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Interior Points in the Core of Two-Sided Matching Markets*

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Theoretical work on two-sided matching markets has focussed on the stable outcomes that are optimal for one side of the market. An innovative exception is a paper by Rochford, who shows how to identify a nonempty set of interior points of the core of the assignment market first studied by Shapley and Shubik. We strengthen Rochfold's results, by showing that such a set of fixed points must reflect the same kind of polarization of interests that characterizes the core, and generalize these results to a wider class of markets, via a powerful algebraic fixed point theorem of Tarski. *Journal of Economic Literature* Classification Numbers: 022, 026.

1. INTRODUCTION

There has been a good deal of recent progress made in understanding *two-sided matching markets*, which model agents as belonging to one of two exogenously specified disjoint sets, say P and Q (e.g., firms and workers), and engaging in bilateral transactions (i.e., if worker i works for firm j then firm j employs worker i). Such models seem to capture some important features of labor markets, and recent progress has been made in applying them to the study of particular markets (see [24, 29]). A wide variety of different models of such markets have been shown to possess a number of striking properties not shared by general markets (see, e.g., [12, 30, 6, 15, 24, 25, 9]; see [27] for a survey).

In each of these models, the set of *stable* outcomes (which essentially coincides with the core of such markets) is always nonempty and reflects a polarization of interests between the two sides of the market. Associated with each side of the market, there is an *optimal* stable outcome that is the

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stable outcome most preferred by every agent on that side of the market and least preferred by every agent on the other side of the market. That is, there is a P-optimal stable outcome such that all agents in P (weakly) prefer it to every other stable outcome, and all agents in Q (weakly) prefer any other stable outcome to it, and there is a Q-optimal stable outcome with the symmetric properties. The best mathematical explanations found to date for this phenomenon have to do with the algebraic structure of the set of stable outcomes, which has been shown to be a complete lattice in almost all of the models exhibiting this polarization of interests (see [16, 30, 3, 4, 14, 26, 28]).

Theoretical work has tended to focus on the P- and Q-optimal stable outcomes, in part because there has been no conceptual framework that easily allowed other stable outcomes to be distinguished. Rochford [19] takes an important step in remedying this situation. In the context of the simple "assignment market" first examined by Shapley and Shubik [30], she defines a "rebargaining" process by which agents (re)negotiate the terms of their contracts in light of the contracts negotiated by all other agents. This rebargaining process takes stable outcomes into other stable outcomes and has as its fixed points what Rochford calls "symmetric pairwise-bargained" (SPB) allocations, which are stable outcomes that are generally different from the P- and Q-optimal stable outcomes. Rochford proves (using Brouwer's fixed point theorem) that the set of SPBs is always nonempty in this model, and that if the rebargaining process starts at either the P- or the Q-optimal stable outcome, then it will converge to an SPB.

The present paper has three goals. First, we will strengthen Rochford's results and show that the set of SPBs also reflects the polarization of interests exhibited by the set of stable outcomes, so that there is a P-optimal SPB such that all agents in P (weakly) prefer it to every other SPB, and all agents in Q (weakly) prefer any other SPB to it, and there is a Q-optimal SPB with the symmetric properties. Furthermore, Rochford's rebargining process converges to the P-optimal SPB when it starts from the P-optimal stable outcome, and to the Q-optimal SPB when its starts from the Q-optimal stable outcome.

Second, we will generalize these results and show how they can be extended to any market whose set of stable outcomes is a complete lattice. This program will be carried out here for the model of Demange and Gale [9], which substantially generalizes the assignment model of Shapley and Shubik [30].

Finally, we will show how these results are tied together by the lattice structure of the set of stable outcomes. Indeed, the set of SPBs is itself a lattice, and this follows from a powerful *algebraic* fixed point theorem (in contrast to *topological* fixed point theorems, such as Brouwer's) due to Alfred Tarski [31], which states the following.

THEOREM 1. *Let C be a complete lattice with respect to some partial order $\geqslant$, and let a be an increasing (order-preserving) function from C to C. Then the set of fixed points of a is nonempty and is itself a complete lattice with respect to the partial order $\geqslant$.*

Overall the results of the paper suggest that since the polarization of interests found in the core persists for (at least) certain classes of subsets of core outcomes, the problem of finding "fair" outcomes in games of this type may be intractable.

