Lecture 5
Epipolar Geometry

Professor Silvio Savarese

Computational Vision and Geometry Lab

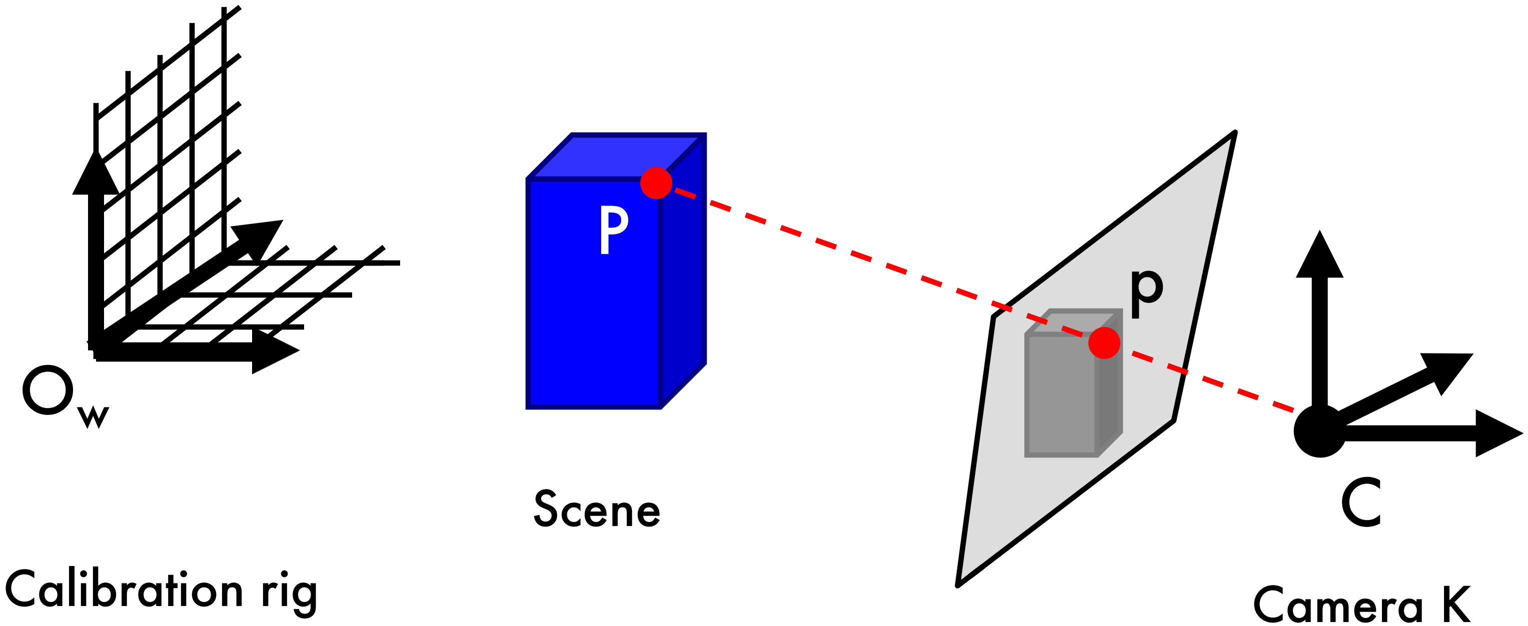
Lecture 5

Epipolar Geometry

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F
- Examples

Reading: [AZ] Chapter: 4 “Estimation – 2D perspective transformations
Chapter: 9 “Epipolar Geometry and the Fundamental Matrix Transformation”
Chapter: 11 “Computation of the Fundamental Matrix F”
[FP] Chapter: 7 “Stereopsis”
Chapter: 8 “Structure from Motion”

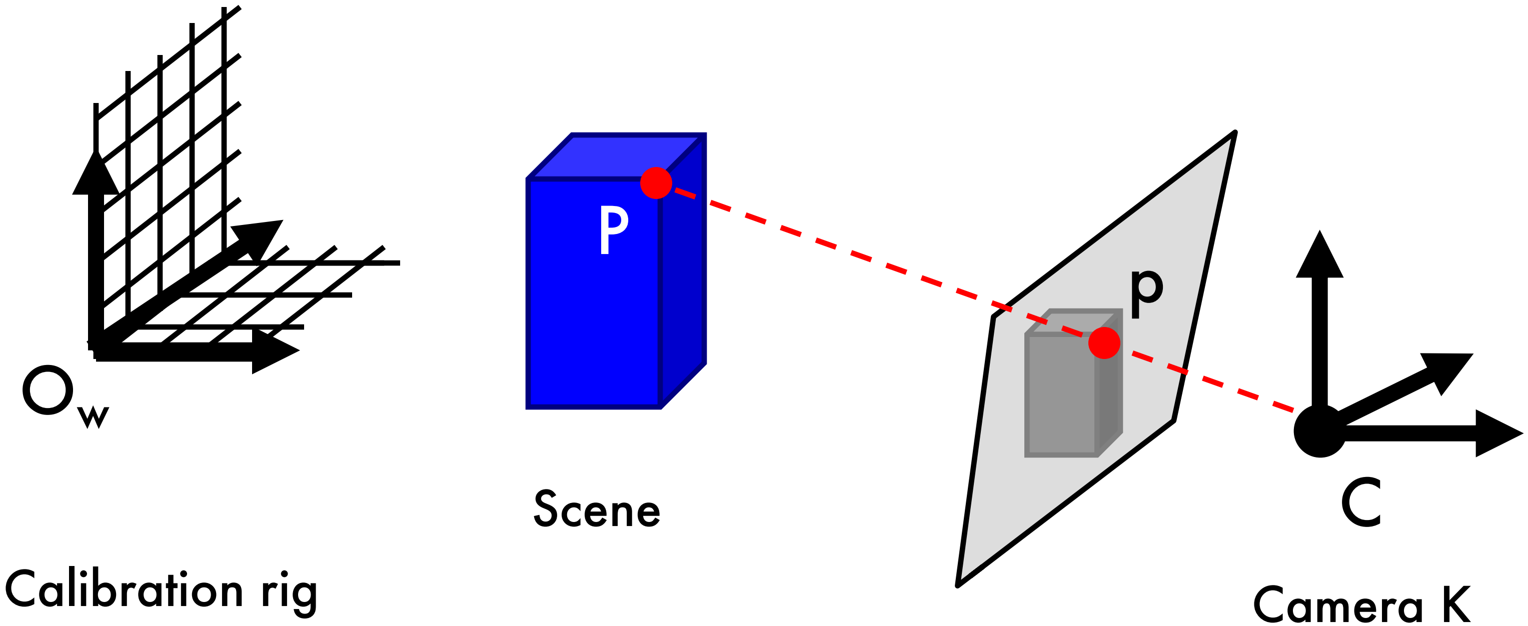
Recovering structure from a single view



From calibration rig	→	location/pose of the rig, K
From points and lines at infinity + orthogonal lines and planes	→	structure of the scene, K

Knowledge about scene (point correspondences, geometry of lines & planes, etc...

Recovering structure from a single view



Why is it so difficult?

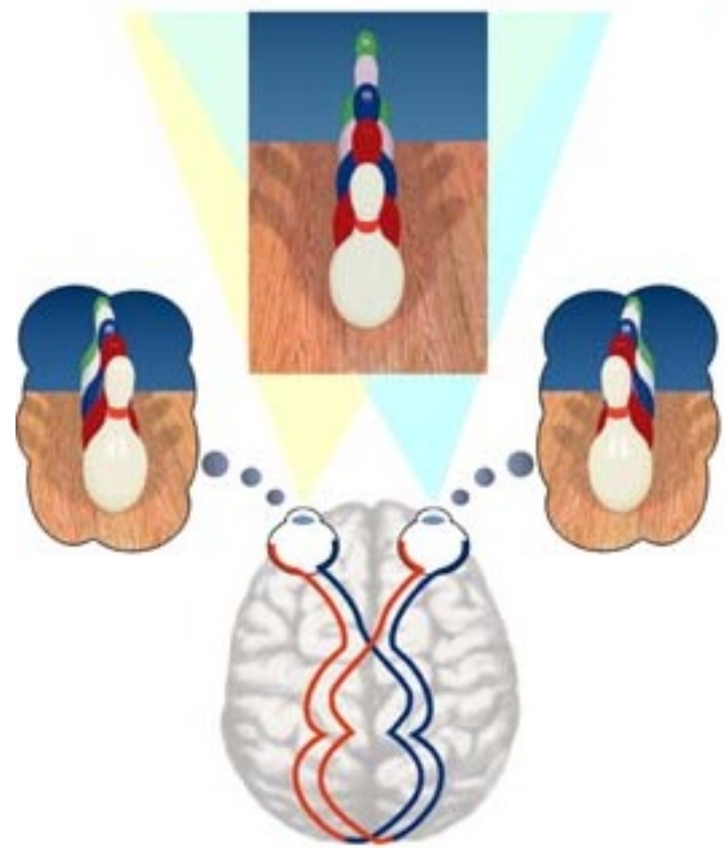
Intrinsic ambiguity of the mapping from 3D to image (2D)

Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)

Courtesy slide S. Lazebnik

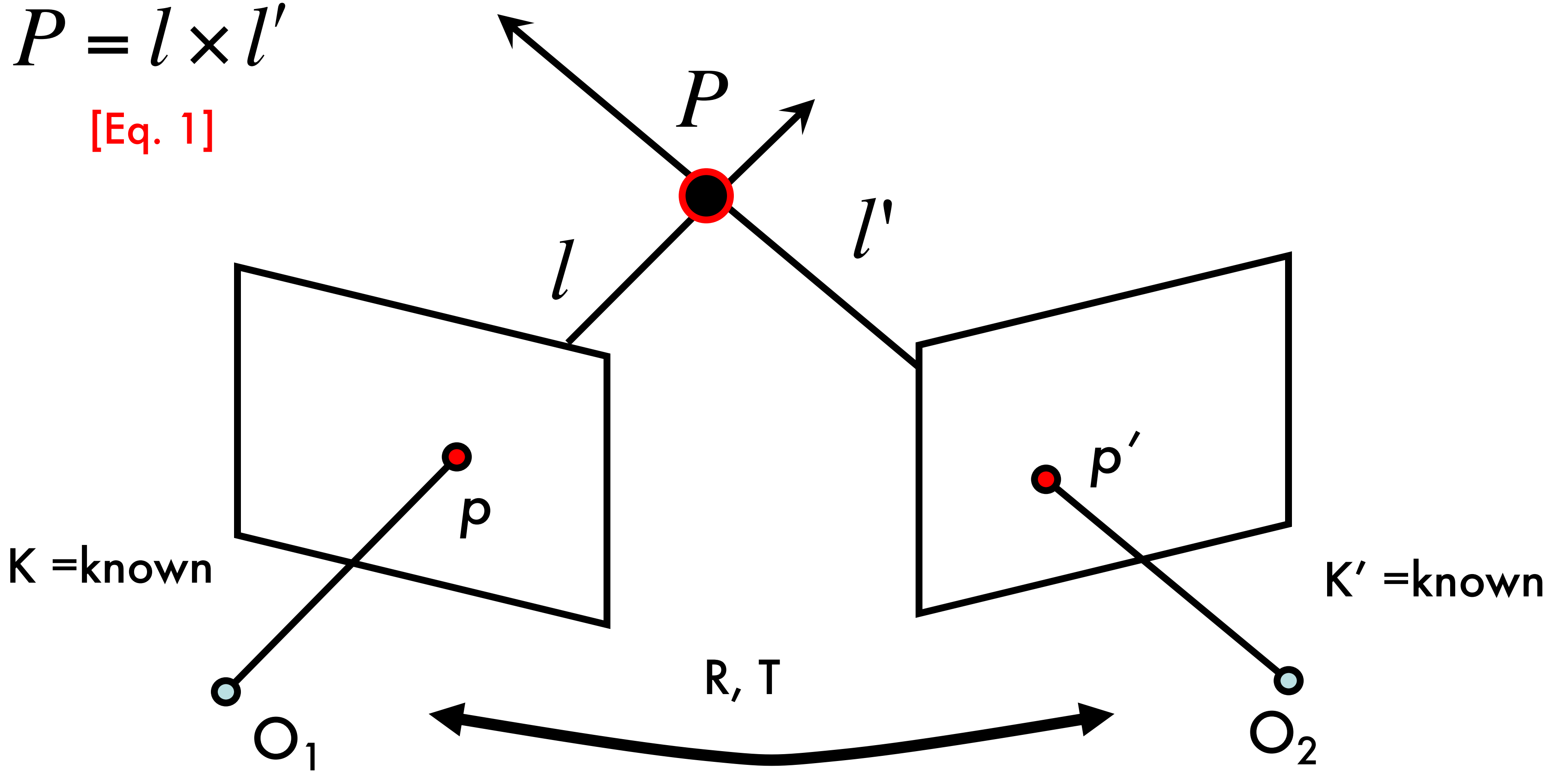
Two eyes help!



Two eyes help!

$$P = l \times l'$$

[Eq. 1]

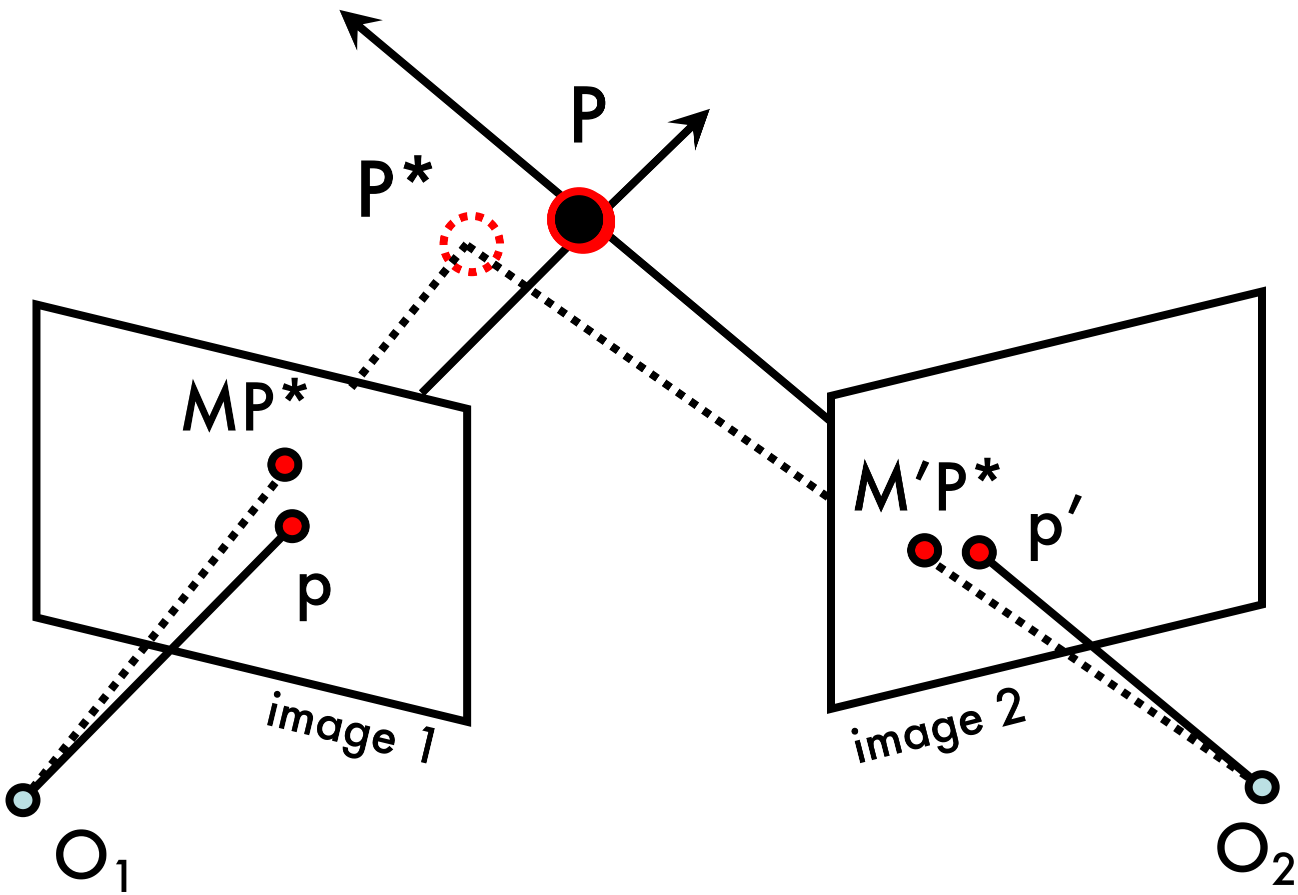


This is called triangulation

Triangulation

- Find P* that minimizes

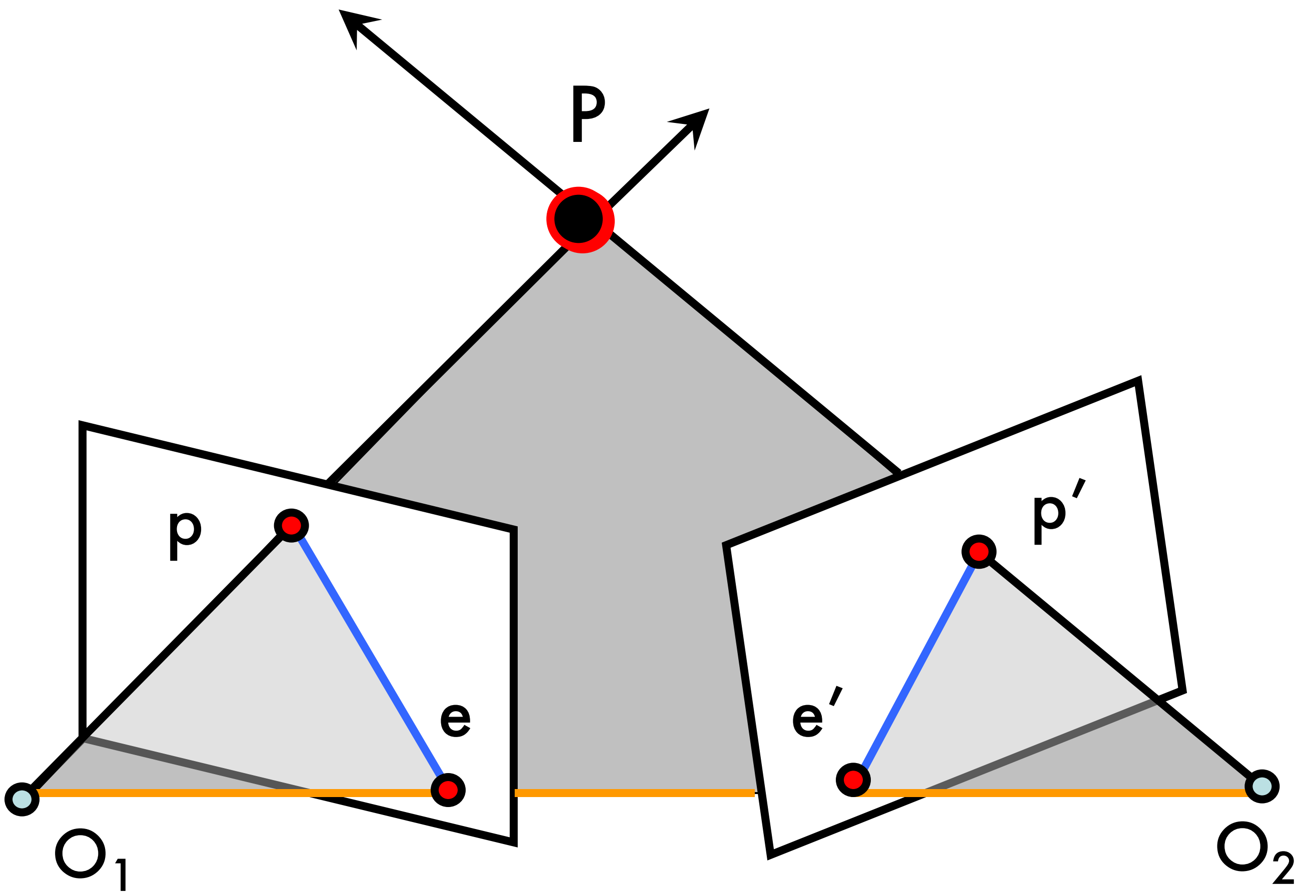
$$d(p, M\ P^*) + d(p', M'P^*)$$ [Eq. 2]



Multi (stereo)-view geometry

- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.
- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.
- **Correspondence:** Given a point **p** in one image, how can I find the corresponding point **p′** in another one?

