability: $Pr = \int_{n:P^{\min}(R,n)>s^*} p(n)$

Independent Outage

- With independent outage cannot use the threshold approach:
  - Any subset of users can be active in each fading state.
- Power allocation must determine how much power to allocate to each state and which users are on in that state.
- Optimal power allocation maximizes the reward for transmitting to a given subset of users for each fading state
  - Reward based on user priorities and outage probabilities.
  - An iterative technique is used to maximize this reward.
  - Solution is a generalized threshold-decision rule.

Minimum-Rate Capacity Region

- Combines ergodic and zero-outage capacity:
  - Minimum rate vector maintained in all fading states.
  - Average rate in excess of the minimum is maximized.
- Delay-constrained data transmitted at the minimum rate at all times.
- Channel variation exploited by transmitting other data at the maximum excess average rate.

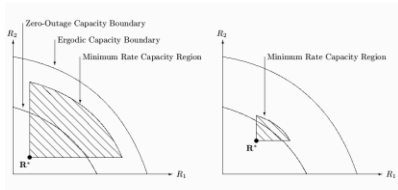
Minimum Rate Constraints

- Define minimum rates $R^* = (R^*_1, \ldots, R^*_M)$:
  - These rates must be maintained in all fading states.
- For a given channel state n:

$$R_j(n) \le B \log\left(1 + \frac{P_j(n)}{n_j B + \sum_{i=1}^{M} P_i(n) 1[n_j > n_i]}\right), \quad R_j(n) \ge R_j^* \ \forall n$$

- R^* must be in zero-outage capacity region
  - Allocate excess power to maximize excess ergodic rate
  - The smaller R^*, the bigger the min-rate capacity region

Comparison of Capacity Regions



- For R^* far from C_{zero} boundary, $C_{min\text{-}rate} \approx C_{ergodic}$
- For R^* close to C_{zero} boundary, $C_{min\text{-}rate} \approx C_{zero} \cap R^*$

Optimal Coding and Power Allocation

- Superposition coding with SIC in usual order (best user decoded last) is optimal.
- Power allocation broken down into two steps:
  - First allocate the minimum power needed to achieve the minimum rates in all fading states.
  - Then optimally allocate the excess power to maximize the excess ergodic rate.
  - Power allocation between users: insights
    - Excess power given to better user impacts interference of worse user but not vice versa
    - Excess power given to better user results in a higher rate increase
    - Power allocation depends on channel state and user priorities

Minimum Rates for Single-User Channels

- Maximize excess ergodic rate:

$$\max E[\log(1+\tfrac{P(n)}{n})] \quad s.t.\ E[P(n)] \le P, \ R(n) \ge R^* \ \forall n$$

- Power required to achieve R^* in state n:

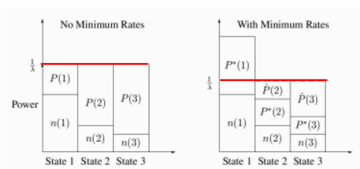
$$P^*(n) = n(e^{R^*} - 1)$$

- Optimal excess power allocation: $P(n)=P^*(n)+\hat{P}(n)$

$$\hat{P}(n) = \begin{cases} \frac{1}{\lambda} - (n + P^*(n)) & n + P^*(n) \le \frac{1}{\lambda} \\ 0 & else \end{cases}$$

Waterfilling to modified noise

Water-filling to Modified Noise for SU Channel



- Without no minimum rate all 3 states are allocated power.
- With a minimum rate the noise level in state i increases by $P^*(i)$
  - Only the two best states are allocated excess power.