2. THE MARKET MODEL

There are two disjoint, finite sets of agents: $P = \{p_1, ..., p_m\}$ and $Q = \{q_1, ..., q_n\}$, called P-agents and Q-agents, respectively. An outcome of the market matches P-agents with Q-agents.

DEFINITION 1. A *matching* μ is a bijection of $P \cup Q$ onto itself of order two (that is $\mu \circ \mu = \text{identity}$) such that if p or q is not *unmatched* ($\mu(p) = p$ or $\mu(q) = q$) then $\mu(p)$ is in Q and $\mu(q)$ is in P.

Each agent is endowed with preferences on agents of the opposite set. These preferences are given by compensation functions f_{pq} and g_{pq}, defined as follows: $f_{pq}(u)$ is the amount of money p must receive in order to achieve utility u if he is matched with q; $g_{pq}(v)$ is analogously defined for q in Q. The f's and g's and their inverses are continuous and strictly increasing functions from R to R. We suppose that for each p and q the utility of being unmatched is given by numbers r_p and s_q. We will denote the set of functions f_{pq} and g_{pq} by f and g. The market is then $M = (P, Q; f, g; r, s)$.

DEFINITION 2. A *feasible payoff* (u, v) of M consists of a vector u, indexed by the components of P, and a vector v, indexed by the components of Q, such that there exists a matching μ with the properties

$$u_p = r_p \; (v_q = s_q) \qquad \text{if} \quad p(q) \quad \text{is self matched} \tag{1}$$

$$f_{pq}(u_p) + g_{pq}(v_q) \leqslant 0, \qquad \text{if} \quad \mu(p) = q. \tag{2}$$

Condition (2) is the requirement that the compensation to the two members of a matched pair is in fact a transfer, what one partner gains, the other loses. We say that the matching μ is *compatible* with the payoff (u, v).

DEFINITION 3. The feasible payoff (u, v) is *stable* if it is individually rational, that is, if

$$u_p \geqslant r_p, \qquad v_q \geqslant s_q \tag{3}$$

and if

$$f_{pq}(u_p)+g_{pq}(v_q)\geqslant 0 \quad \text{for all} \quad (p,q) \quad \text{in} \quad P\times Q. \tag{4}$$

If condition (4) did not hold then, since f_{pq} and g_{pq} are continuous and increasing, we could choose $u'_p > u_p$ and $v'_q > v_q$ such that $f_{pq}(u'_p)+g_{pq}(v'_q)\leqslant 0$, producing a feasible payoff and giving higher utility to both p and q; so the pair (p, q) would block the payoff (u, v).

The set of all stable payoffs will be denoted by C, since it equals the core of this market M when the rules of the market are that any P-agent and Q-agent who both agree may conclude a match, and each agent has the right to remain unmatched. U will denote the set of vectors u such that (u, v) is in C for some v. Quinzii [18] and Gale [11] present general results that show that C is always nonempty. Consider now the partial order on payoffs $x=(u, v)$ and $x'=(u', v')$ defined by $x \geqslant_P x$ if $u_p \geqslant u'_p$ for all p in P. It is shown in [9] that the core is a compact lattice under this partial order. (Since C is contained in $\mathbf{R}^{m+n}$, and the partial order $\geqslant_P$ is the natural order on $\mathbf{R}^{m+n}$, the fact that C is a compact lattice implies that it is a complete lattice, so that Tarski's theorem will apply.) Further, if $x \geqslant_P x'$ then $x' \geqslant_Q x$, meaning that $v_q \leqslant v'_q$ for all q in Q. These observations are also contained in [9]. As a corollary, since every compact lattice has a smallest and a largest element, there is a P-optimal stable payoff $(\bar{u}, \underline{v})$ with the property that for any stable (u,v), $\bar{u}\geqslant u$ and $\underline{v}\leqslant v$. There is also a Q-optimal stable payoff $(\underline{u}, \bar{v})$ such that for any stable (u, v), $\underline{u}\leqslant u$ and $\bar{v}\geqslant v$.

We will denote by C_μ the set of stable payoffs that are compatible with μ, and by Σ the set of matchings μ that are compatible to some stable payoff. If μ is in Σ we will call μ a *stable matching*. Similarly, $\mathbf{U}_\mu$ denotes the payoffs to P-agents compatible with a stable matching μ.