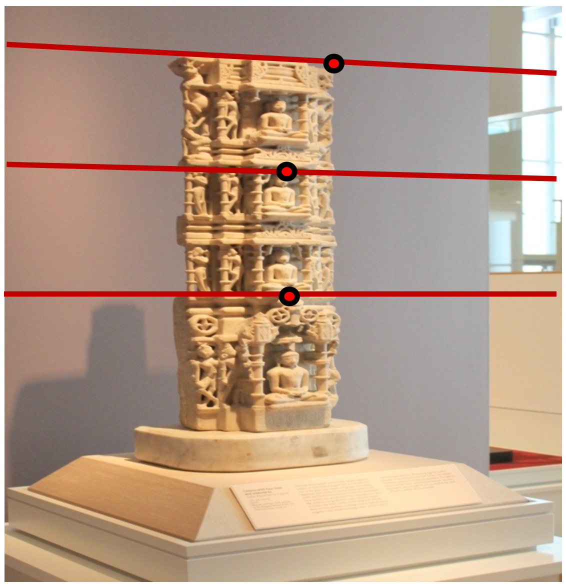
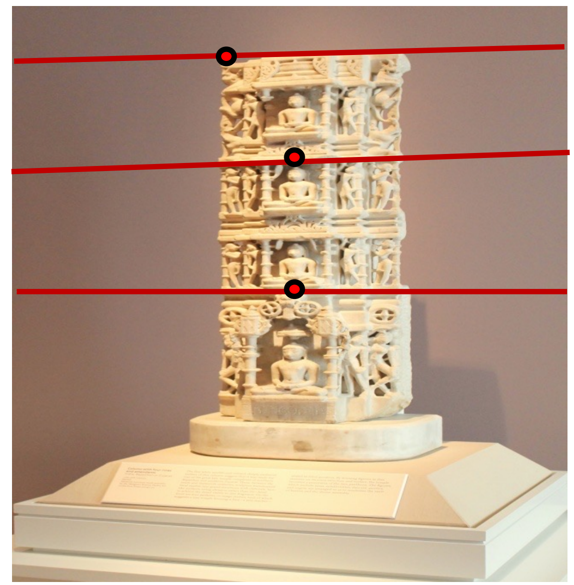
Epipolar geometry



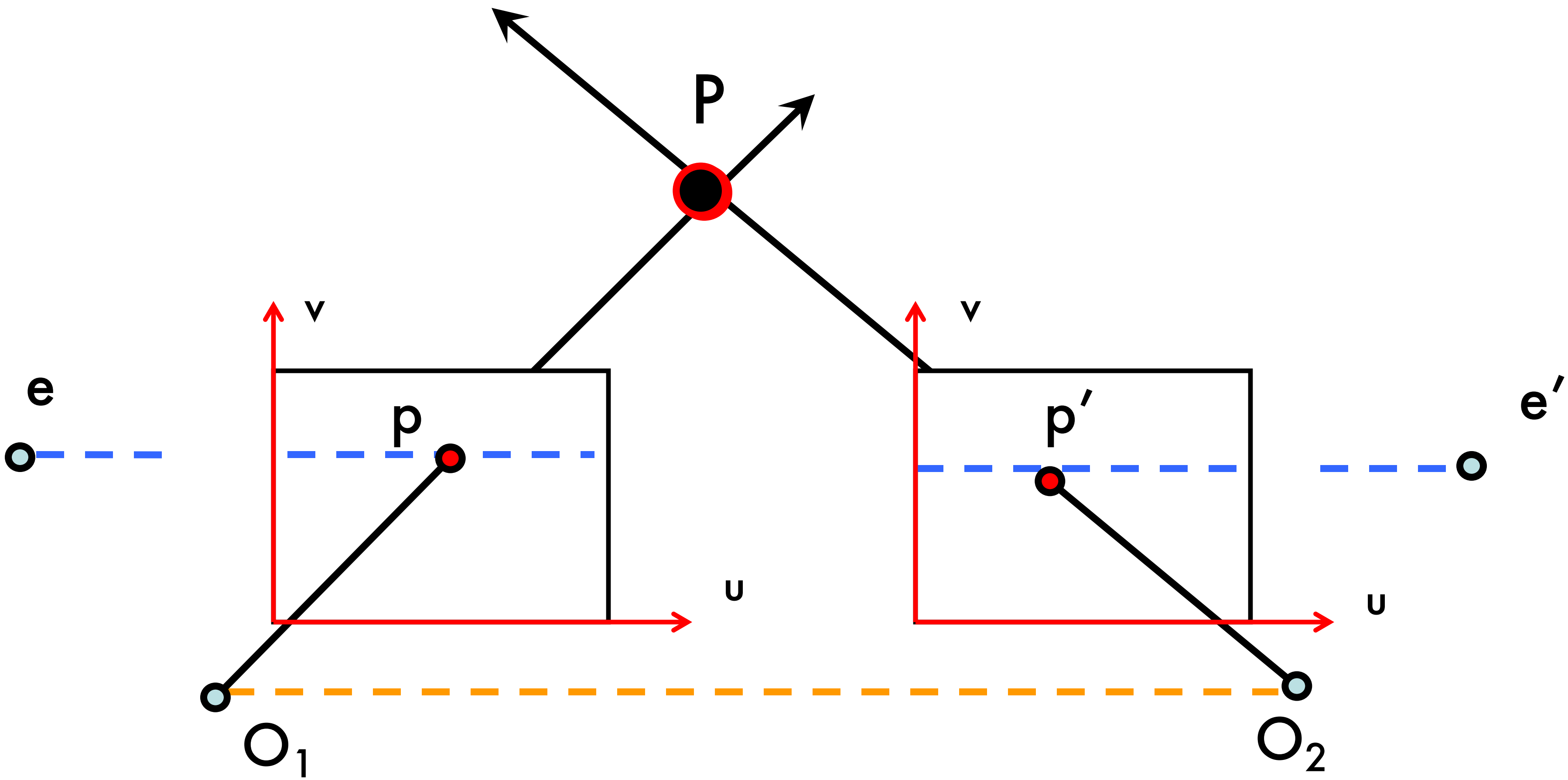
- Epipolar Plane
- Baseline
- Epipolar Lines
- Epipoles e, e′

  = intersections of baseline with image planes
  = projections of the other camera center

Example of epipolar lines

Example: Parallel image planes

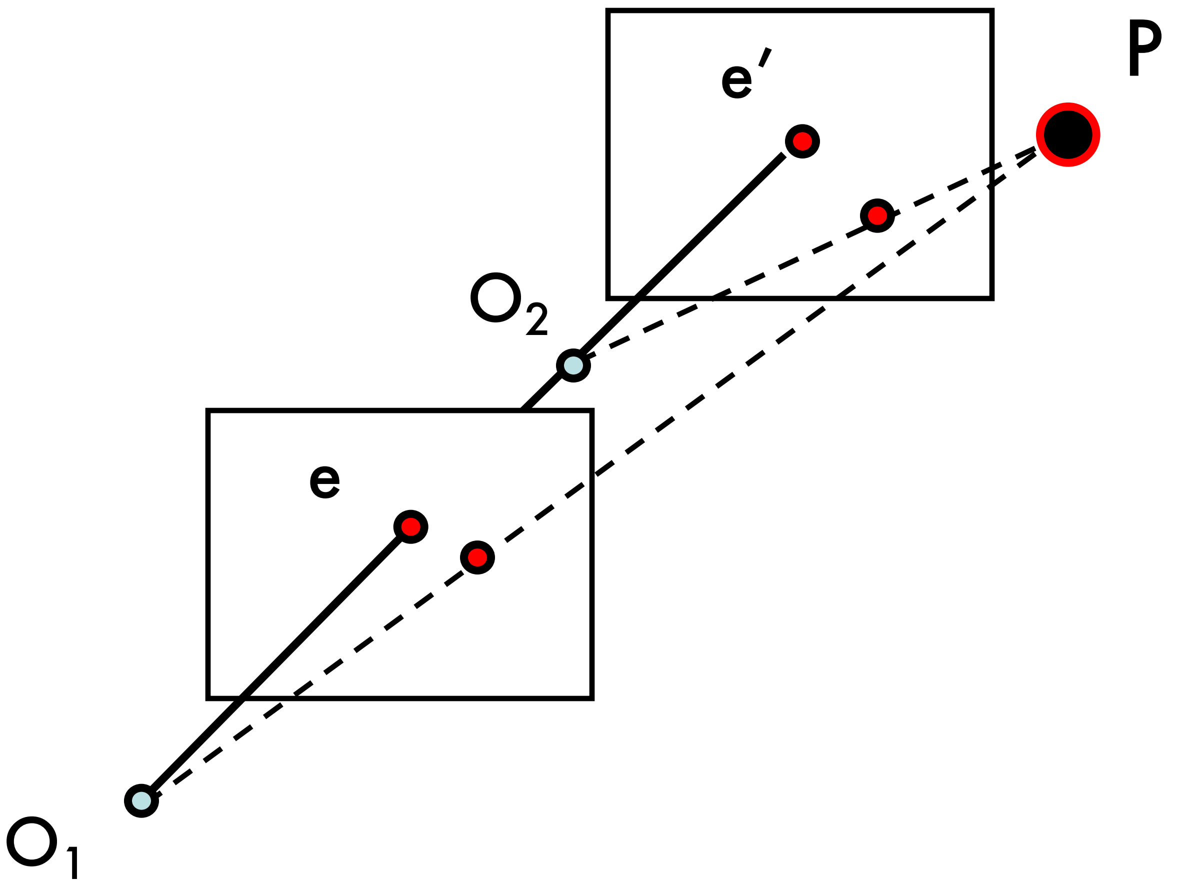


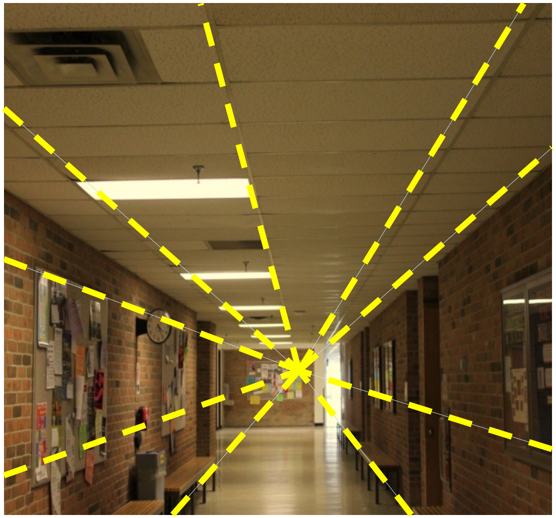
- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to u axis

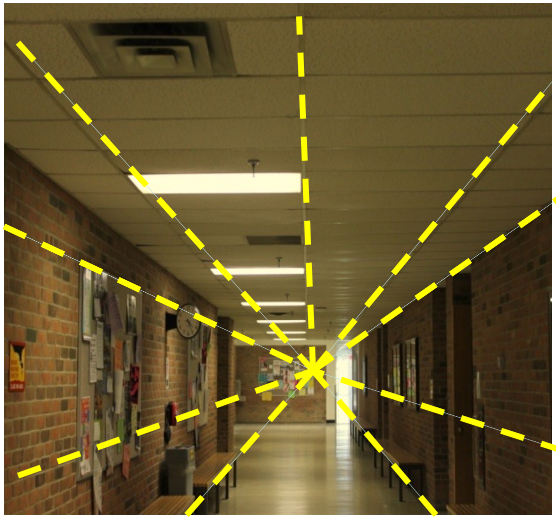
Example: Parallel Image Planes

Example: Forward translation

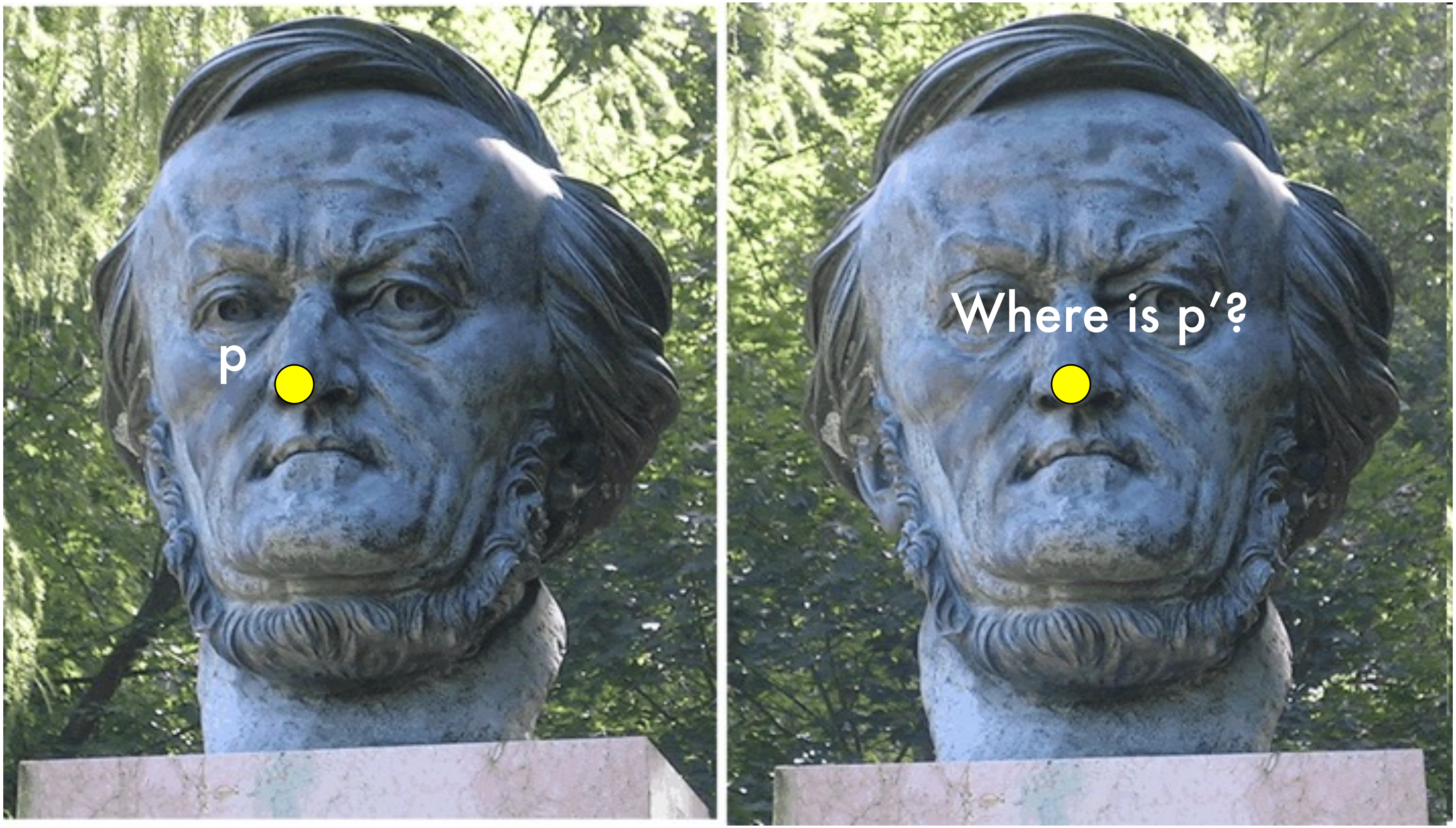






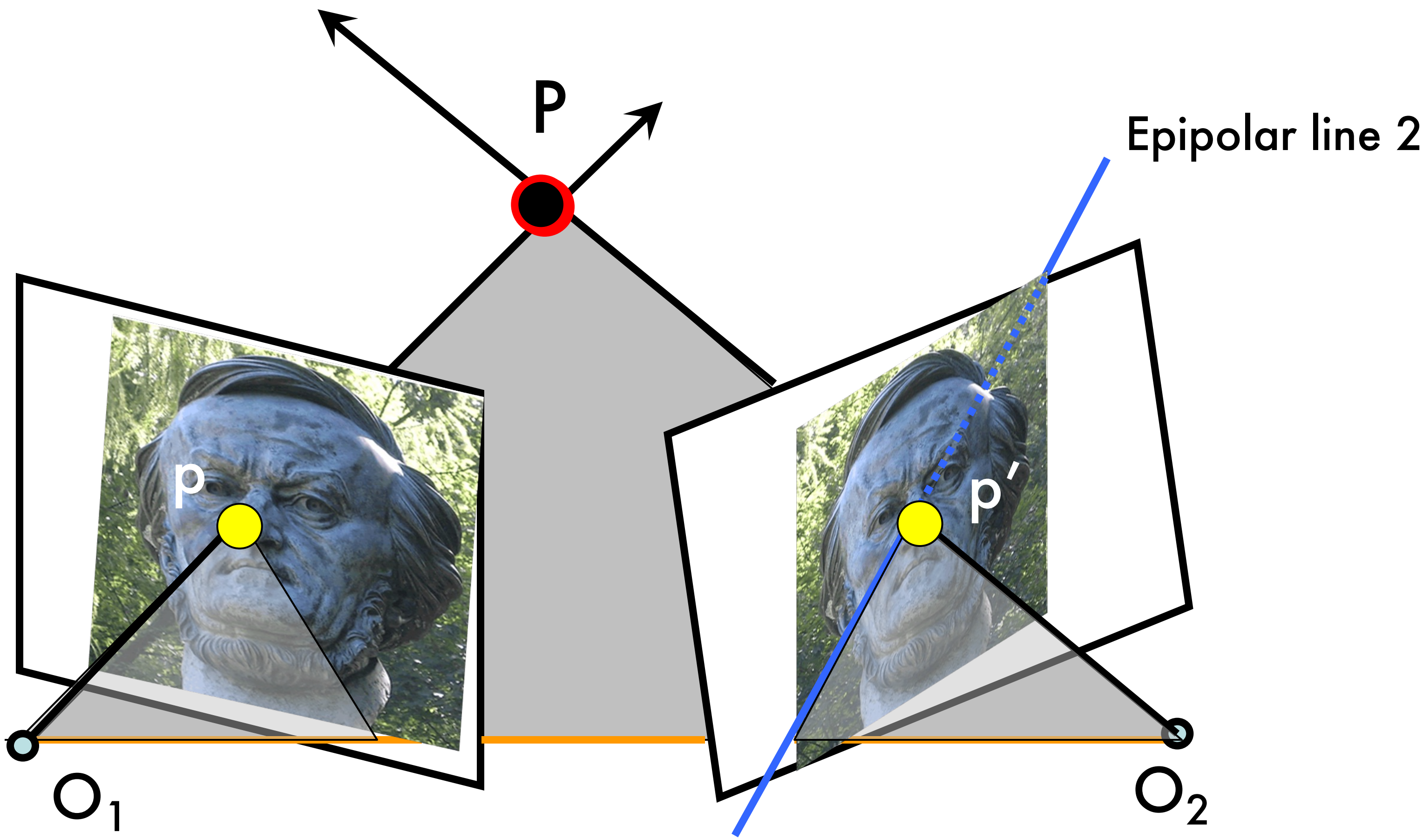
- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)