Two-User Broadcast Channel with Minimum Rates

- Min-rate capacity region boundary defined by:

$$\max_{P(n)} E_n[\mu_1 R_1(n) + \mu_2 R_2(n)] \quad s.t.$$
$$E_n[P_1(n) + P_2(n)] \le P, \quad R_i(n) \ge R_i^* \ \forall n$$

- Minimum power required in state n ($n_2 > n_1$):

$$P_1^* = n_1(e^{R_1^*} - 1), \quad P_2^* = (P_1^* + n_2)(e^{R_2^*} - 1)$$

- Total excess power to allocate over all states

$$\hat{P} = P - E_n[P_1^*(n) + P_2^*(n)]$$

Modified Problem

- Optimize relative to excess power ($n_2 > n_1$):

$$\max_{P(n)} E_n\left[\mu_1 \log\left(1+\frac{\hat{P}_1(n)+P_1^*(n)}{n_1}\right)+\mu_2 \log\left(1+\frac{\hat{P}(n)-\hat{P}_1(n)+P_2^*(n)}{n_2+\hat{P}_1(n)+P_1^*(n)}\right)\right] \quad s.t.$$

$$E_n[\hat{P}(n)] \le \hat{P}, \quad 0 \le \hat{P}_1(n) \le \hat{P}(n)e^{-R_2^*} \quad \forall n$$

- Excess power allocation:
  - Optimize excess power $\hat{P}(n)$ allocated to state n
  - Divide $\hat{P}(n)=\hat{P}_1(n)+\hat{P}_2(n)$ between the two users
  - Solved via two dimensional Lagrangian or greedy algorithm

Total Excess Power Allocation

- Optimal allocation of *excess* power to state n is a multilevel water-filling:

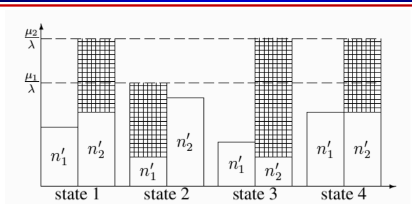
$$\hat{P}(\mathbf{n}) = \max\left(\frac{\mu_1}{\lambda} - n_1', \frac{\mu_2}{\lambda} - n_2', 0\right)$$

where n_1' and n_2' are effective noises:

$$\begin{cases} n_1' = (P_1^*(\mathbf{n}) + n_1)e^{R_2^*}, \; n_2' = (P_1^*(\mathbf{n}) + n_2)e^{R_2^*} & n_1 < n_2 \\ n_1' = (P_2^*(\mathbf{n}) + n_1)e^{R_1^*}, \; n_2' = (P_2^*(\mathbf{n}) + n_2)e^{R_1^*} & n_1 \ge n_2 \end{cases}$$

and the water-level λ satisfies the power constraint

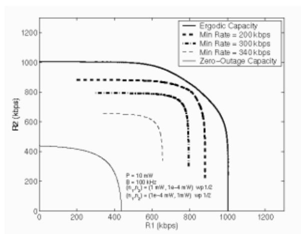
Multi-User Water-filling



- Identical to the optimal power allocation scheme for ergodic capacity with modified noise and power constraint.
- Once $\hat{P}(n)$ known, division between users straightforward.
  - Depends on user priorities and effective noises

Min-Rate Capacity Region: Large Deviation in User Channels

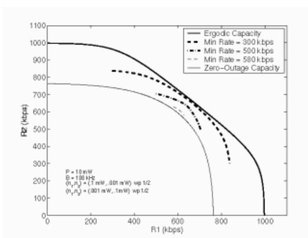
P = 10 mW,
B = 100 KHz



Symmetric channel with 40 dB difference in noises in each fading state (user 1 is 40 dB stronger in 1 state, and vice versa).

Min-Rate Capacity Region: Smaller Deviation

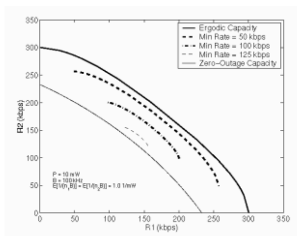
P = 10 mW,
B = 100 KHz



Symmetric channel with 20 dB difference in noises in each fading state (user 1 is 20 dB stronger in 1 state, and vice versa).

Min-Rate Capacity Region: Severe Rician Fading

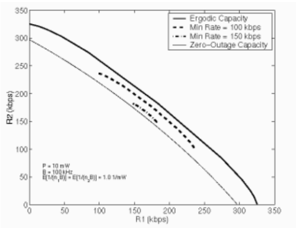
P = 10 mW,
B = 100 KHz



Independent Rician fading with K=1 for both users (severe fading, but not as bad as Rayleigh).

