## Problem Set \#5

Due May 10
This week the homework is just review problems for the midterm next Wednesday. We'll post solutions Monday night, so you will have time on Tuesday to get answers to any questions. You still have to hand the homework in, but you will get full credit even if you don't do all the problems.

There is no lab for this week.

1. Find the inverse Laplace transforms of the following functions
(a) $\frac{s^{2}}{(s+1)^{2}}$
(b) $\frac{4}{s^{3}+4 s}$
2. A system is described by the differential equation

$$
y^{\prime \prime}(t)+2 y^{\prime}(t)+10 y(t)=x^{\prime \prime \prime}(t)+2 x^{\prime \prime}(t)+x^{\prime}(t)
$$

where the initial conditions are all zero, $y^{\prime \prime}(0)=0, y^{\prime}(0)=0$, and $y(0)=0$. The input is the unit step

$$
x(t)=u(t) .
$$

Find $y(t)$. Note that $x\left(0^{-}\right)=x^{\prime}\left(0^{-}\right)=x^{\prime \prime}\left(0^{-}\right)=0$, since the unit step is zero to the left of the origin.
3. A system is described by the differential equation

$$
y^{\prime \prime}(t)+6 y^{\prime}(t)+13 y(t)=x^{\prime \prime \prime}(t)+6 x^{\prime \prime}(t)+17 x^{\prime}(t)
$$

where the initial conditions are all zero, $y^{\prime \prime}(0)=0, y^{\prime}(0)=0$, and $y(0)=0$. The input is the unit step

$$
x(t)=u(t) .
$$

Find $y(t)$.
4. A system has a transfer function

$$
H(s)=\frac{s^{3}+5 s^{2}+10 s+5}{s^{2}+4 s+5}
$$

Find the output of the system $y(t)$ if the input is the unit step

$$
x(t)=u(t) .
$$

## 5. Systems and Stability

a) The system equation for a system is

$$
y^{\prime \prime}(t)+y^{\prime}(t)-2 y(t)=x(t)-x^{\prime}(t)
$$

Find the transfer function $H(s)$. Factor the denominator, and simplify your answer.
b) Is this system stable?
c) Find the step response when $y\left(0^{-}\right)=y^{\prime}\left(0^{-}\right)=0$.
d) Find $y(t)$ if $y(0)=0, y^{\prime}\left(0^{-}\right)=1$, and the input is zero.
6. DFT

Determine whether these statements are true or false, and provide a brief supporting argument.
a) A periodic signal $x[n]$ has a period $N$. It is convolved with a non-zero sequence $h[n]$ to create an output sequence $y[n]$

$$
y[n]=x[n] * h[n] .
$$

The assertion is that $y[n]$ is either zero, or is also periodic, with period no less than $N$.
b) We are given two N -sample sequences $x[n]$ and $y[n]$, and we know that $y[n]$ is the circular convolution of $x[n]$ with an unknown sequence $h[n]$

$$
y[n]=x[n] \mathbb{1} h[n]
$$

The assertion is that given $y[n]$ and $x[n]$, we can determine $h[n]$.
7. DFT An N-sample possibly complex signal $x[n]$ is bounded, so that $|x[n]|<1$ for all $n$.
a) What is the largest value possible for $|X[k]|$, the magnitude of the DFT of $x[n]$ ?
b) Find an expression for all of the $x[n]$ sequences which achieve this maximum.
8. We want to implement the linear convolution of a 15,600 -point sequence with an FIR impulse response that is 101 points long. The convolution is to be implemented by using DFTs and inverse DFTs of length 256.

In class, we learned about a method called overlap-and-add that allows us to use the DFT to take a linear convolution. If we use this method to convolve the long sequence with the FIR filter response, how many DFTs and inverse DFTs are required to convolve the entire sequence?
9. Discrete Time Filters You had an ideal lowpass filter $H_{1}\left(e^{j \Omega}\right)$ with cutoff frequency $\Omega_{c}$ whose frequency response is plotted below.



Unfortunately, we have lost every other sample of the filter's impulse response. We will consider 3 replacement filters.
a) Solution (a): throw away the missing samples.

$$
h_{a}[n]=h_{1}[2 n]
$$

Plot the frequency response $H_{a}\left(e^{j \Omega}\right)$ of this new filter.
b) Solution (b): replace the missing samples with zeros.

$$
h_{b}[n]= \begin{cases}h_{1}[n] & n \text { even } \\ 0 & n \text { odd }\end{cases}
$$

Plot the frequency response $H_{b}\left(e^{j \Omega}\right)$ of this new filter.
c) Solution (c): linearly interpolate to replace the missing samples.

$$
h_{c}[n]= \begin{cases}h_{1}[n] & n \text { even } \\ \frac{1}{2}\left(h_{1}[n-1]+h_{1}[n+1]\right) & n \text { odd }\end{cases}
$$

Sketch the frequency response $H_{c}\left(e^{j \Omega}\right)$ of this new filter.
10. Input Signal Recovery.

An instrument with an impulse response

$$
h(t)=e^{-t}-e^{-2 t}
$$

is used to measure a signal $x(t)$, producing a corrupted measurement $y(t)$ that is the convolution

$$
y(t)=\int_{0}^{t} h(\tau) x(t-\tau) d \tau
$$

This means that the measurement $y$ is a "smoothed" or "smeared" version of the input signal $x(t)$. The smoothing function $h$ is plotted below.


You can assume that $x$ and $y$ are smooth, i.e., they have derivatives of all orders. You can also assume that all initial conditions are zero.
(a) Find a transfer function $H_{\text {inv }}(s)$ of a system that takes $y(t)$ and recovers the original input $x(t)$ :
(b) Find the impulse response $h_{i n v}(t)$ of this system.

## 11. Simplified Tracer Uptake Dynamics

Background: A common problem in the study of biological and chemical systems concerns two groups of molecules in separate environments, or compartments, that are in exchange with each other. A tracer can be introduced into one compartment. By studying the dynamics of the tracer concentration in the second compartment, or elimination from the first compartment, we can infer the rate of exchange between the two.

Here, one compartment is the capillary bed and another is the surrounding tissue. We make the simplifying assumption that the exchange rate is proportional to the concentration difference, and that the two compartments have the same volume (which is not actually true). The tracer also washes out of the capillary bed, again at a rate proportional to its concentration. This can be illustrated schematically as:


Here $y_{1}$ is the concentration in the capillary bed in $\mathrm{mg} / 1, y_{2}$ is the concentration in the surrounding tissue, $R$ is the rate of tracer exchange between the two, and $C$ is the rate at which the tracer is cleared from the capillary bed. In medical diagnostic imaging, cancerous tissue is often identified by its higher exchange rate.
The Problem: The rate of change of tracer concentration in the two compartments is given by

$$
\begin{aligned}
y_{1}^{\prime}(t) & =x(t)+R\left(y_{2}(t)-y_{1}(t)\right)-C y_{1}(t), \\
y_{2}^{\prime}(t) & =R\left(y_{1}(t)-y_{2}(t)\right) .
\end{aligned}
$$

The initial conditions are $y_{1}(0)=0$ and $y_{2}(0)=0$, meaning there is no tracer in the system initially. The coefficients $R$ and $C$ are both positive real numbers.
(a) Eliminate $y_{1}$ and derive a differential equation for $y_{2}$.
(b) Assume the exchange rate is $R=3$, the clearance rate is $C=8$, and that the input $x(t)$ is an impulse $\delta(t)$. Solve for $y_{2}(t)$, and sketch the result.