Epipolar Constraint

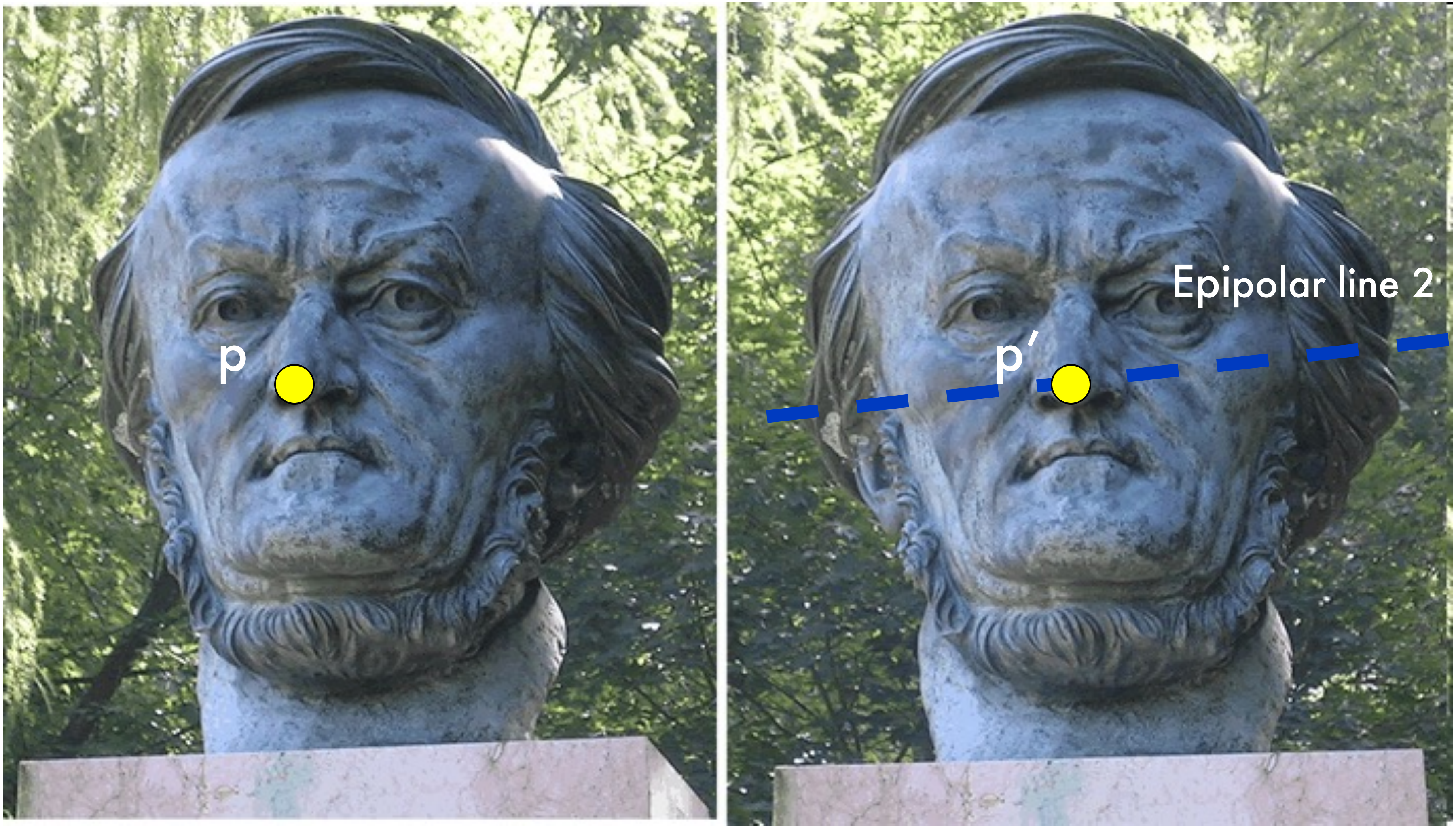


- Two views of the same object
- Given a point on left image, how can I find the corresponding point on right image?

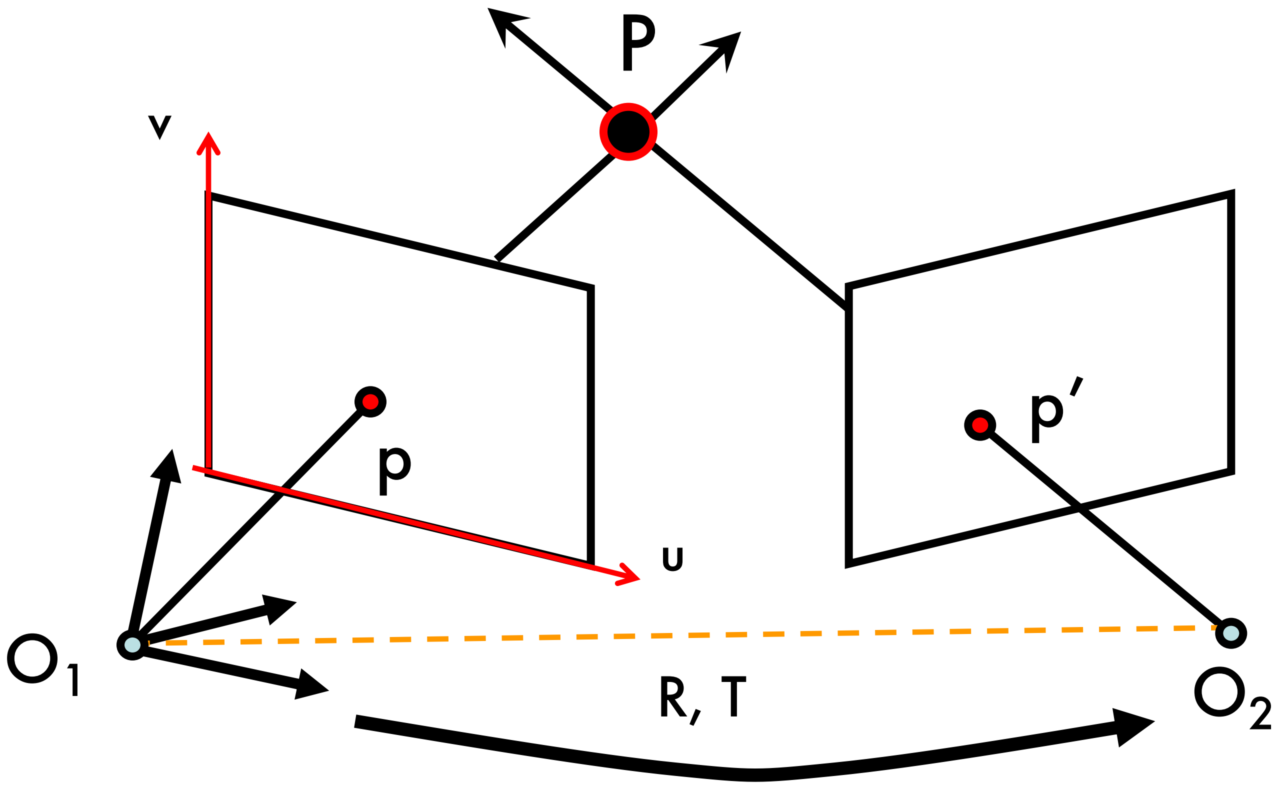
Epipolar geometry



Epipolar Constraint



Epipolar Constraint



$$M = K\begin{bmatrix} I & 0 \end{bmatrix}$$

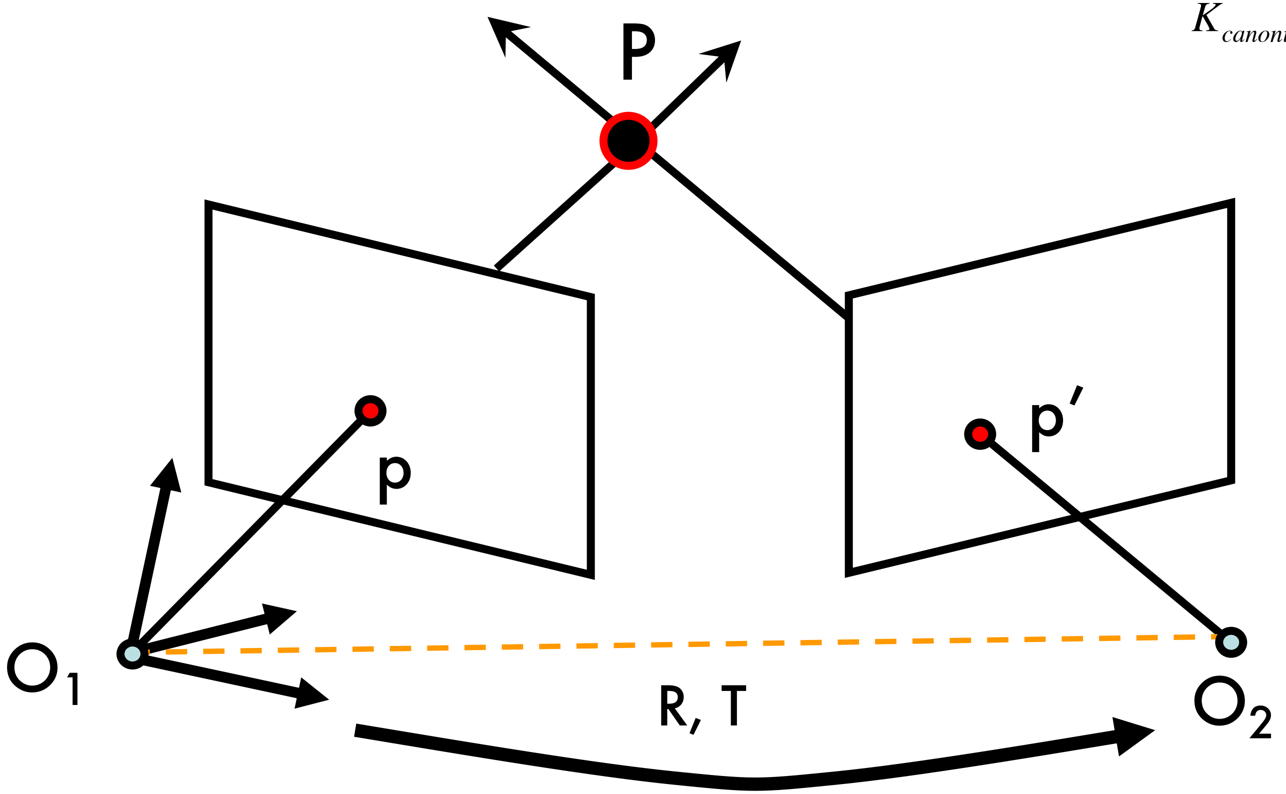
$$M' = K'\begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$M\ P = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = p$$ [Eq. 3]

$$M'\ P = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = p'$$ [Eq. 4]

Epipolar Constraint

$$K_{canonical} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$M = K\begin{bmatrix} I & 0 \end{bmatrix}$$

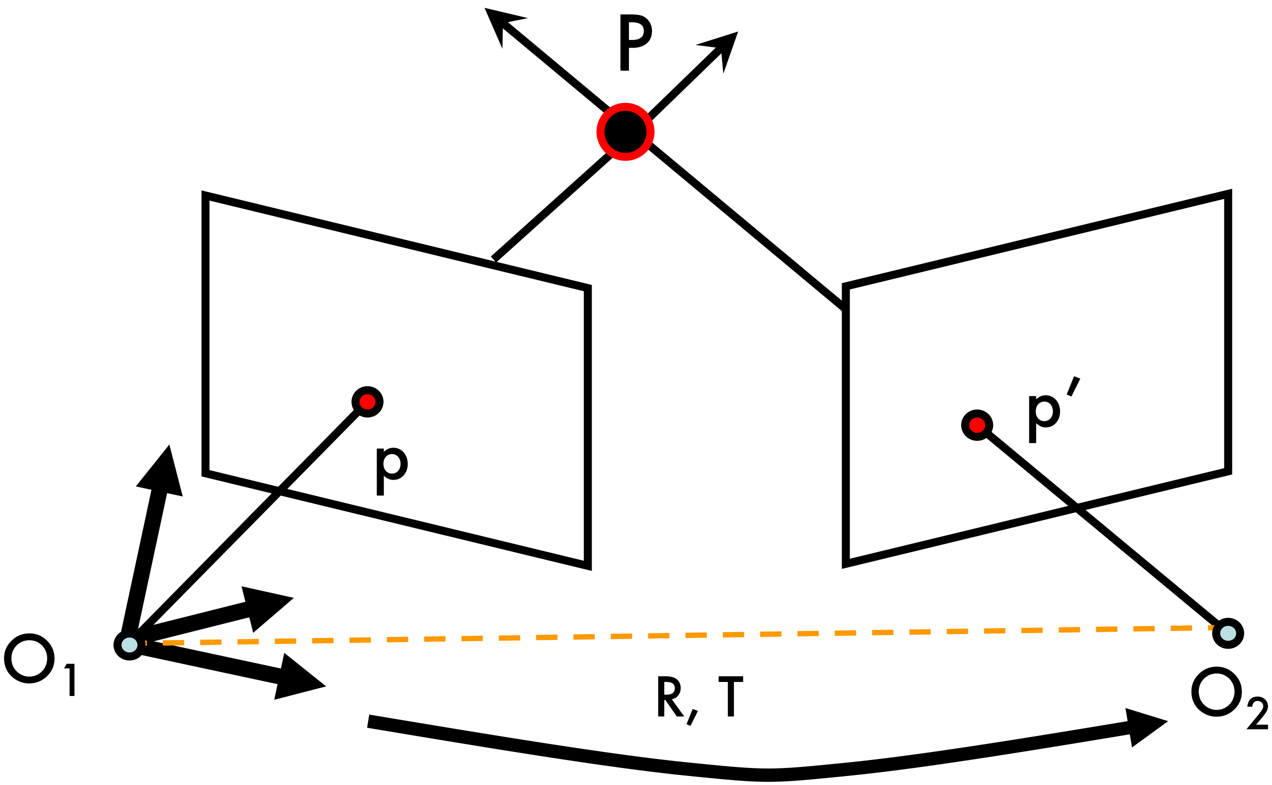
K = K′ are known
(canonical cameras)

$$M' = K'\begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$M = \begin{bmatrix} I & 0 \end{bmatrix}$$ [Eq. 5]

$$M' = \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$ [Eq. 6]

Epipolar Constraint



$$p^T \cdot [T \times (R\, p')] = 0 \rightarrow p^T \cdot [T_\times] \cdot R\, p' = 0$$

[Eq. 8]

[Eq. 9]

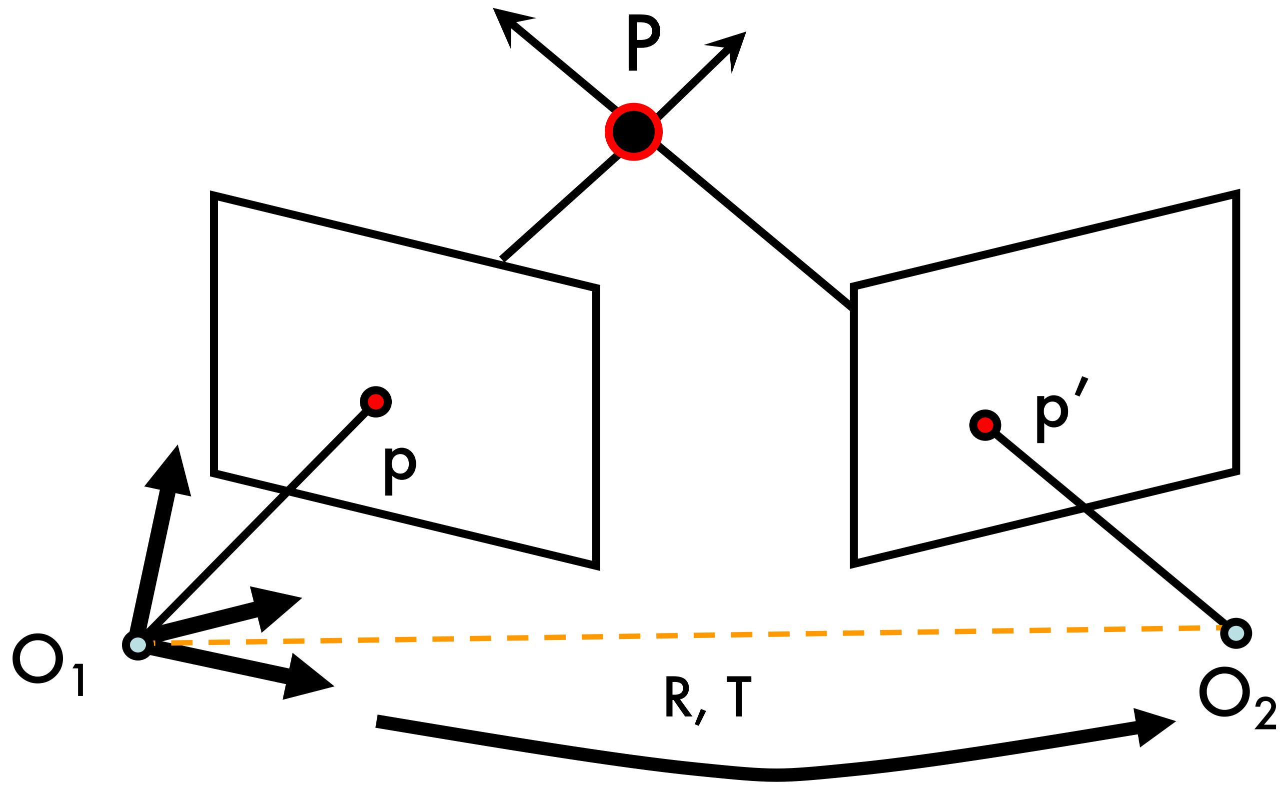
Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

$$\mathbf{a} = [\mathrm{a_x} \;\; \mathrm{a_y} \;\; \mathrm{a_z}]^\mathrm{T}$$

$$\mathbf{b} = [\mathrm{b_x} \;\; \mathrm{b_y} \;\; \mathrm{b_z}]^\mathrm{T}$$

Epipolar Constraint



$$p^T \cdot \left[T \times (R\, p')\right] = 0 \rightarrow p^T \cdot \boxed{[T_\times] \cdot R}\, p' = 0$$