Min-Rate Capacity Region: Mild Rician Fading

P = 10 mW,
B = 100 KHz

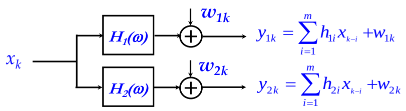


Independent Rician fading with K=5 for both users.

Broadcast Channels with ISI

- ISI introduces memory into the channel
- The optimal coding strategy decomposes the channel into parallel broadcast channels
  - Superposition coding is applied to each subchannel.
- Power must be optimized across subchannels and between users in each subchannel.

Broadcast Channel Model



- Both H_1 and H_2 are finite IR filters of length m.
- The w_{1k} and w_{2k} are correlated noise samples.
- For $1<k\leq n$, we call this channel the n-block discrete Gaussian broadcast channel (n-DGBC).
- The channel capacity region is $C=(R_1,R_2)$.

Circular Channel Model

- Define the zero padded filters as:

$$\{\tilde{h}_i\}_{i=1}^{n} = (h_1,\ldots,h_m,0,\ldots,0)$$

- The n-Block Circular Gaussian Broadcast Channel (n-CGBC) is defined based on circular convolution:

$$\tilde{y}_{1k} = \sum_{i=0}^{n-1} \tilde{h}_{1i} x_{((k-i))} + w_{1k} = x_i \otimes h_{1i} + w_{1k}$$

$$0 \leq k < n$$

$$\tilde{y}_{2k} = \sum_{i=0}^{n-1} \tilde{h}_{2i} x_{((k-i))} + w_{2k} = x_i \otimes h_{2i} + w_{2k}$$

where $((\cdot))$ denotes addition modulo n.

Equivalent Channel Model

- Taking DFTs of both sides yields

$$\tilde{Y}_{1j} = \tilde{H}_{1j} X_j + W_{1j}$$

$$0 \leq j < n$$

$$\tilde{Y}_{2j} = \tilde{H}_{2j} X_j + W_{2j}$$

- Dividing by $\tilde{H}$ and using additional properties of the DFT yields

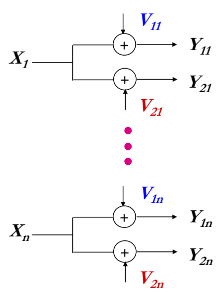
$$Y'_{1j} = X'_j + V'_{1j}$$

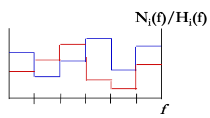
$$0 \leq j < n$$

$$Y'_{2j} = X'_j + V'_{2j}$$

where $\{V'_{1j}\}$ and $\{V'_{2j}\}$ are independent zero-mean Gaussian random variables with $\sigma_{lj}^2 = n(N_l(2\pi j/n)/|\tilde{H}_{lj}|^2, l = 1,2.$

Parallel Channel Model





Channel Decomposition

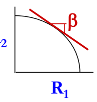
- The n-CGBC thus decomposes to a set of n parallel discrete memoryless degraded broadcast channels with AWGN.
  - Can show that as n goes to infinity, the circular and original channel have the same capacity region
- The capacity region of parallel degraded broadcast channels was obtained by El-Gamal (1980)
  - Optimal power allocation obtained by Hughes-Hartogs('75).
- The power constraint $\sum_{i=0}^{n-1} E[x_i^2] \le nP$ on the original channel is converted by Parseval's theorem to $\sum_{i=0}^{n-1} E[(X_i')^2] \le n^2 P$ on the equivalent channel.