The assignment market [30] is the special case of this market (up to renormalization of the right hand side of Conditions (2) and (4)) that arises when the functions f_{pq} and g_{pq} are all linear and of the form $f_{pq}(u)=u-a_{pq}$ and $g_{pq}(v)=v+b_{pq}$, for nonnegative a_{pq} and b_{pq}. In the assignment market, if a matching μ is compatible to some stable payoff, then it is compatible to every stable payoff. This is no longer true in the more general market considered here (see the Appendix for an example). The fact that multiple stable matchings must be considered will be the source of most of the technical difficulties encountered in generalizing results from the assignment market.

Except for Proposition 5, the results presented in the remainder of this section are mostly special cases of results proved in [9]. We will state them without proofs.

PROPOSITION 1. *Let (u, v) and (u', v') be stable payoffs with matchings μ and μ'. Let $P^1=\{p\in P;\ u'_p>u_p\}$, $P^2=\{p\in P; u_p>u'_p\}$, $P^0=\{p\in P;$*

$u_p = u'_p\}$. Define Q^1, Q^2, and Q^0 analogously (i.e., $Q^1 = \{q \in Q;\ v'_q > v_q\}$, etc.). Then $\mu'(P^1) = \mu(P^1) = Q^2$ and $\mu'(P^2) = \mu(P^2) = Q^1$. Furthermore, every agent in P^0 who is matched (under either μ or μ') is matched with an agent in Q^0, and vice versa.

The proposition states that everyone who prefers one of two stable outcomes is matched (at either outcome) with someone who prefers the other outcome.

COROLLARY 1. *If p (q) is unmatched under some stable matching μ, then for every stable payoff (u, v) we have that $u_p = r_p$ ($v_q = s_q$).*

PROPOSITION 2. *The set* **U** *is a compact lattice*

PROPOSITION 3. *The set C_μ forms a compact lattice.*

COROLLARY 2. *The set* $\mathbf{U}_\mu$ *is a compact lattice.*

Define $\underline{u}(r) = \inf\{u \in U;\ u \geqslant r\}$, for all r in $[\underline{u}, \bar{u}]$. Since **U** is a compact lattice, $\underline{u}(r)$ is in U.

PROPOSITION 4. *The function $\underline{u}(r)$ is an increasing function of r.*

The following is an unpublished result of D. Gale, and will be proved in the Appendix.

PROPOSITION 5. *The function $\underline{u}(r)$ is a continuous function of r.*

3. SYMMETRICALLY PAIRWISE-BARGAINED OUTCOMES

In this section we define, for the model considered above, the bargaining and rebargaining functions introduced by Rochford [19], which determine the set of symmetrically pairwise-bargained outcomes. Rochford defines SPBs to be core outcomes that are fixed points of a bargaining process in which each agent's payoff is determined by a symmetric bargaining solution (see [22]) when the "threats" (or disagreement payoffs) are determined by the payoffs of the other agents.

For convenience we will use the notation $h_{ij} \equiv f_{ij}^{-1}(-g_{ij})$, so $h_{ij}(v_j) = u_i$ is the utility of agent i when matched to agent j with a money transfer that gives j a utility of v_j.

DEFINITION 4. The *threats* of p in P and q in Q at $x=(u, v)$ in C_μ are

$$t_p(x, \mu)=\max_{q \neq \mu(p)} \{h_{pq}(v_q), r_p\}$$

$$t_q(x, \mu)=\max_{p \neq \mu(q)} \{h_{pq}^{-1}(u_p), s_q\}.$$

That is, each agent's threat at an outcome x with matching μ is the maximum utility he could receive if he were matched with another agent, with a money transfer that would maintain that agent's current utility. The stability of $x=(u, v)$ implies that $u_p \geqslant t_p(x, \mu)$ and $v_q \geqslant t_q(x, \mu)$; i.e., no agent's threat at a stable outcome x exceeds his current utility.

The bargaining model is simple: if p and q are matched at $x \in C$, then they split equally the surplus that remains after each of them receives his threat utility. (This surplus is always nonnegative, since no agent's threat exceeds his current utility.[1]) Agent p, for example, would receive the sum of $f_{pq}(t_p(x))$, which is the monetary compensation he needs to maintain his threat utility when matched to q, and one-half of the surplus $(f_{pq}(u)+g_{pq}(v)-[f_{pq}(t_p(x))+g_{pq}(t_q(v))])$. But $f_{pq}(u)+g_{pq}(v)=0$, so this leads to the following definition.