[Eq. 8] [Eq. 9]

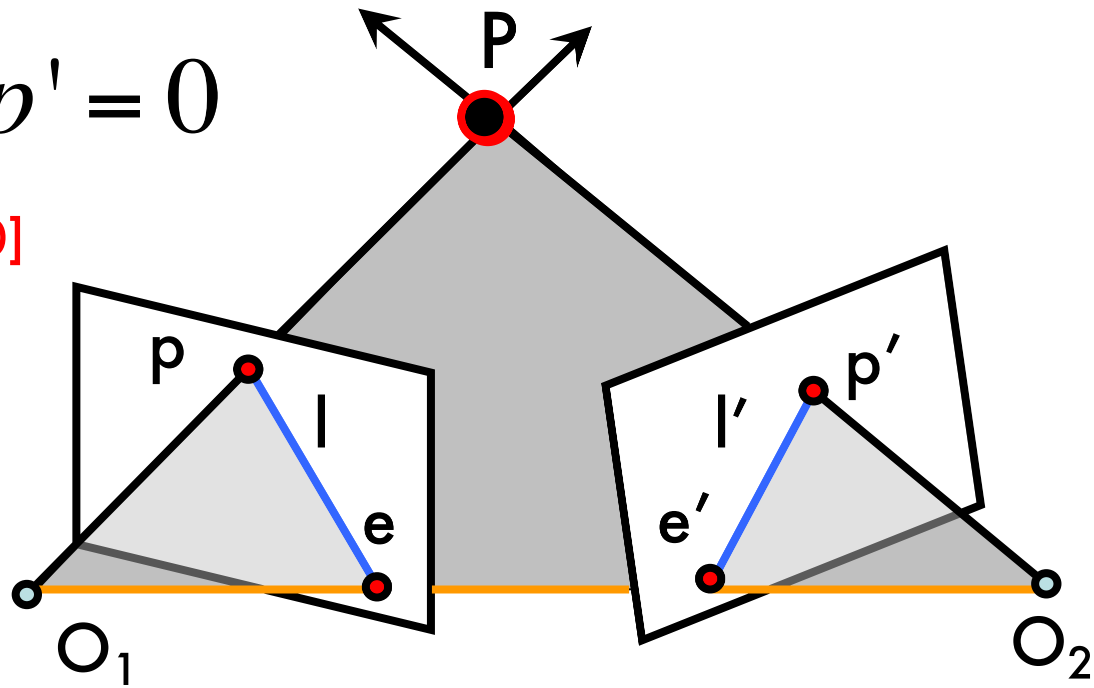
E = Essential matrix

(Longuet-Higgins, 1981)

Epipolar Constraint

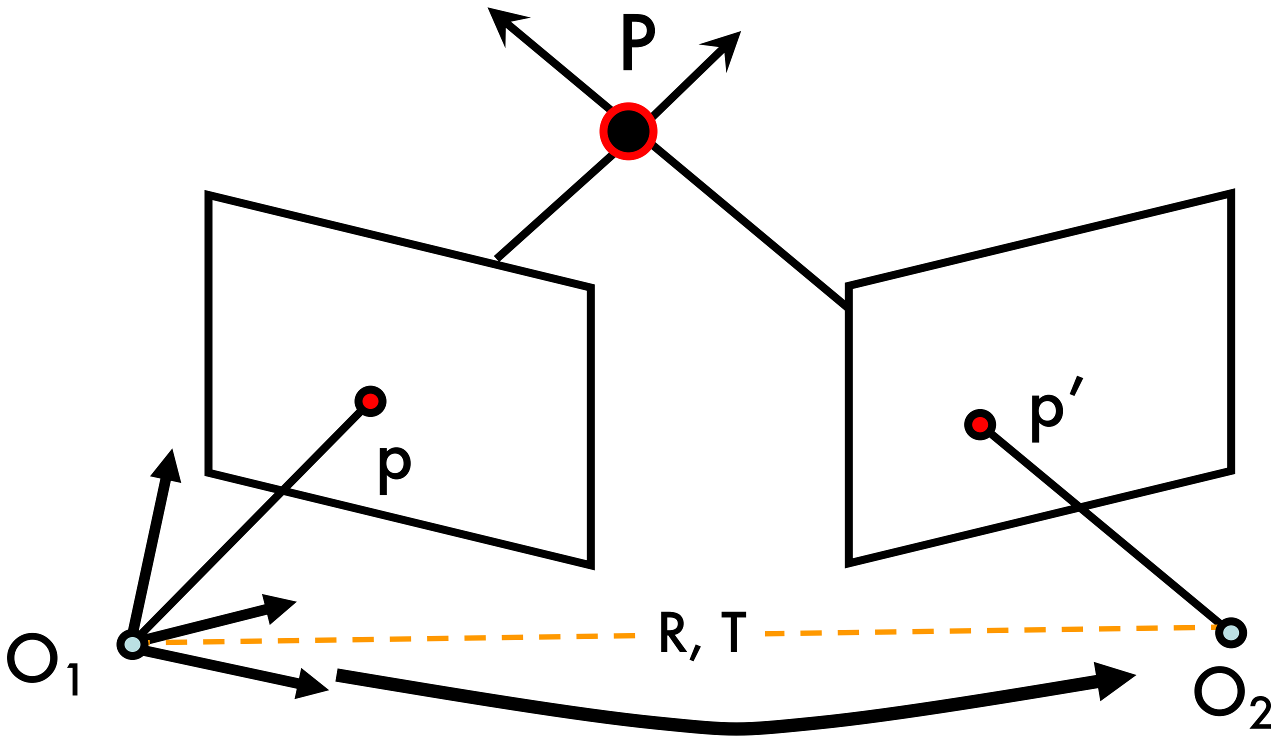
$$p^T \cdot E\, p' = 0$$

[Eq. 10]



- l = E p′ is the epipolar line associated with p′
- l′ = E^T p is the epipolar line associated with p
- E e′ = 0 and E^T e = 0
- E is 3x3 matrix; 5 DOF
- E is singular (rank two)

Epipolar Constraint



$$M = K\begin{bmatrix} I & 0 \end{bmatrix} \qquad M' = K'\begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$p_c = K^{-1}\, p \quad \text{[Eq. 11]} \qquad p'_c = K'^{-1}\, p' \quad \text{[Eq. 12]}$$

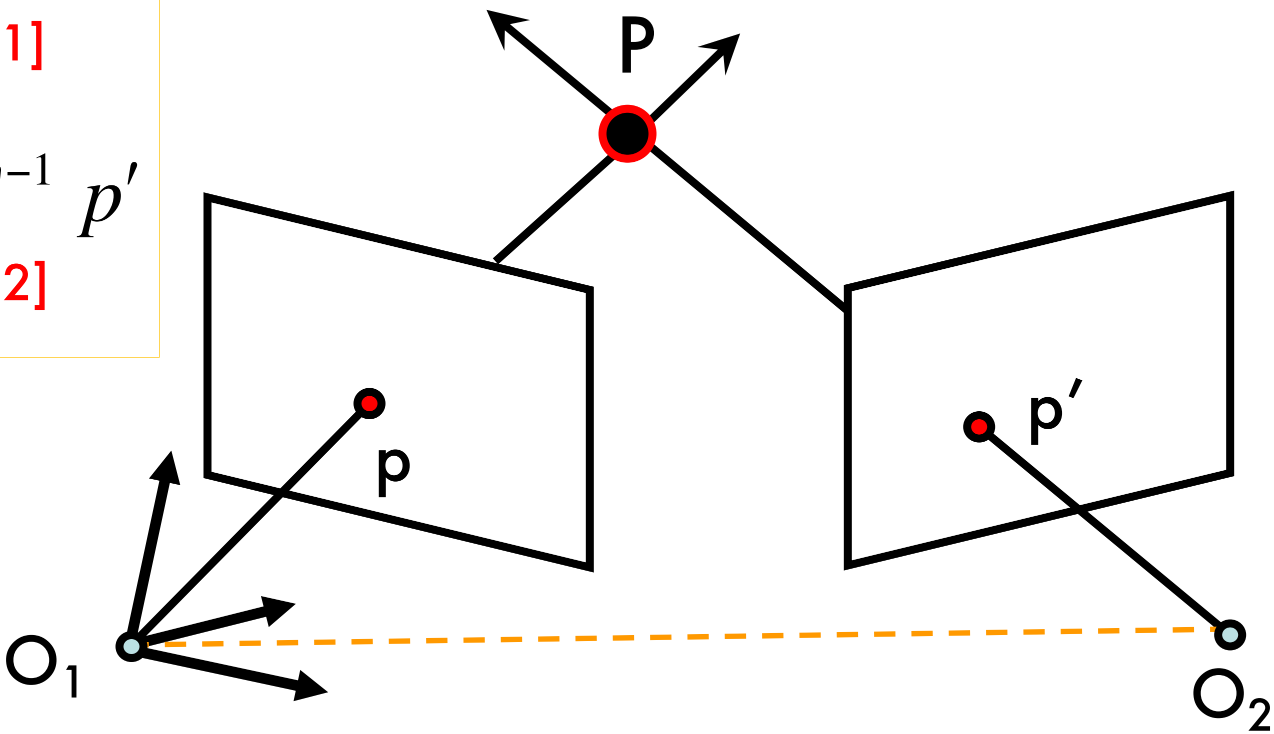
Epipolar Constraint

$$p_c = K^{-1}\ p$$

[Eq. 11]

$$p'_c = K'^{-1}\ p'$$

[Eq. 12]



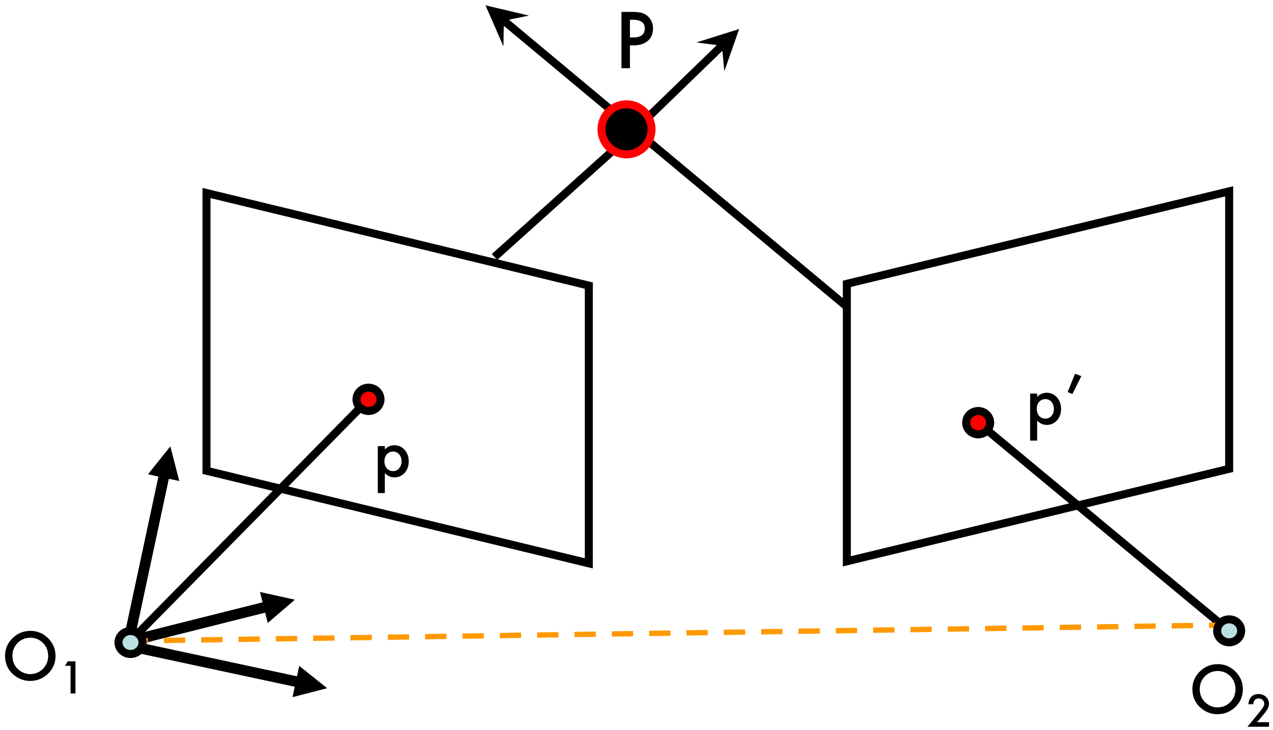
[Eq.9]

$$p_c^T \cdot [T_\times] \cdot R\ p'_c = 0 \rightarrow (\mathrm{K}^{-1}\ \mathrm{p})^{\mathrm{T}} \cdot [\mathrm{T}_\times] \cdot \mathrm{R}\ \mathrm{K}'^{-1}\ \mathrm{p}' = 0$$

$$\mathrm{p}^{\mathrm{T}} \boxed{\mathrm{K}^{-\mathrm{T}} \cdot [\mathrm{T}_\times] \cdot \mathrm{R}\ \mathrm{K}'^{-1}}\ \mathrm{p}' = 0 \rightarrow \mathrm{p}^{\mathrm{T}} \boxed{\mathrm{F}}\, \mathrm{p}' = 0$$

[Eq. 13]

Epipolar Constraint



[Eq. 13] $\boxed{p^T F p' = 0}$

$$F = K^{-T} \cdot [T_\times] \cdot R\, K'^{-1}$$

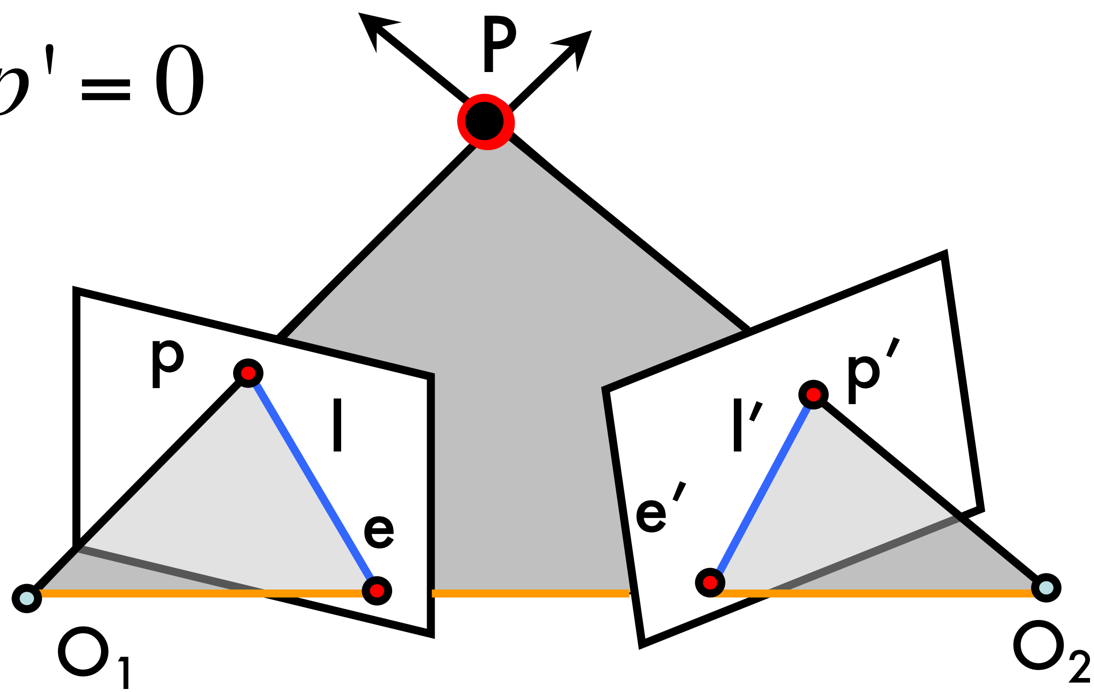
[Eq. 14]

F = Fundamental Matrix

(Faugeras and Luong, 1992)

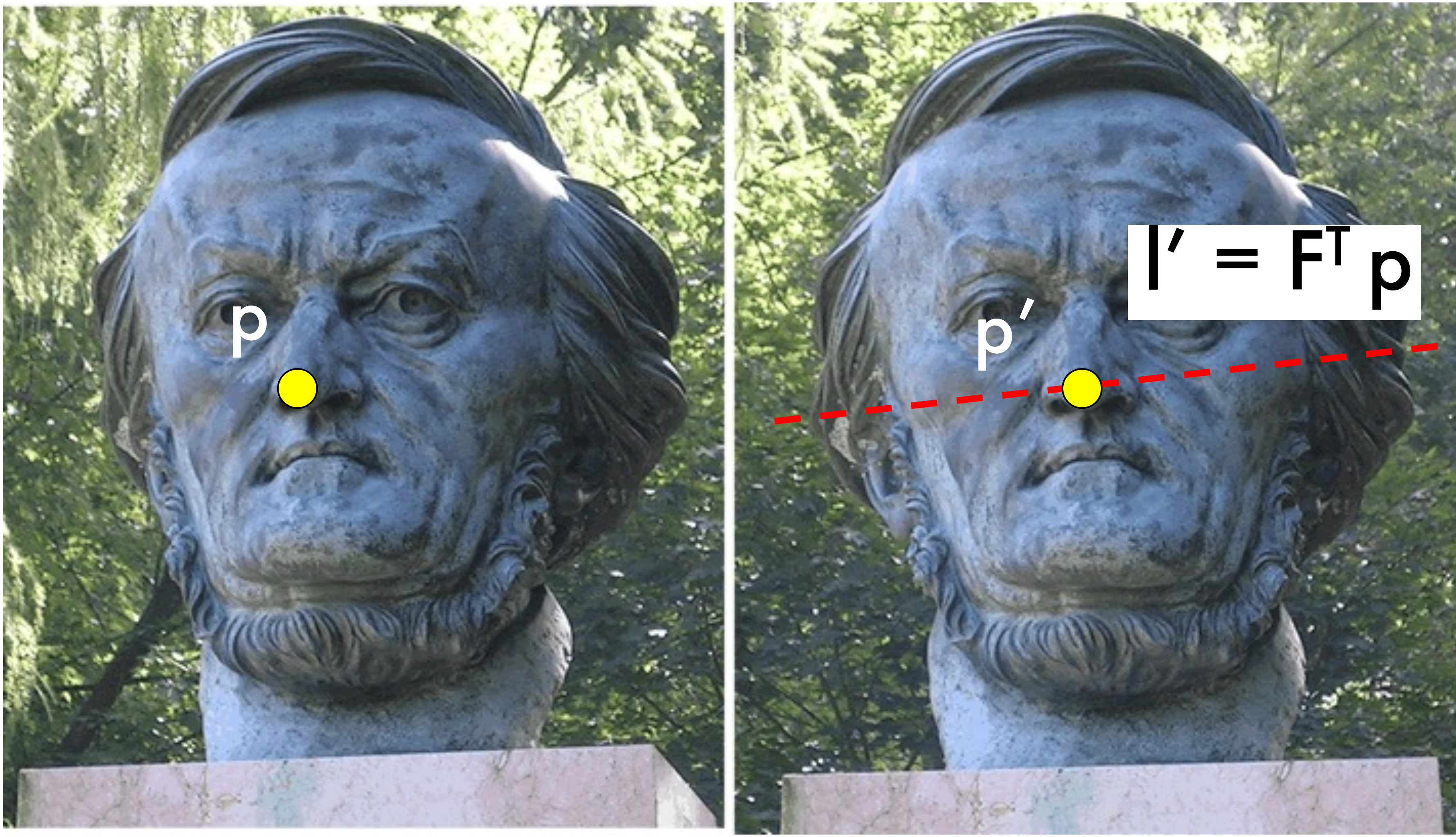
Epipolar Constraint

$$p^T \cdot F\ p' = 0$$



- $l = F\,p'$ is the epipolar line associated with p'
- $l' = F^T p$ is the epipolar line associated with p
- $F\,e' = 0$ and $F^T e = 0$
- F is 3x3 matrix; 7 DOF
- F is singular (rank two)

Why F is useful?



- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, we can compute the corresponding epipolar line in the second imag

Why F is useful?

- F captures information about the epipolar geometry of 2 views + camera parameters

- MORE IMPORTANTLY: F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

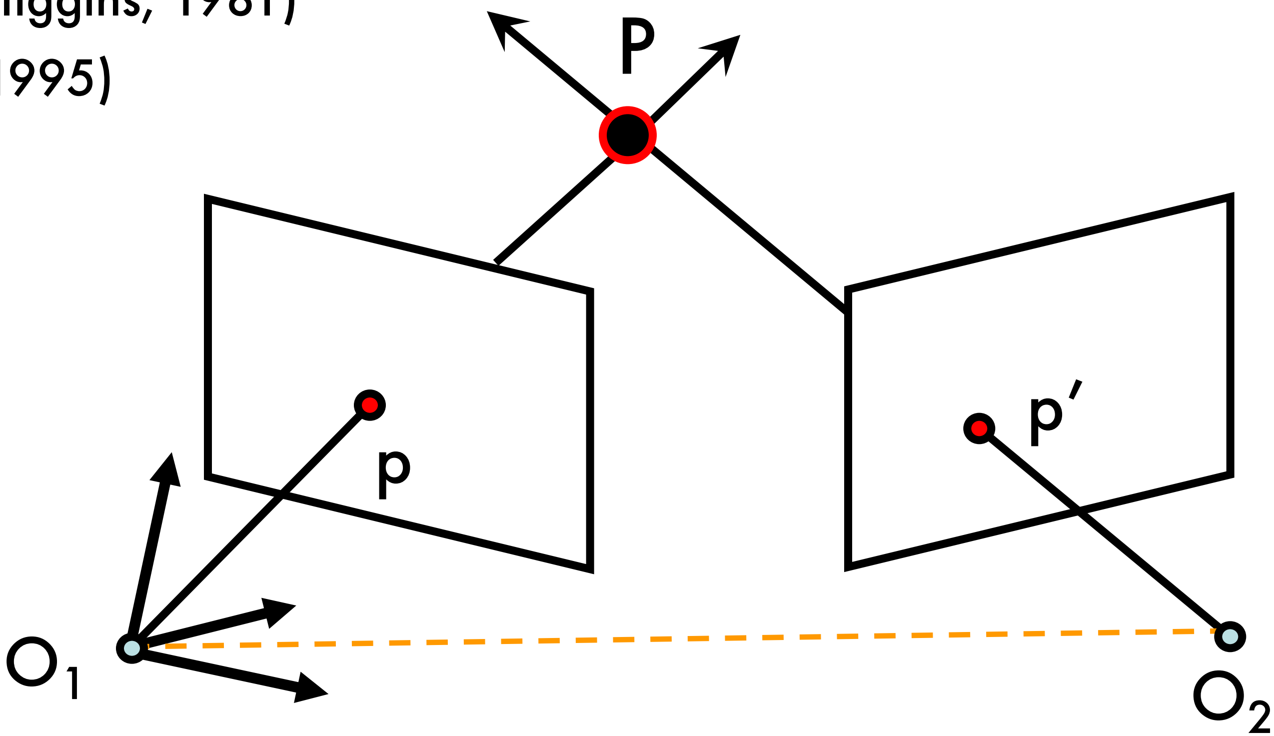
- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching

Estimating F

The Eight-Point Algorithm

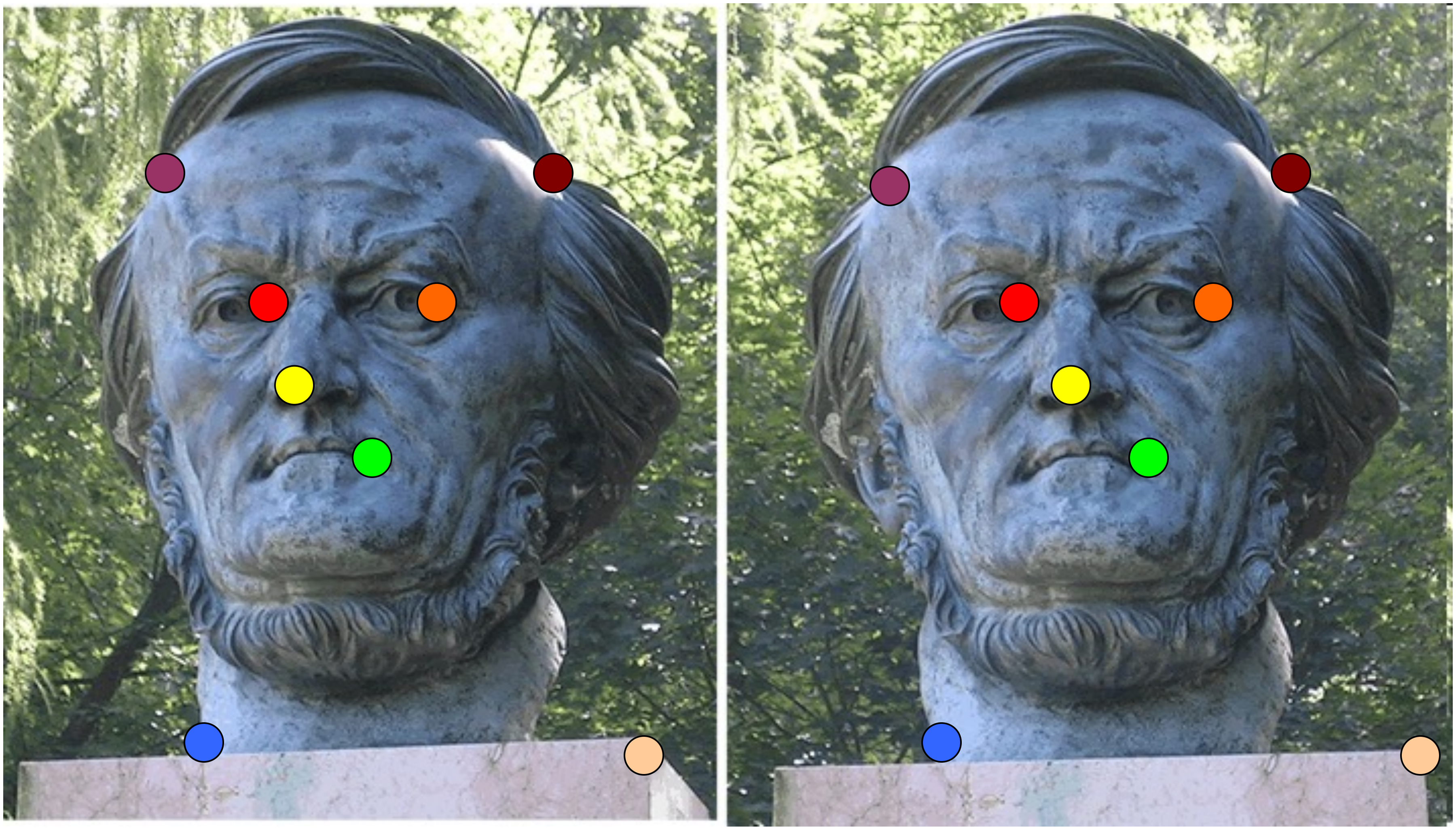
(Longuet-Higgins, 1981)

(Hartley, 1995)



$$p^T F p' = 0$$

Estimating F



Estimating F

[Eq. 13] $\mathrm{p}^{\mathrm{T}}\ \mathrm{F}\ \mathrm{p}' = 0$ ➡ $p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$

$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

➡ $$(uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

Estimating F

$$\left(u_i u'_i, u_i v'_i, u_i, v_i u'_i, v_i v'_i, v_i, u'_i, v_i', 1\right)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Estimating F

$$\mathbf{W}\underbrace{\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}}_{\mathbf{f}} = 0$$

[Eqs. 15]

- Homogeneous system $\mathbf{W}\,\mathbf{f} = 0$
- Rank 8 $\longrightarrow$ A non-zero solution exists (unique)
- If N>8 $\longrightarrow$ Lsq. solution by SVD! $\longrightarrow \hat{\mathrm{F}}$

$$\|\mathbf{f}\| = 1$$

$\hat{F}$ satisfies: $p^T \hat{F} p' = 0$

and estimated $\hat{F}$ may have full rank ($\det(\hat{F}) \neq 0$)

But remember: fundamental matrix is Rank2

Find F that minimizes $\left\| F - \hat{F} \right\| = 0$

Frobenius norm (*)

Subject to det(F)=0

SVD (again!) can be used to solve this problem

(*) Sq. root of the sum of squares of all entries

Find F that minimizes $\left\| \mathrm{F} - \hat{\mathrm{F}} \right\| = 0$

Frobenius norm (*)

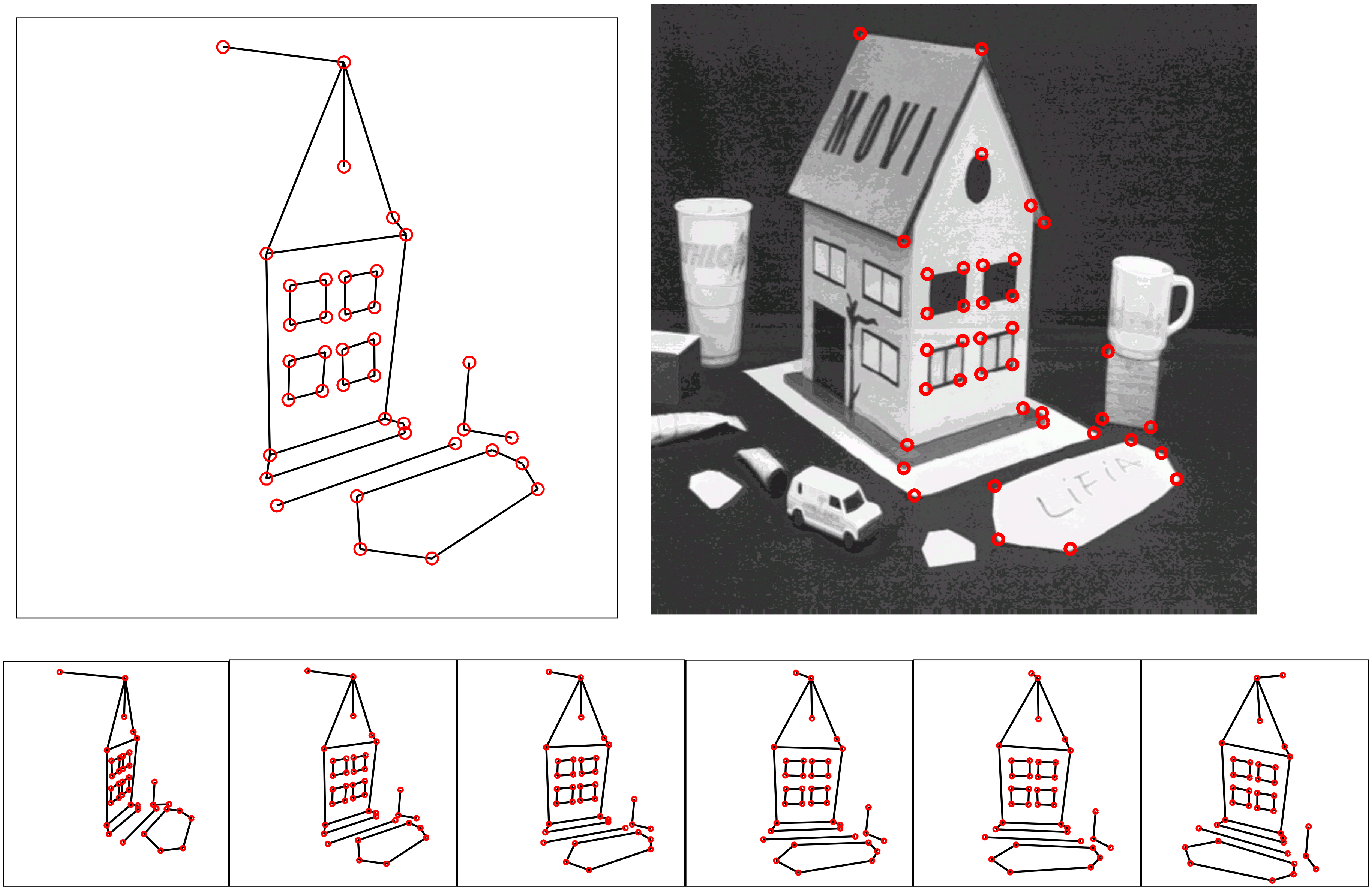
Subject to det(F)=0

$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

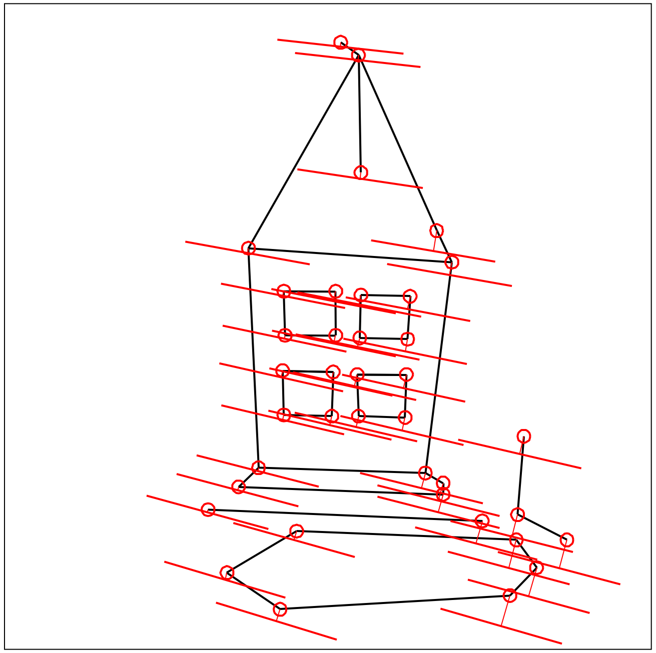
Where:

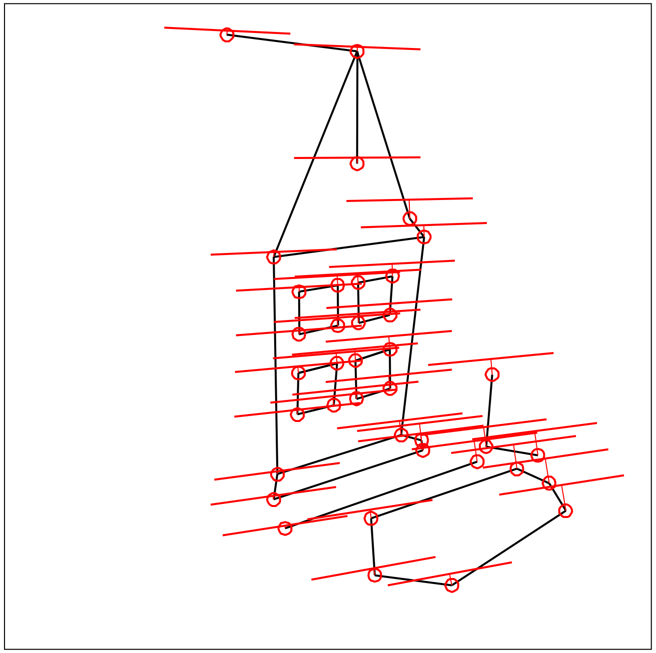
$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$

[HZ] pag 281, chapter 11, “Computation of F”



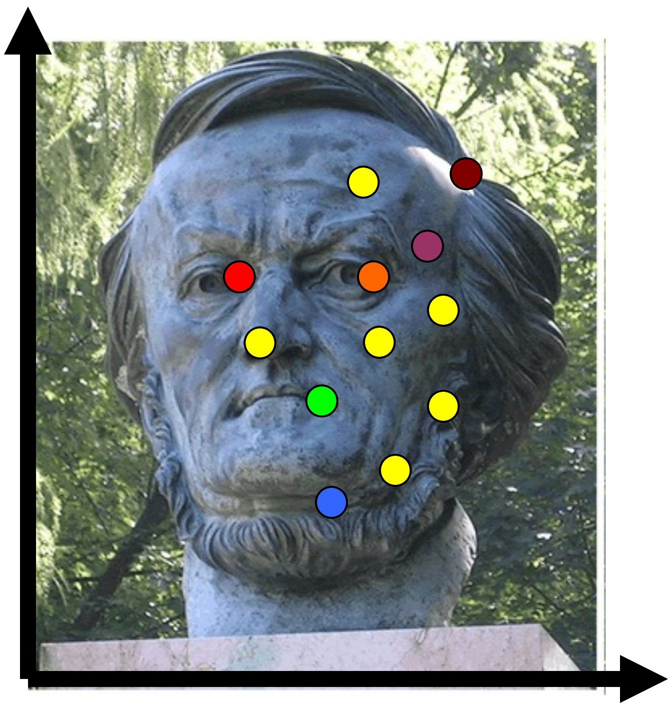
Data courtesy of R. Mohr and B. Boufama.





Mean errors:
10.0pixel
9.1pixel

Problems with the 8-Point Algorithm



$$\mathbf{W}\,\mathbf{f} = 0, \quad \|\mathbf{f}\| = 1 \xrightarrow{\text{Lsq solution by SVD}} \mathrm{F}$$

- Recall the structure of W:
  - do we see any potential (numerical) issue?