Capacity Region of Parallel Set

- Achievable Rates (no common information)

$$\Big\{R_1 \le .5 \sum_{j:\sigma_{1j}<\sigma_{2j}} \log\left(1+\frac{\alpha_j P_j}{\sigma_{1j}}\right) + .5 \sum_{j:\sigma_{1j}\ge\sigma_{2j}} \log\left(1+\frac{\alpha_j P_j}{(1-\alpha_j)P_j+\sigma_{1j}}\right),$$
$$R_2 \le .5 \sum_{j:\sigma_{1j}<\sigma_{2j}} \log\left(1+\frac{(1-\alpha_j)P_j}{\alpha_j P_j+\sigma_{2j}}\right) + .5 \sum_{j:\sigma_{1j}\ge\sigma_{2j}} \log\left(1+\frac{(1-\alpha_j)P_j}{\sigma_{2j}}\right),$$
$$0 \le \alpha_j \le 1, \sum P_j \le n^2 P\Big\}$$

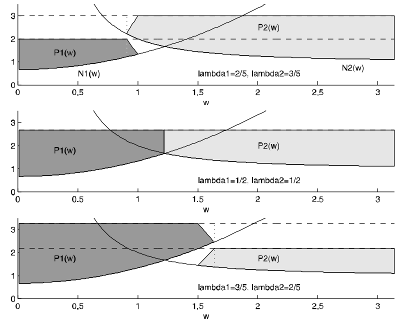
- Capacity Region
  - For $0<\beta\le\infty$ find $\{\alpha_j\}$, $\{P_j\}$ to maximize $R_1+\beta R_2+\lambda \sum P_j$.
  - Let $(R_1^*,R_2^*)_{n,\beta}$ denote the corresponding rate pair.
  - $C_n=\{(R_1^*,R_2^*)_{n,\beta} : 0<\beta\le\infty\}$, $C=\liminf_n \frac{1}{n} C_n$.



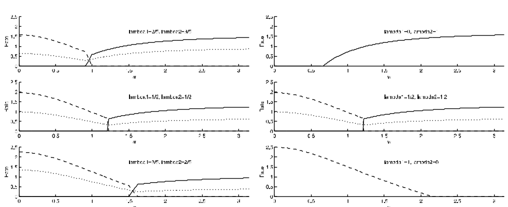
Limiting Capacity Region

$$\Big\{R_1 \le .5 \int_{f:H_1(f)>H_2(f)} \log\left(1+\frac{\alpha(f)P(f)|H_1(f)|^2}{.5N_0}\right) + .5 \int_{f:H_1(f)\le H_2(f)} \log\left(1+\frac{\alpha_j P_j}{(1-\alpha_j)P_j+\sigma_{1j}}\right),$$
$$R_2 \le .5 \int_{f:H_1(f)>H_2(f)} \log\left(1+\frac{(1-\alpha(f))P(f)}{\alpha(f)P(f)+.5N_0/|H_2(f)|^2}\right) + .5 \int_{f:H_1(f)\le H_2(f)} \log\left(1+\frac{(1-\alpha(f))P(f)|H_2(f)|^2}{.5N_0}\right),$$
$$0 \le \alpha(f) \le 1, \quad \int P(f)df \le P\Big\}$$

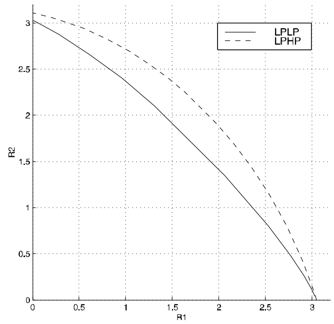
Optimal Power Allocation: Two Level Water Filling



Capacity vs. Frequency



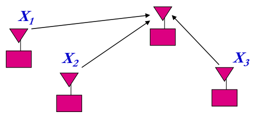
Capacity Region



Multiple Access Channel

- Multiple transmitters
  - Transmitter i sends signal X_i with power P_i
- Common receiver with AWGN of power N_0B
- Received signal:

$$Y = \sum_{i=1}^{M} X_i + N$$



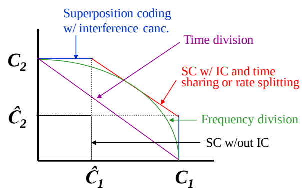
MAC Capacity Region

- Closed convex hull of all $(R_1,\ldots,R_M)$ s.t.

$$\sum_{i\in S} R_i \le B\log\left[1+\sum_{i\in S} P_i / N_0 B\right], \quad \forall S \subseteq \{1,\ldots,M\}$$

  - For all subsets of users, rate sum equals that of 1 superuser with sum of powers from all users
- Power Allocation and Decoding Order
  - Each user has its own power (no power alloc.)
  - Decoding order depends on desired rate point

