Stanford University, Management Science and Engineering (and ICME)
CME 338   Large-Scale Numerical Optimization
Instructor: Michael Saunders   Spring 2019

Homework 2, Due Monday April 29
http://stanford.edu/class/cme338/homework.html

Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and $b \in \mathbb{R}^n$ ($b \neq 0$). Let the kth Krylov subspace be $\mathcal{K}_k \equiv \text{span}\{b, Ab, A^2 b, \ldots, A^{k-1}b\}$, and let (T_k, H_k, V_k) denote the matrices obtained by applying the Lanczos process to (A, b) for k steps, as defined in the notes. Assume that we use exact arithmetic and that the process terminates with $\beta_{k+1} = 0$ for some $k = \ell \leq n$.

Exercise 1

If A has $m < n$ distinct eigenvalues, show that the Lanczos process for (A, b) terminates with $\ell \leq m$. (If you wish, you may prove that the dimension of the Krylov subspaces $\mathcal{K}_k(A, b)$ cannot exceed m.)

Exercise 2

Show that if A is changed to $A - \sigma I$ for some scalar shift σ, T_k becomes $T_k - \sigma I$ and V_k is unaltered.

Exercise 3

Using $AV_k = V_{k+1}H_k$ for $k < \ell$ and the definitions of $\{\alpha_k, \beta_k, v_k\}$, show that the columns of V_k are orthonormal ($k \leq \ell$).

Exercise 4

Show that the following subproblems are equivalent for defining how CG chooses x_k when A is positive definite:

(1)   minimize $\frac{1}{2}x_k^T A x_k - b^T x_k$ such that $x_k \in \mathcal{K}_k$
(2)   minimize $\|r_k\|_{A^{-1}}$ such that $x_k \in \mathcal{K}_k$
(3)   find x_k such that $x_k \in \mathcal{K}_k$ and $r_k \perp \mathcal{K}_k$,

where $r_k = b - Ax_k$ and $\|w\|_{A^{-1}} = \sqrt{w^T A^{-1} w}$ for all $w \in \mathbb{R}^n$.

Exercise 5 (extra credit)

Prove that T_ℓ is nonsingular if and only if $b \in \text{range}(A)$.
Deduce that $\beta_{\ell+1} = 0$ is a "lucky breakdown" for CG, MINRES, and SYMMLQ if $b \in \text{range}(A)$.