Problems with the 8-Point Algorithm

$$\mathbf{Wf} = 0$$

$$\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

- Highly un-balanced (not well conditioned)
- Values of W must have similar magnitude
- This creates problems during the SVD decomposition

HZ pag 108

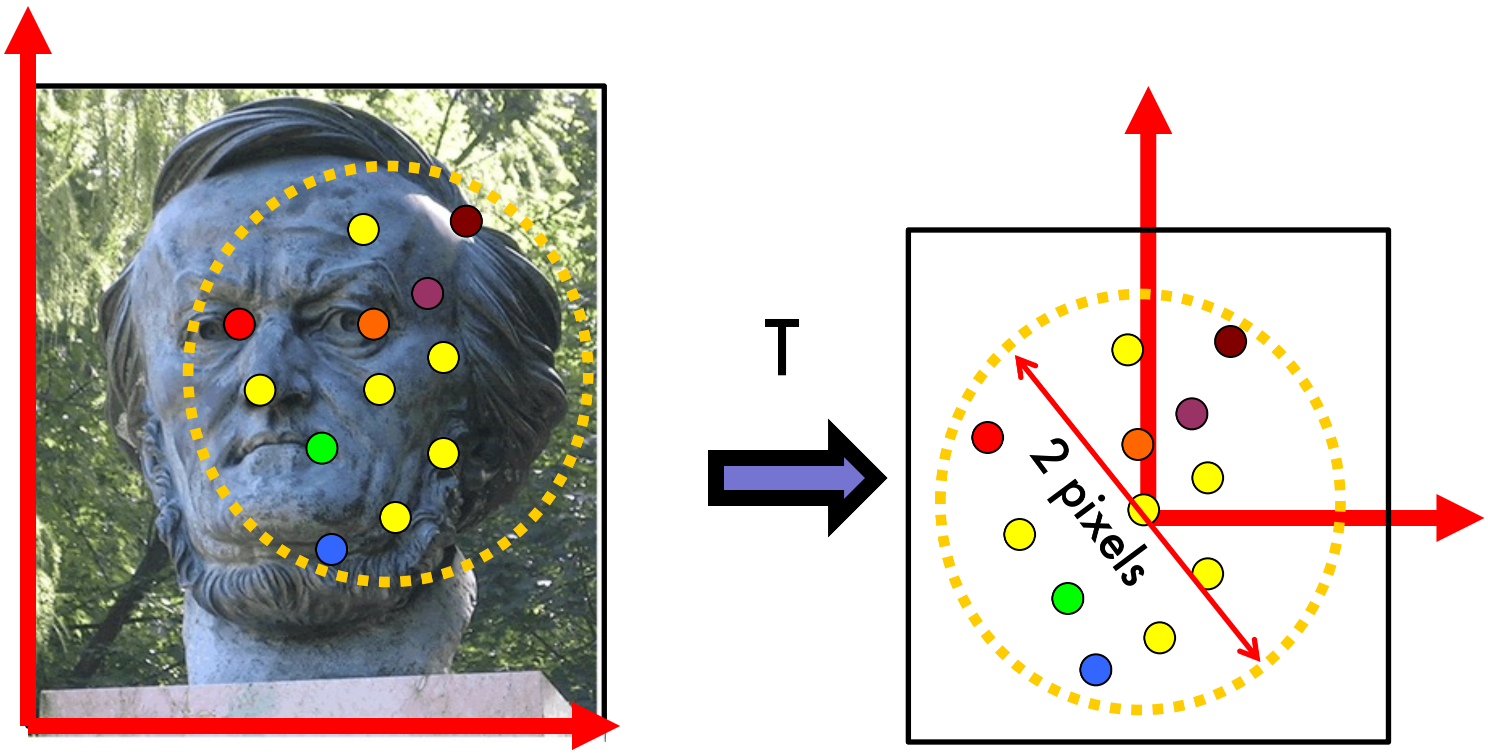
Normalization

IDEA: Transform image coordinates such that the matrix **W** becomes better conditioned (**pre-conditioning**)

For each image, apply a transformation T (translation and scaling) acting on image coordinates such that:

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels

Example of normalization

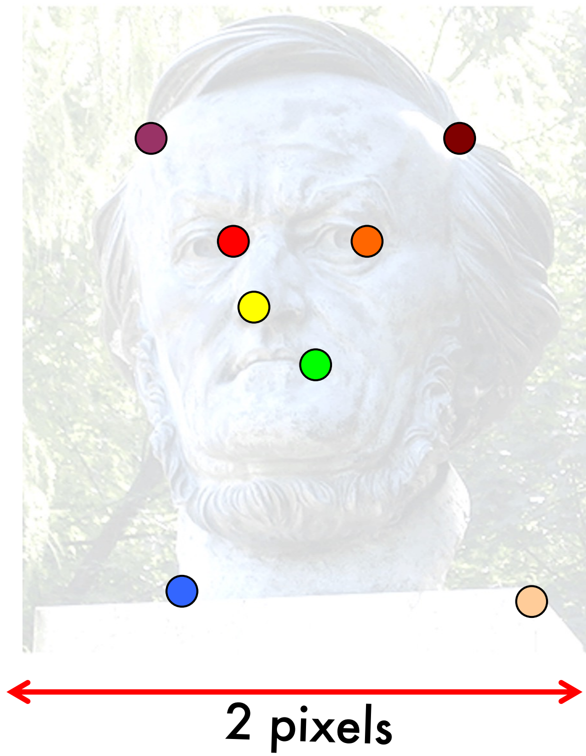


Coordinate system of the image before applying T

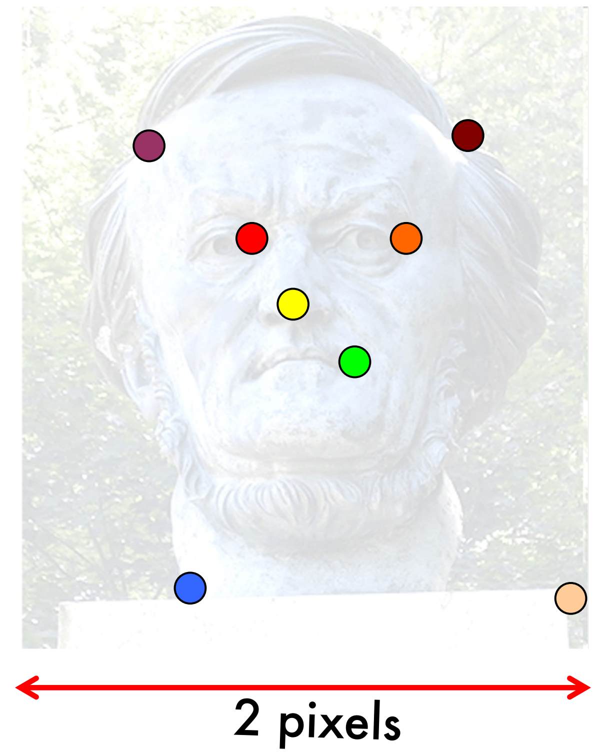
Coordinate system of the image after applying T

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels

Normalization



$$q_i = T\ p_i$$



$$q_i' = T'\ p_i'$$

The Normalized Eight-Point Algorithm

0. Compute T and T′ for image 1 and 2, respectively

1. Normalize coordinates in images 1 and 2:

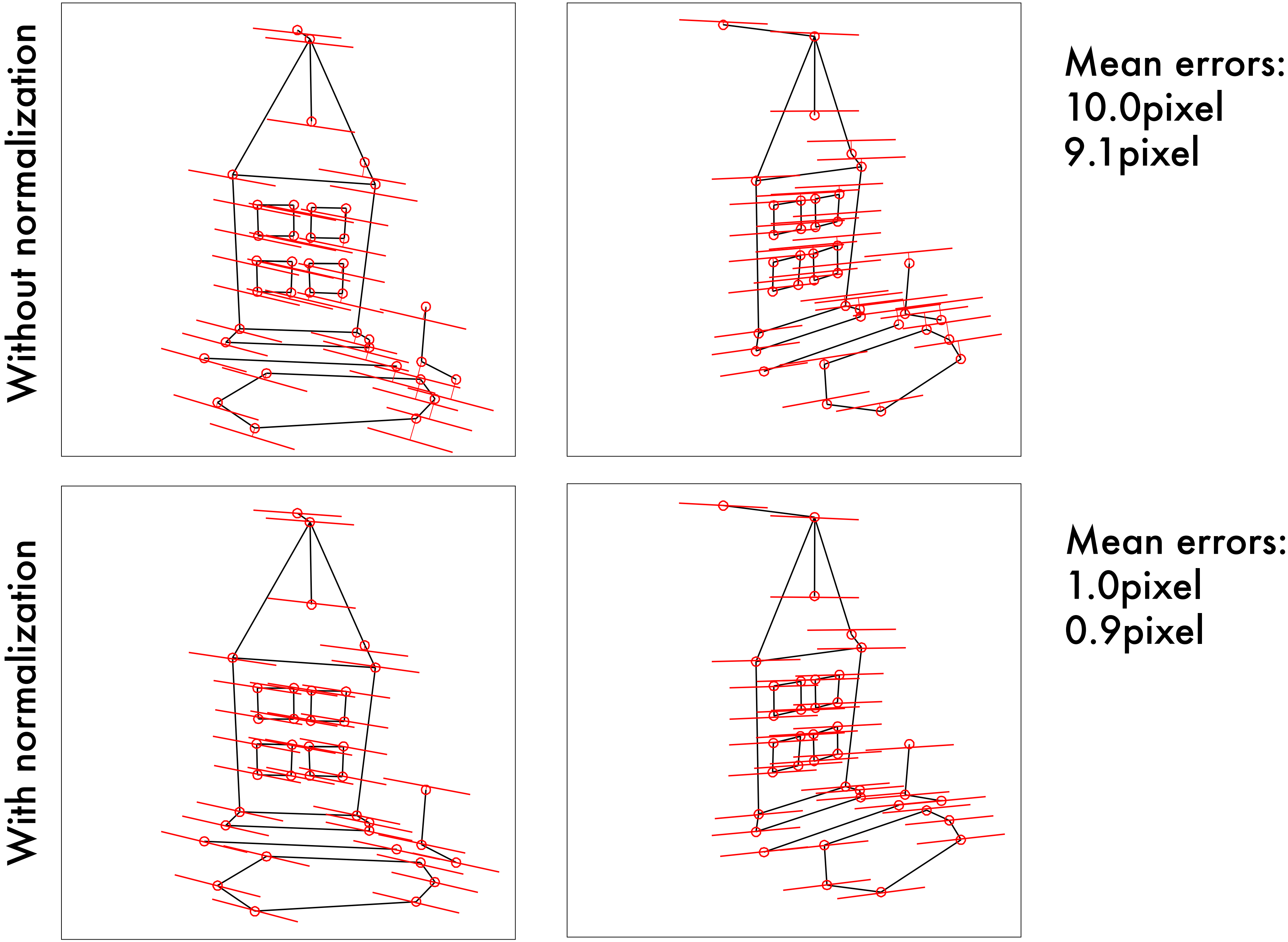
$$q_i = T\ p_i \qquad q'_i = T'\ p'_i$$

2. Use the eight-point algorithm to compute $\hat{F}_q$ from the corresponding points q_i and q'_i .

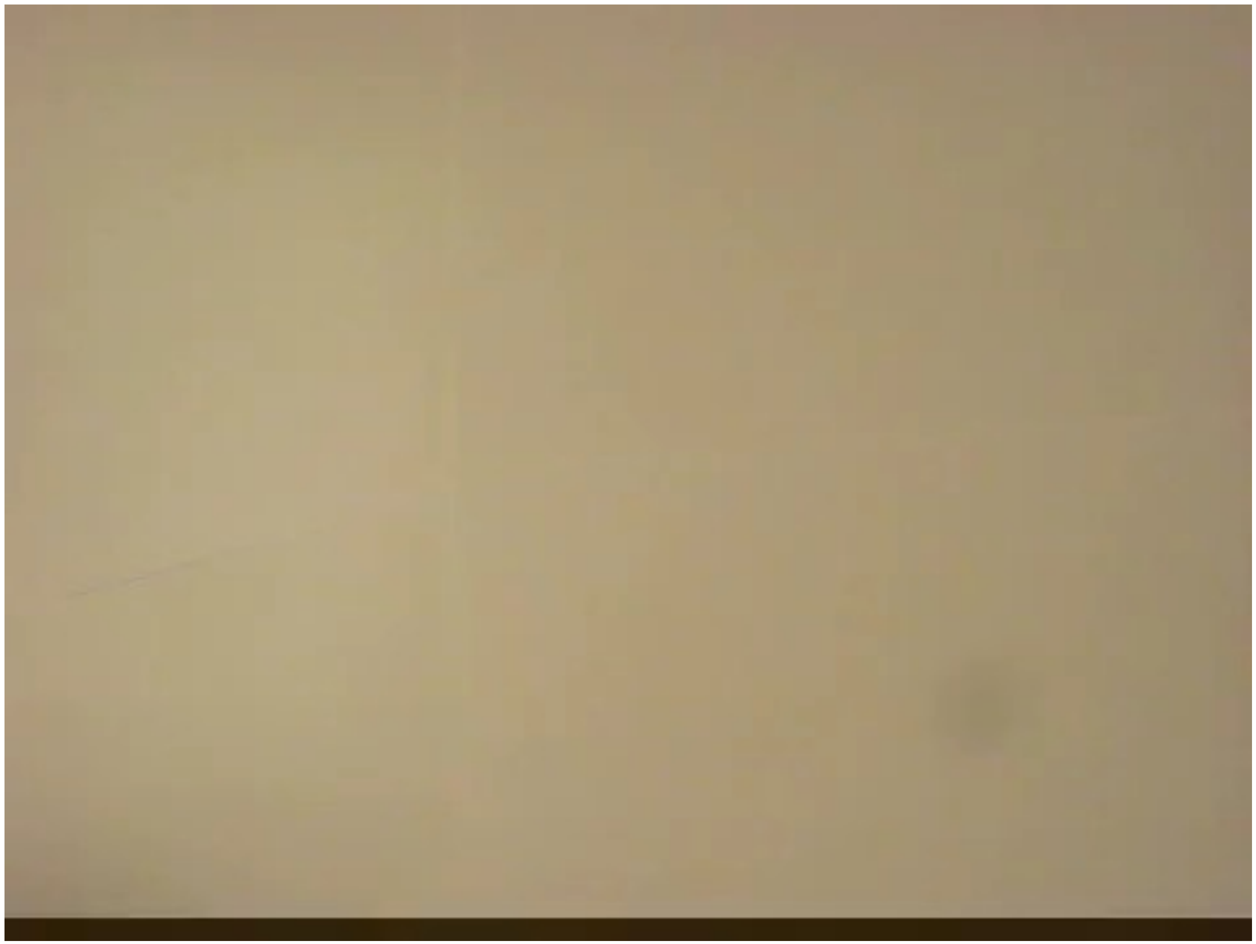
1. Enforce the rank-2 constraint: $\rightarrow \mathrm{F_q}$ such that:

$$\begin{cases} \mathrm{q^T\ F_q\ q'} = 0 \\ \det(\mathrm{F_q}) = 0 \end{cases}$$

2. De-normalize F_q: $F = T^T F_q\ T'$

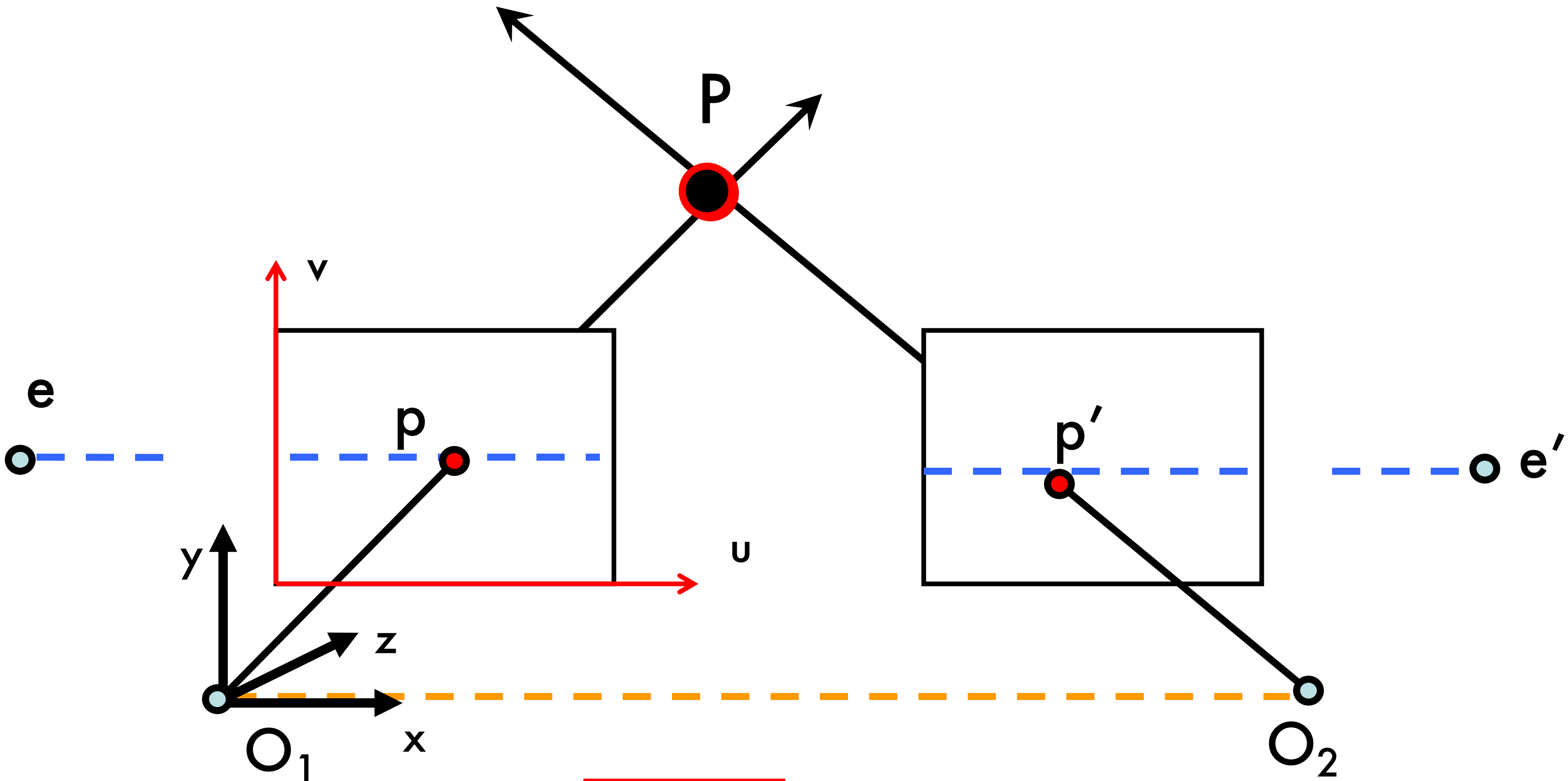

Without normalization
With normalization
Mean errors:
10.0pixel
9.1pixel
Mean errors:
1.0pixel
0.9pixel

The Fundamental Matrix Song



Next lecture:
Stereo systems

Example: Parallel image planes



$K_1 = K_2$ = known

x parallel to O_1O_2

$\boxed{E=?}$

Hint :