Two-User Region



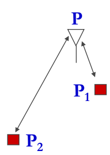
$$C_i = B\log\left[1+\frac{P_i}{N_0B}\right], i=1,2$$

$$\hat{C}_1 = B\log\left[1+\frac{P_1}{N_0B+P_2}\right], \qquad \hat{C}_2 = B\log\left[1+\frac{P_2}{N_0B+P_1}\right],$$

Fading and ISI

- MAC capacity under fading and ISI determined using similar techniques as for the BC
- In fading, can define ergodic, outage, and minimum rate capacity similar as in BC case
  - Ergodic capacity obtained based on AWGN MAC given fixed fading, averaged over fading statistics
  - Outage can be declared as common, or per user
- MAC capacity with ISI obtained by converting to equivalent parallel MAC channels over frequency

Comparison of MAC and BC



- Differences:
  - Shared vs. individual power constraints
  - Near-far effect in MAC
- Similarities:
  - Optimal BC "superposition" coding is also optimal for MAC (sum of Gaussian codewords)
  - Both decoders exploit successive decoding and interference cancellation

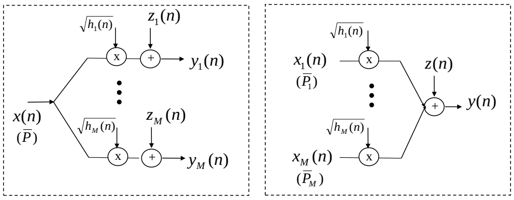
MAC-BC Capacity Regions

- MAC capacity region known for many cases
  - Convex optimization problem
- BC capacity region typically only known for (parallel) degraded channels
  - Formulas often not convex
- Can we find a connection between the BC and MAC capacity regions?

Duality

Dual Broadcast and MAC Channels

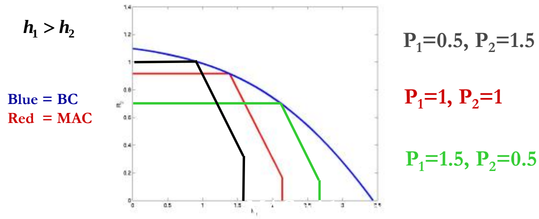
Gaussian BC and MAC with *same* channel gains and *same* noise power at each receiver



Broadcast Channel (BC) Multiple-Access Channel (MAC)

The BC from the MAC

$$C_{MAC}(P_1,P_2;h_1,h_2) \subseteq C_{BC}(P_1+P_2;h_1,h_2)$$

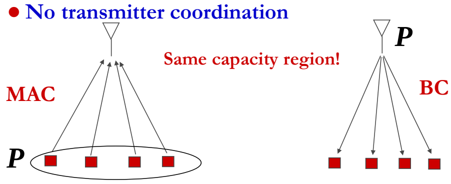


$$\boxed{C_{BC}(P;h_1,h_2) = \bigcup_{0 \le P_1 \le P} C_{MAC}(P_1, P-P_1;h_1,h_2)}$$

Sum-Power MAC

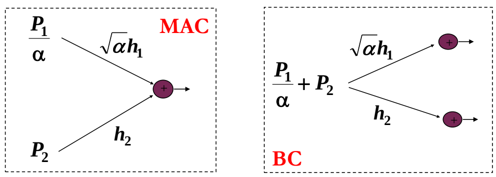
$$\boxed{C_{BC}(P;h_1,h_2) = \bigcup_{0 \le P_1 \le P} C_{MAC}(P_1, P-P_1;h_1,h_2) \equiv C_{MAC}^{Sum}(P;h_1,h_2)}$$

- MAC with sum power constraint
  - Power pooled between MAC transmitters
  - No transmitter coordination

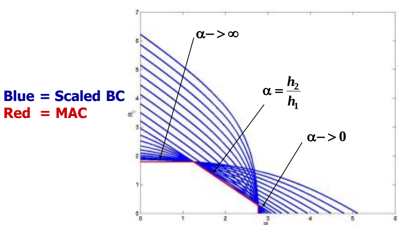


BC to MAC: Channel Scaling

- Scale channel gain by √α, power by 1/α
- MAC capacity region unaffected by scaling
- Scaled MAC capacity region is a subset of the scaled BC capacity region for any α
- MAC region inside scaled BC region for any scaling



The BC from the MAC

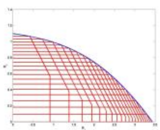


$$\boxed{C_{MAC}(P_1,P_2;h_1,h_2) = \bigcap_{\alpha>0} C_{BC}\left(\frac{P_1}{\alpha}+P_2;\sqrt{\alpha}h_1,h_2\right)}$$

Duality: Constant AWGN Channels

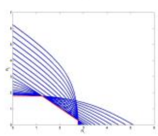
- BC in terms of MAC

$$C_{BC}(P;h_1,h_2) = \bigcup_{0 \le P_1 \le P} C_{MAC}(P_1, P-P_1;h_1,h_2)$$



- MAC in terms of BC

$$C_{MAC}(P_1,P_2;h_1,h_2) = \bigcap_{\alpha>0} C_{BC}\left(\frac{P_1}{\alpha}+P_2;\alpha h_1,h_2\right)$$



What is the relationship between the optimal transmission strategies?

Transmission Strategy Transformations

- Equate rates, solve for powers

$$R_1^M = \log(1+\frac{h_1^2 P_1^M}{h_2 P_2^M + \sigma^2}) = \log(1+\frac{h_1^2 P_1^B}{\sigma^2}) = R_1^B$$

$$R_2^M = \log(1+\frac{h_2^2 P_2^M}{\sigma^2}) = \log(1+\frac{h_2^2 P_2^B}{h_2^2 P_1^B + \sigma^2}) = R_2^B$$

- Opposite decoding order
  - Stronger user (User 1) decoded last in BC
  - Weaker user (User 2) decoded last in MAC

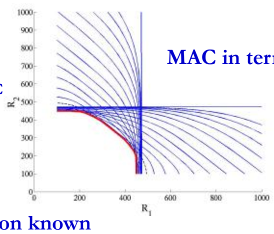
Duality Applies to Different Fading Channel Capacities

- Ergodic (Shannon) capacity: maximum rate averaged over all fading states.
- Zero-outage capacity: maximum rate that can be maintained in all fading states.
- Outage capacity: maximum rate that can be maintained in all nonoutage fading states.
- Minimum rate capacity: Minimum rate maintained in all states, maximize average rate in excess of minimum

Explicit transformations between transmission strategies

Duality: Minimum Rate Capacity

Blue = Scaled BC
Red = MAC



MAC in terms of BC

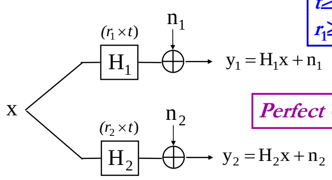
- BC region known
- MAC region can only be obtained by duality

What other capacity regions can be obtained by duality?

Broadcast MIMO Channels

Broadcast MIMO Channel

t≥1 TX antennas
r_1≥1, r_2≥1 RX antennas



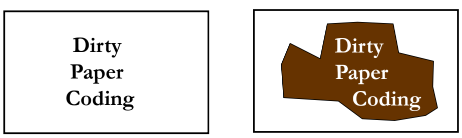
Perfect CSI at TX and RX

$$n_1 \sim N(0, I_{r_1}) \quad n_2 \sim N(0, I_{r_2})$$

Non-degraded broadcast channel

Dirty Paper Coding (Costa'83)

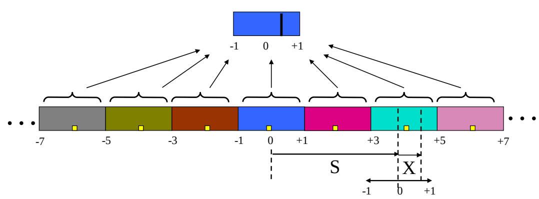
- Basic premise
  - If the interference is known, channel capacity same as if there is no interference
  - Accomplished by cleverly distributing the writing (codewords) and coloring their ink
  - Decoder must know how to read these codewords