$\boldsymbol{R} = I \qquad \boldsymbol{T} = (T, 0, 0)$

Essential matrix for parallel images

$$\mathbf{E} = [\mathbf{T}_\times] \cdot \mathbf{R}$$

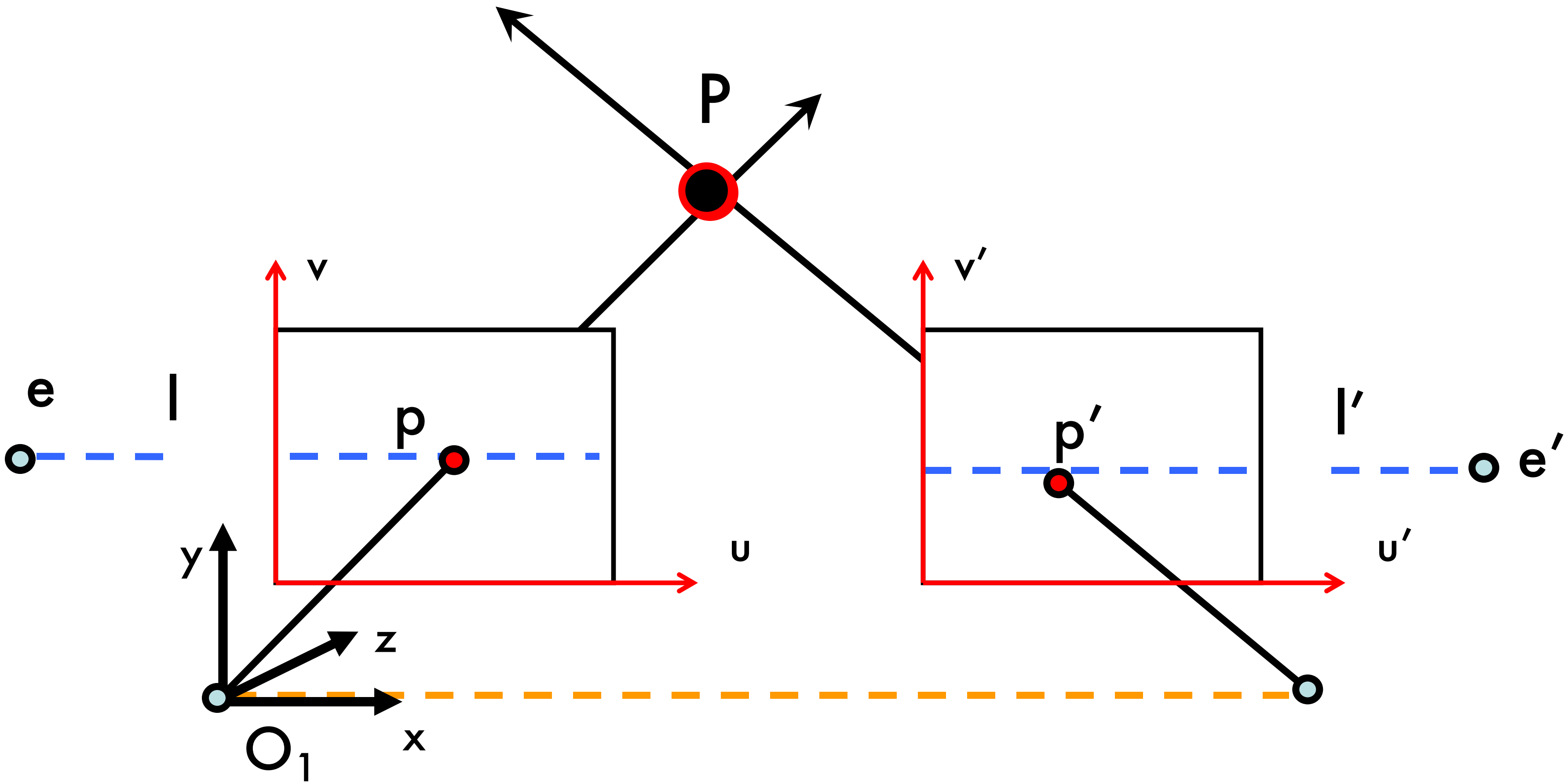
$$\mathbf{E} = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix} \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$$

[Eq. 20]

$$\mathbf{T} = [\ \mathrm{T}\ \ 0\ \ 0\ ]$$

$$\mathbf{R} = \mathbf{I}$$

Example: Parallel image planes

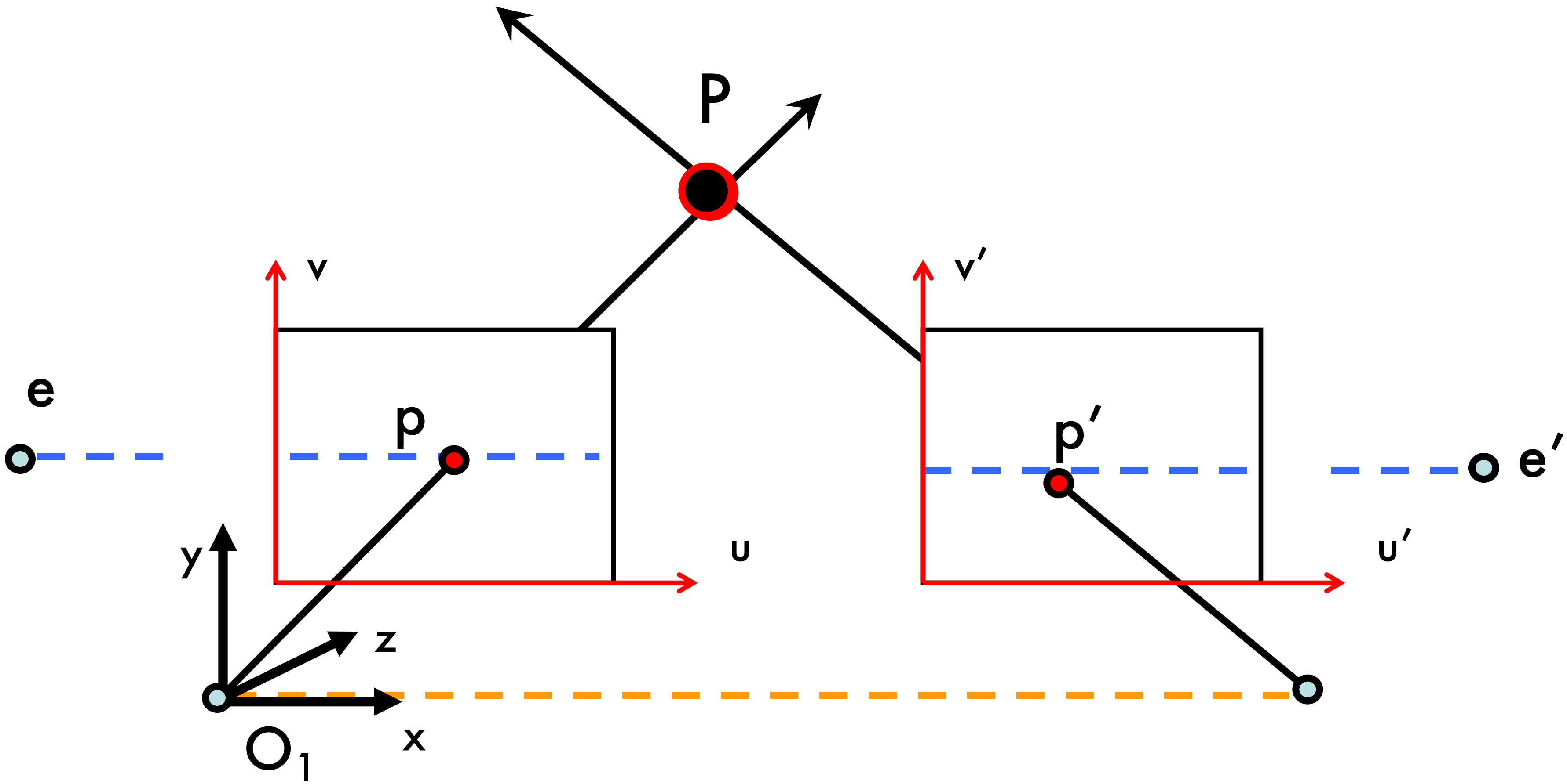


What are the directions of epipolar lines?

$$l = \mathrm{E}\, p' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -T \\ T v' \end{bmatrix}$$

horizontal!

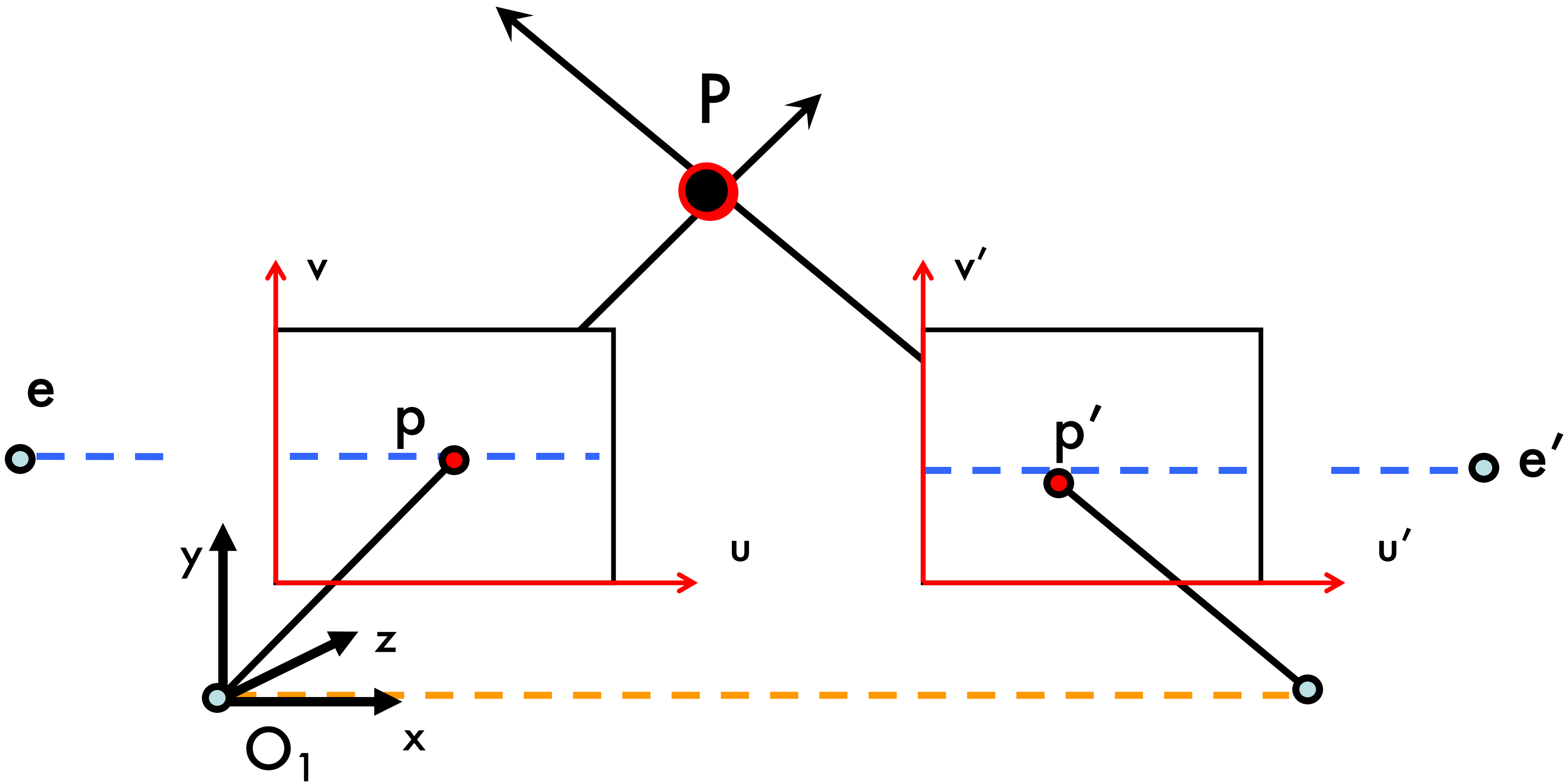
Example: Parallel image planes



How are p and p′ related?

$$p^T \cdot E\, p' = 0$$

Example: Parallel image planes

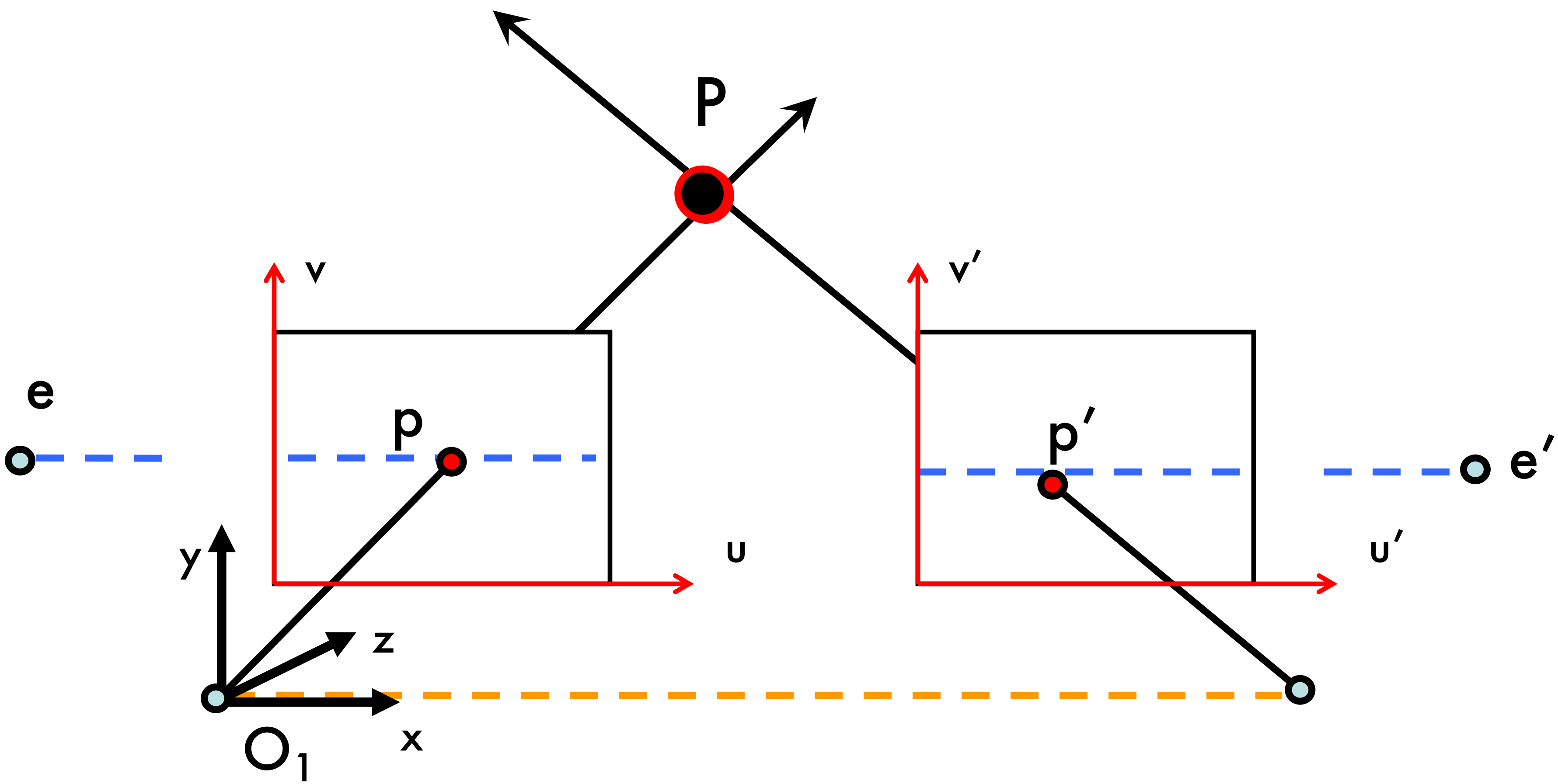


How are p and p′ related?

$$\Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0 \Rightarrow Tv = Tv'$$

$$\Rightarrow v = v'$$

Example: Parallel image planes



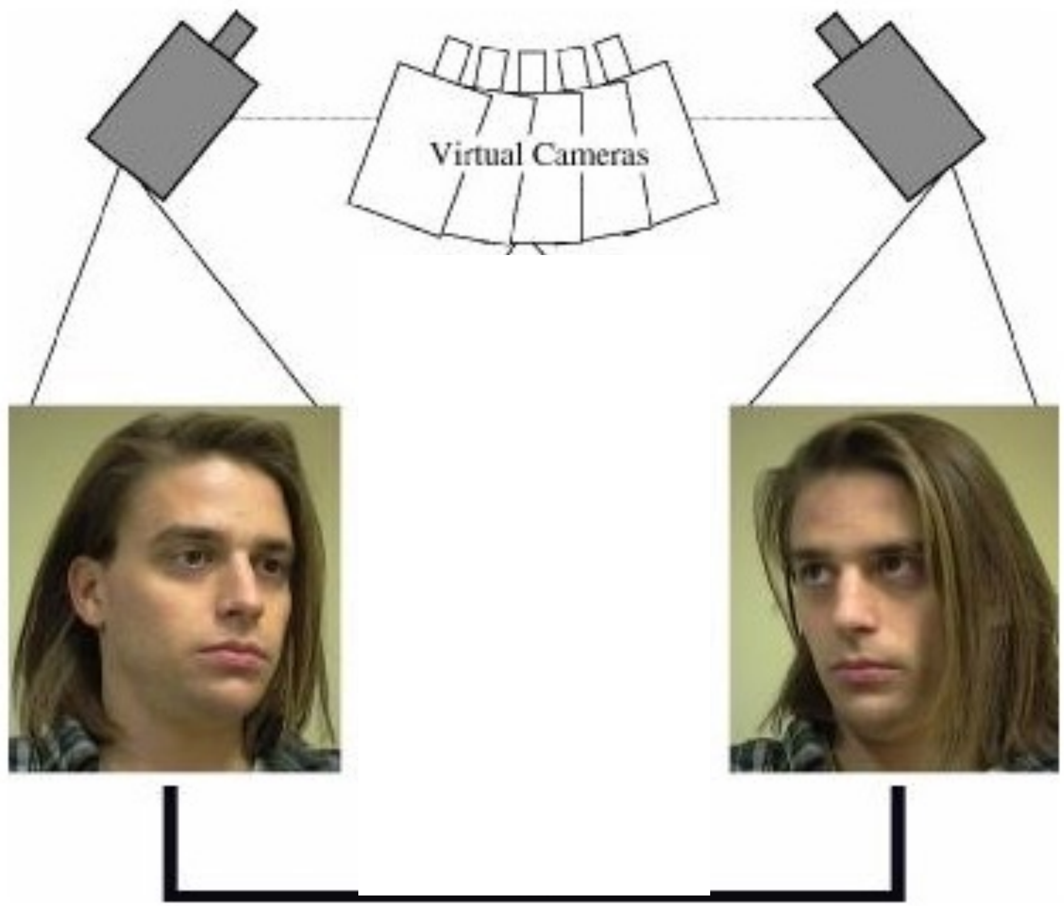
Rectification: making two images "parallel"

Why it is useful?