Clean Channel Dirty Channel

Modulo Encoding/Decoding

- Received signal Y=X+S, -1≤X≤1
  - S known to transmitter, not receiver
- Modulo operation removes the interference effects
  - Set X so that $\lfloor Y \rfloor_{[-1,1]}$=desired message (e.g. 0.5)
  - Receiver demodulates modulo [-1,1]



Capacity Results

- Non-degraded broadcast channel
  - Receivers not necessarily "better" or "worse" due to multiple transmit/receive antennas
  - Capacity region for general case unknown
- Pioneering work by Caire/Shamai (Allerton'00):
  - Two TX antennas/two RXs (1 antenna each)
  - Dirty paper coding/lattice precoding (achievable rate)
    - Computationally very complex
  - MIMO version of the Sato upper bound
  - Upper bound is achievable: capacity known!

Dirty-Paper Coding (DPC) for MIMO BC

- Coding scheme:
  - Choose a codeword for user 1
  - Treat this codeword as interference to user 2
  - Pick signal for User 2 using "pre-coding"
- Receiver 2 experiences no interference:

$$R_2 = \log(\det(I + H_2 \Sigma_2 H_2^T))$$

- Signal for Receiver 2 interferes with Receiver 1:

$$R_1 = \log\left(\frac{\det(I + H_1(\Sigma_1 + \Sigma_2)H_1^T)}{\det(I + H_1 \Sigma_2 H_1^T)}\right)$$

- Encoding order can be switched
- DPC optimization highly complex

Does DPC achieve capacity?

- DPC yields MIMO BC achievable region.
  - We call this the dirty-paper region
- Is this region the capacity region?
- We use duality, dirty paper coding, and Sato's upper bound to address this question
- First we need MIMO MAC Capacity

MIMO MAC Capacity

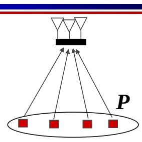
- MIMO MAC follows from MAC capacity formula

$$C_{MAC}(P_1,\dots,P_k) = \bigcup \left\{ (R_1,\dots,R_k) : \sum_{k\in S} R_k \le \log_2 \det\left[I + \sum_{k\in S} H_k Q_k H_k^H\right], \quad \forall S \subseteq \{1,\dots,K\} \right\}$$

- Basic idea same as single user case
  - Pick some subset of users
  - The sum of those user rates equals the capacity as if the users pooled their power
- Power Allocation and Decoding Order
  - Each user has its own power (no power alloc.)
  - Decoding order depends on desired rate point

MIMO MAC with sum power

- MAC with sum power:
  - Transmitters code independently
  - Share power



$$C_{MAC}^{Sum}(P) = \bigcup_{0 \le P_1 \le P} C_{MAC}(P_1, P - P_1)$$

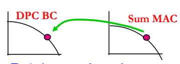
- Theorem: Dirty-paper BC region equals the dual sum-power MAC region

$$\boxed{C_{BC}^{DPC}(P) = C_{MAC}^{Sum}(P)}$$

Transformations: MAC to BC

- Show any rate achievable in sum-power MAC also achievable with DPC for BC:

$$C_{BC}^{DPC}(P) \supseteq C_{MAC}^{Sum}(P)$$

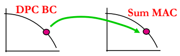


- A sum-power MAC strategy for point ($R_1,\dots R_N$) has a given input covariance matrix and encoding order
- We find the corresponding PSD covariance matrix and encoding order to achieve ($R_1,\dots,R_N$) with DPC on BC
  - The rank-preserving transform "flips the effective channel" and reverses the order
  - Side result: beamforming is optimal for BC with 1 Rx antenna at each mobile

Transformations: BC to MAC

- Show any rate achievable with DPC in BC also achievable in sum-power MAC:

$$C_{BC}^{DPC}(P) \subseteq C_{MAC}^{Sum}(P)$$



- We find transformation between optimal DPC strategy and optimal sum-power MAC strategy
  - "Flip the effective channel" and reverse order