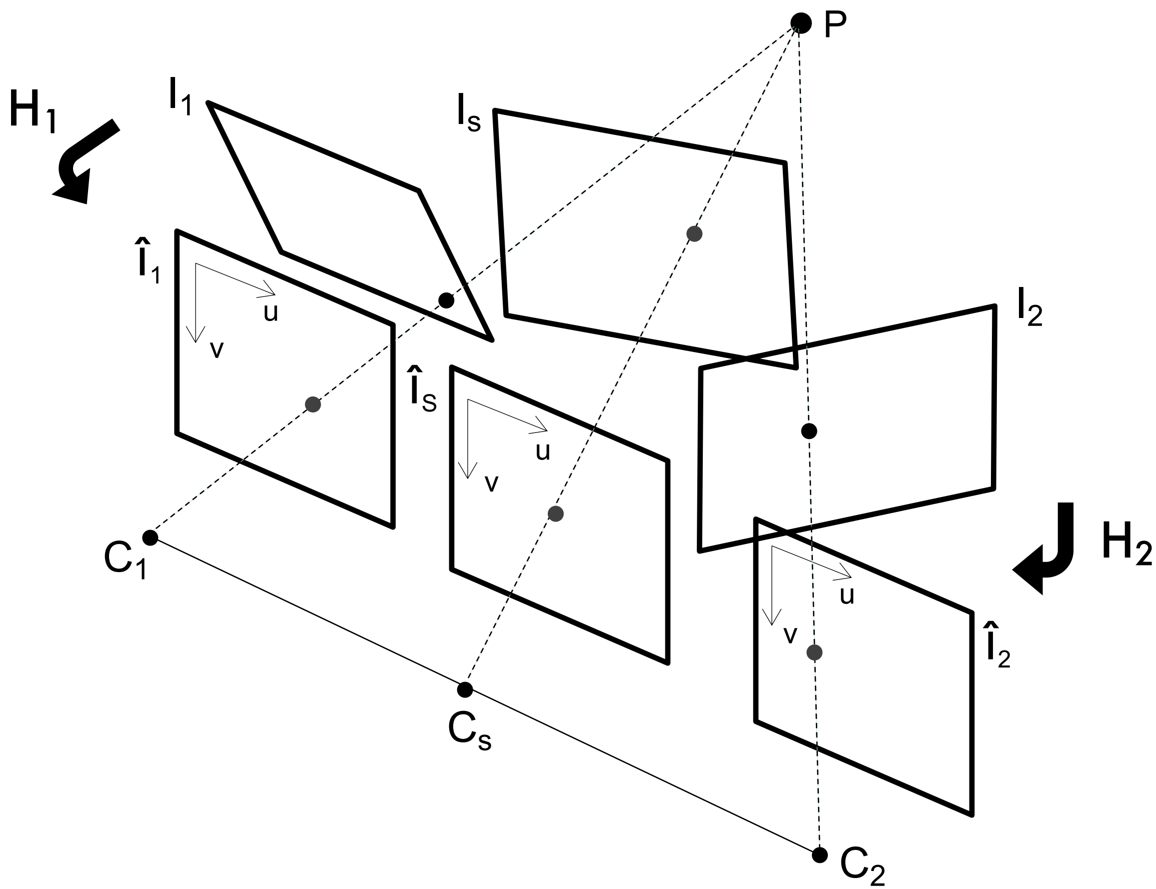
- Epipolar constraint → v = v′
- New views can be synthesized by linear interpolation

Application: view morphing

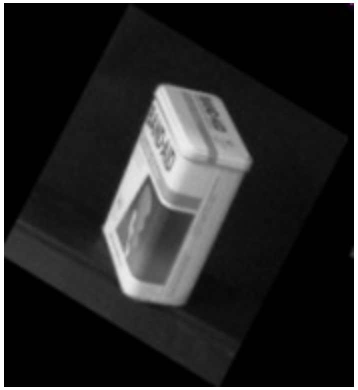
S. M. Seitz and C. R. Dyer, *Proc. SIGGRAPH 96*, 1996, 21-30

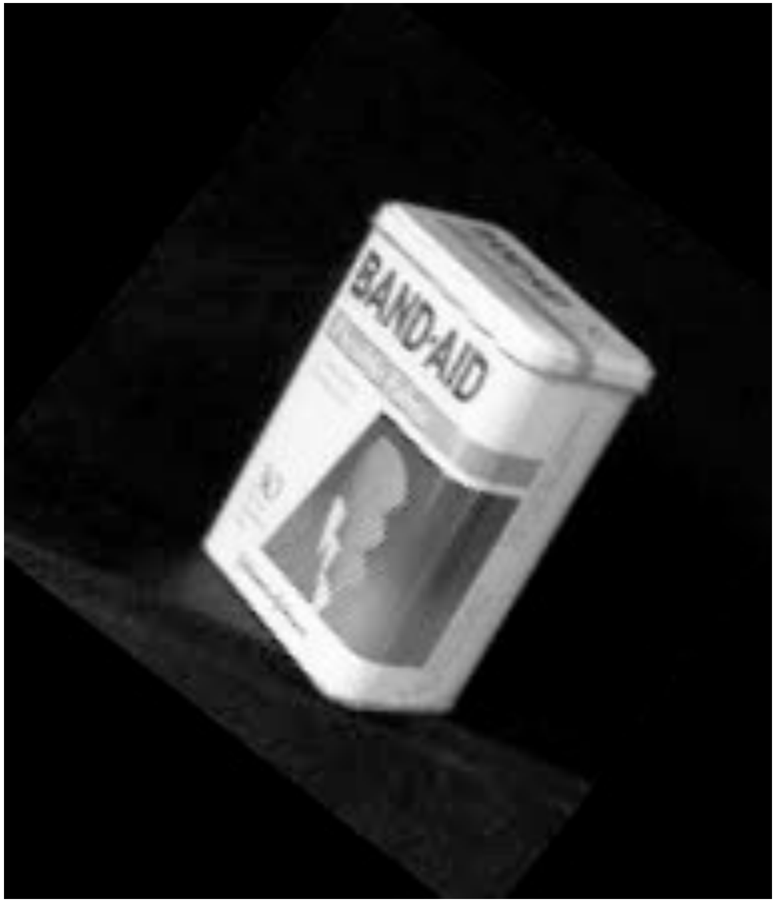


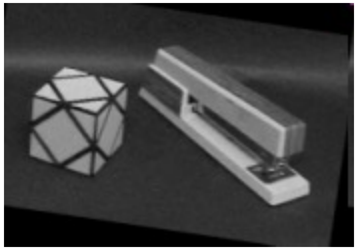
Rectification

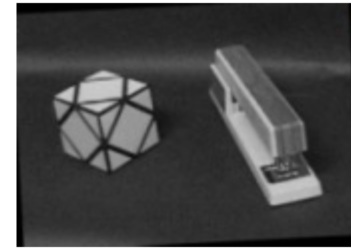


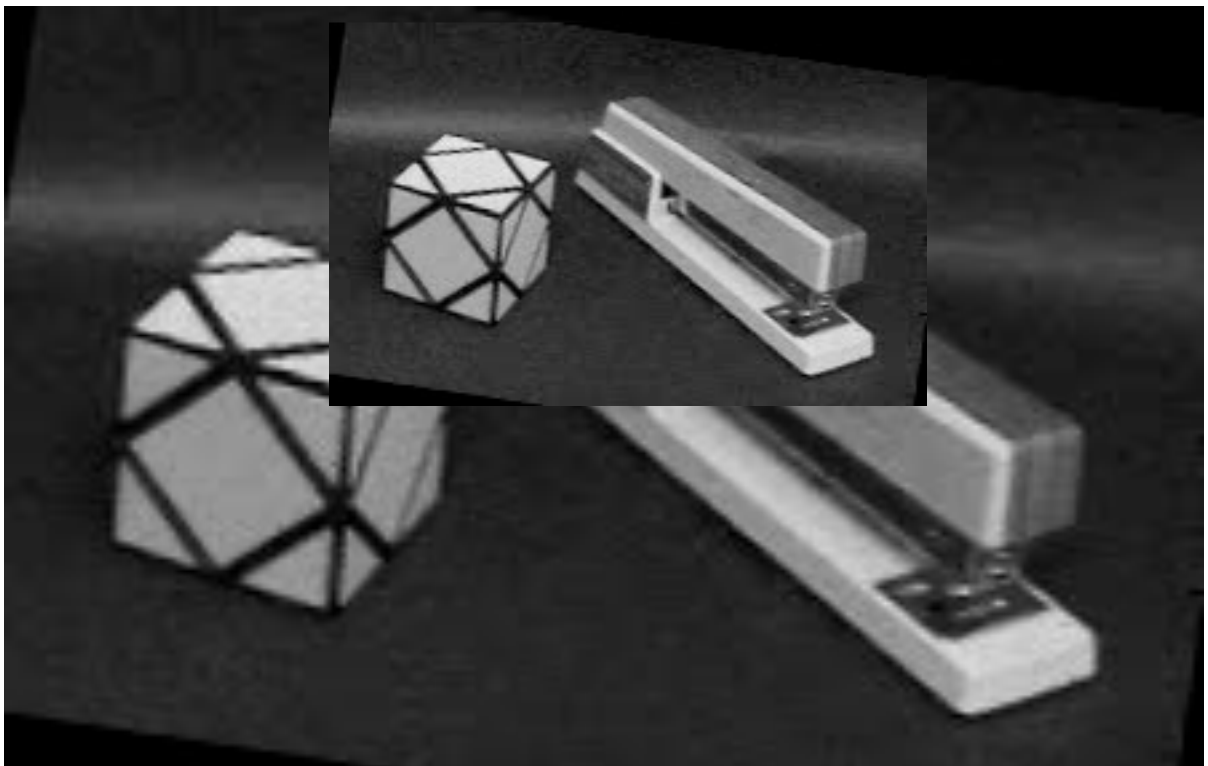

BAND-AID

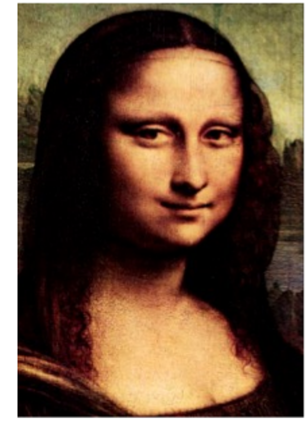

BAND-AID


BAND-AID

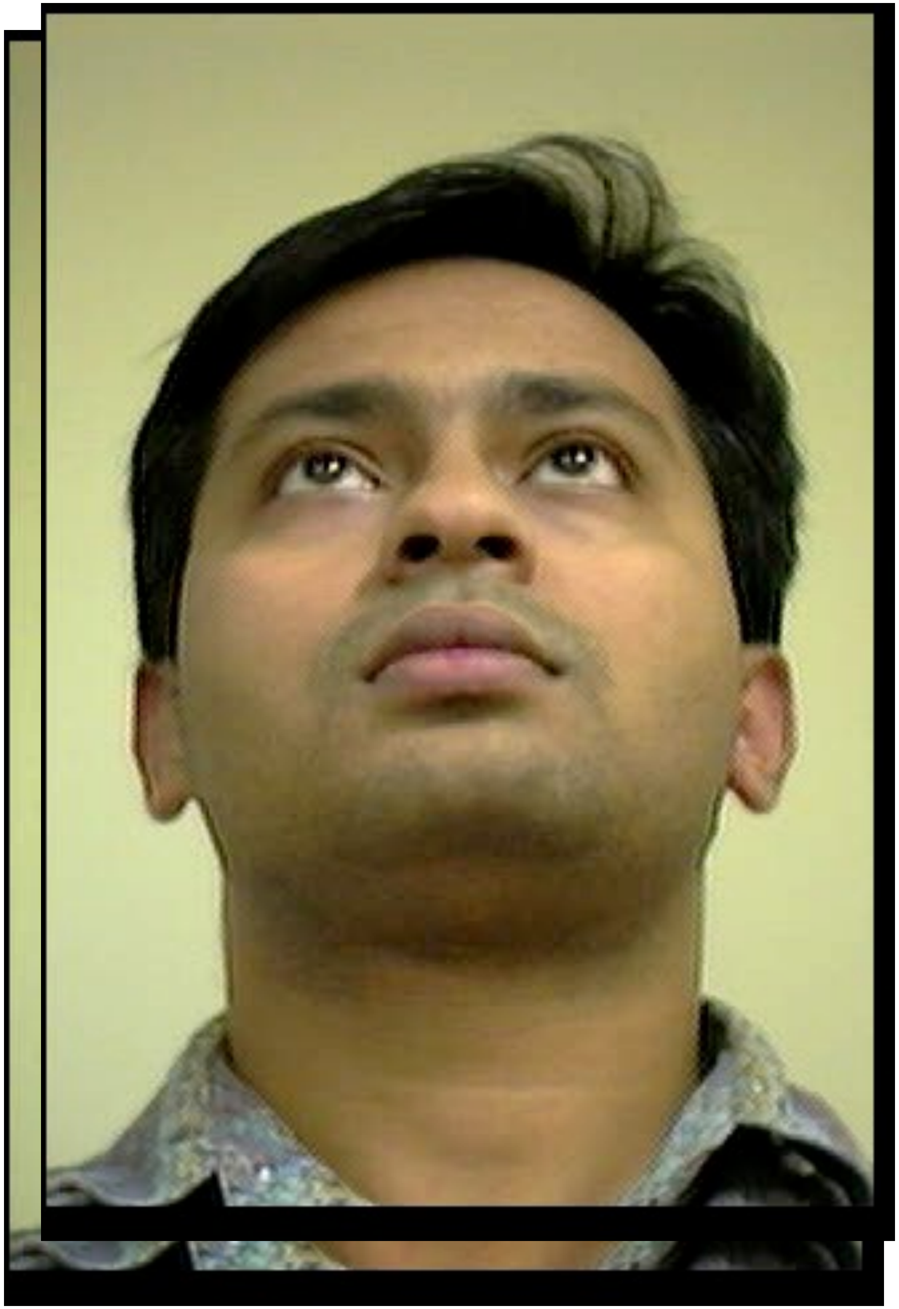






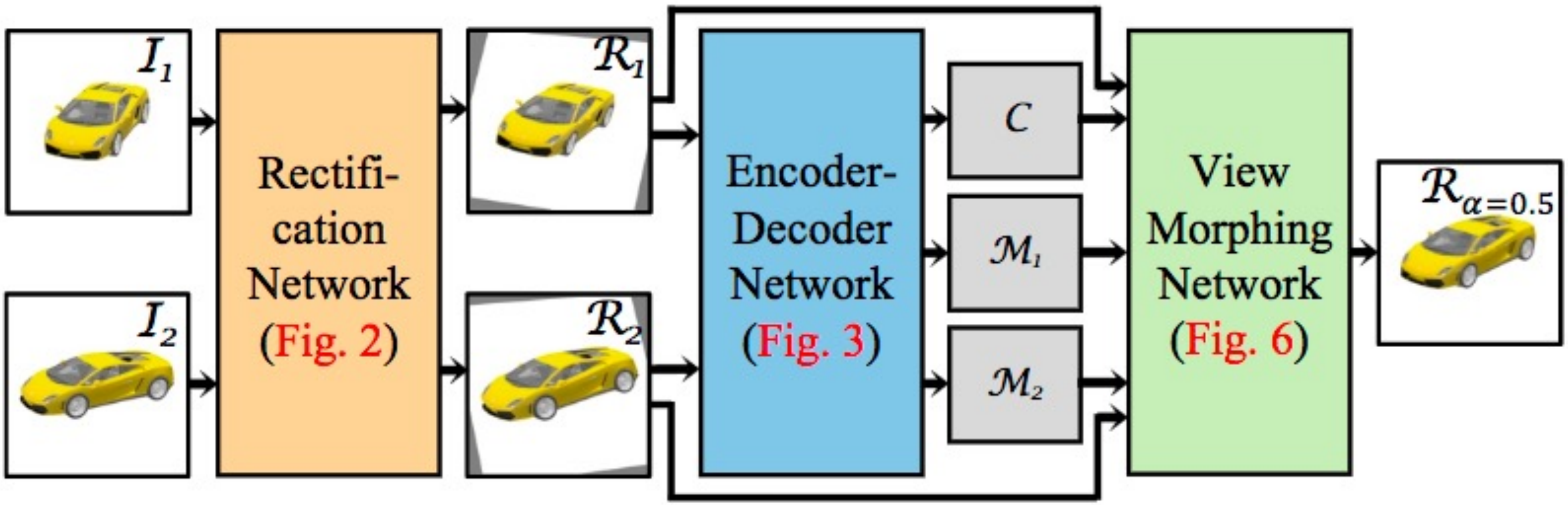


From its reflection!



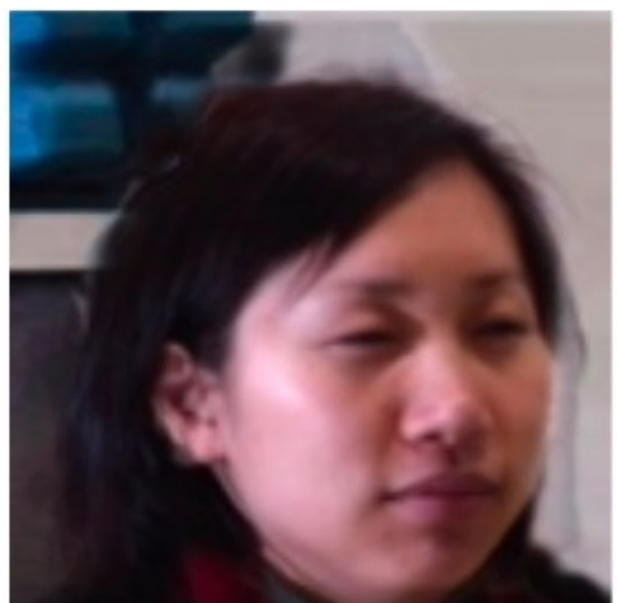
Deep view morphing

D. Ji, J. Kwon, M. McFarland, S. Savarese, CVPR 2017



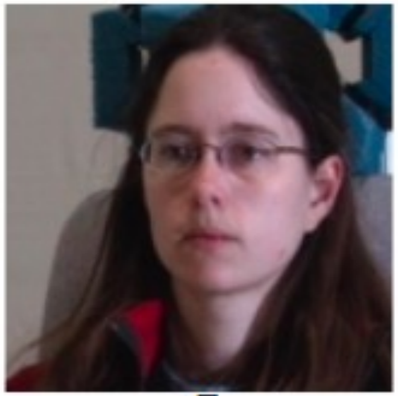
Deep view morphing

D. Ji, J. Kwon, M. McFarland, S. Savarese, CVPR 2017



Deep view morphing

D. Ji, J. Kwon, M. McFarland, S. Savarese, CVPR 2017



I_1

GT

I_2