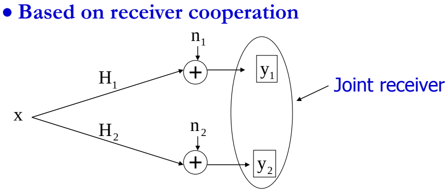
Computing the Capacity Region

$$\boxed{C_{BC}^{DPC}(P) = C_{MAC}^{Sum}(P)}$$

- Hard to compute DPC region (Caire/Shamai'00)
- "Easy" to compute the MIMO MAC capacity region
  - Obtain DPC region by solving for sum-power MAC and applying the theorem
  - Fast iterative algorithms have been developed
  - Greatly simplifies calculation of the DPC region and the associated transmit strategy

Sato Upper Bound on the BC Capacity Region

- Based on receiver cooperation



- BC sum rate capacity ≤ Cooperative capacity

$$C_{BC}^{\text{sumrate}}(P,H) \le \max_{\Sigma_x} \frac{1}{2}\log|I + H\Sigma_x H^T|$$

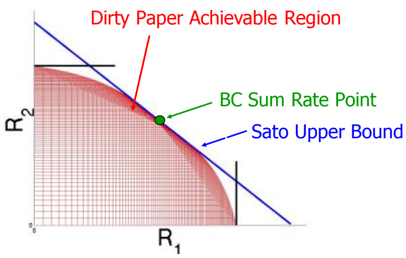
The Sato Bound for MIMO BC

- Introduce noise correlation between receivers
- BC capacity region unaffected
  - Only depends on noise marginals
- Tight Bound (Caire/Shamai'00)
  - Cooperative capacity with <u>worst-case</u> noise correlation

$$C_{BC}^{\text{sumrate}}(P,H) \le \inf_{\Sigma_z} \max_{\Sigma_x} \frac{1}{2}\log|I + \Sigma_z^{-1/2} H\Sigma_x H^T \Sigma_z^{-1/2}|$$

- Explicit formula for worst-case noise covariance
- By Lagrangian duality, cooperative BC region equals the sum-rate capacity region of MIMO MAC

MIMO BC Capacity Bounds



Does the DPC region equal the capacity region?

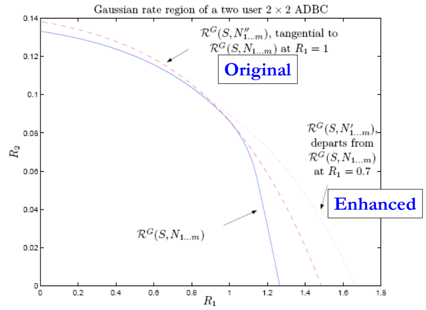
Full Capacity Region

- DPC gives us an achievable region
- Sato bound only touches at sum-rate point
- Bergman's entropy power inequality is not a tight upper bound for nondegraded broadcast channel
- A tighter bound was needed to prove DPC optimal
  - It had been shown that if Gaussian codes optimal, DPC was optimal, but proving Gaussian optimality was open.
- Breakthrough by Weingarten, Steinberg and Shamai
  - Introduce notion of <u>enhanced channel</u>, applied Bergman's converse to it to prove DPC optimal for MIMO BC.

Enhanced Channel Idea

- The aligned and degraded BC (AMBC)
  - Unity matrix channel, noise innovations process
  - Limit of AMBC capacity equals that of MIMO BC
  - Eigenvalues of some noise covariances go to infinity
  - Total power mapped to covariance matrix constraint
- Capacity region of AMBC achieved by Gaussian superposition coding and successive decoding
  - Uses entropy power inequality on enhanced channel
  - Enhanced channel has less noise variance than original
  - Can show that a power allocation exists whereby the enhanced channel rate is inside original capacity region
- By appropriate power alignment, capacities equal

Illustration



Main Points

- Shannon capacity gives fundamental data rate limits for multiuser wireless channels
- Fading multiuser channels optimize at each channel instance for maximum average rate
- Outage capacity has higher (fixed) rates than with no outage.
- OFDM is near optimal for broadcast channels with ISI
- Duality connects BC and MAC channels
  - Used to obtain capacity of one from the other
- Capacity of broadcast MIMO channel obtained using duality and the notion of an enhanced channel