DEFINITION 5. A symmetrically pairwise-bargained (SPB) payoff is a stable payoff $x=(u, v)$ (with compatible matching μ) such that $u_p=b_p(x)$ and $v_q=b_q(x)$, for all (p, q) in $P \times Q$, where

$b_p(x)=f_{pq}^{-1}[f_{pq}(t_p(x))-\frac{1}{2}(f_{pq}(t_p(x))+g_{pq}(t_q(x)))]$, if $\mu(p)=q$, and
$b_p(x)=r_p$ if p is self matched.
$b_q(x)=g_{pq}^{-1}[g_{pq}(t_q(x))-\frac{1}{2}(f_{pq}(t_p(x))+g_{pq}(t_q(x)))]$, if $\mu(q)=p$, and
$b_q(x)=s_q$, if q is self matched.

The following proposition says that the threats do not depend on the particular matching μ. This implies that $b_p(x)$ and $b_q(x)$ do not depend on the particular matching μ which is compatible with $x=(u, v)$. The set of SPB payoffs can therefore be defined simply as $\text{SPB} \equiv \{x \in C: u_p=b_p(x), v_q=b_q(x), \text{ for all } (p,q) \in P \times Q\}$.

[1] For our purposes here, this simple bargaining model can be regarded as a representative of any bargaining process in which the bargaining responds positively to the threats of the players, and in which the threats of a player vary inversely with the fortunes of his potential alternative partners. For a family of bargaining models in which the threats are formulated in a more consistent manner, see [1]. For a discussion of bargaining environments in which a player's bargaining outcome might *not* respond to his threats, see the illuminating discussion presented in [2].

PROPOSITION 6. *Let* $x=(u, v)$ *be in* $C_\mu \cap C_{\mu'}$. *Then*

(a) $t_p(x, \mu)=t_p(x, \mu')=t_p(x)$, *for all* p *in* P, *and* $t_p(x)=u_p$ *if* $\mu'(p) \neq \mu(p)$.

(b) $t_q(x, \mu)=t_q(x, \mu')=t_q(x)$, *for all* q *in* Q, *and* $t_q(x)=v_q$ *if* $\mu'(q) \neq \mu(q)$.

Note that the proposition also states that an agent's threat equals his payoff when there is a choice of partners with whom he can receive the payoff at a stable match.

Proof. We will prove (a). The proof of (b) is analogous. There are two cases:

Case 1. $\mu'(p)=\mu(p)$. The result follows directly from Definition 4.

Case 2. $\mu'(p) \neq \mu(p)$. Then,

$$u_p \geqslant t_p(x, \mu) \geqslant h_{p\mu'(p)}(v_{\mu'(p)})=u_p \tag{5}$$

and

$$u_p \geqslant t_p(x, \mu') \geqslant h_{p\mu(p)}(v_{\mu(p)})=u_p. \tag{6}$$

By (5) and (6) it follows that $u_p=t_p(x, \mu')=t_p(x)$, and the proof is complete.

Two of our main results can now be stated.

THEOREM 2. $SPB \neq \phi$.

THEOREM 3. *SPB forms a complete lattice under the partial order* $\geqslant_P$ *whose* sup *is P-optimal and whose* inf *is Q-optimal.*

Outline of the proof. Following [19], we will introduce a "rebargaining function" a to assist in the proof. Theorems 2 and 3 will follow immediately from Tarski's theorem once we establish that the function a has the following properties.

(1) $a: C \to C$

(2) $a(x)=x$ if and only if x is an SPB

(3) a is an increasing function (i.e., order-preserving in the partial order $\geqslant_P$).

(Rochford [19] established properties 1 and 2 as well as the continuity of a for the simple assignment market, in order to apply Brouwer's Theorem. In that market, property 3 is almost immediate. In the more general market considered here, the problem of establishing property 3 will be considerably harder, because of the need to consider multiple stable matchings.)

It will be simplest to first define the rebargaining function a, and then to explain the recursion on which it depends. Fix the order of the P-agents to be $p_1, \ldots, p_m$. For all $i = 1, \ldots, m$ let $S_i = \{p_j \in P; j < i\}$ and let $x = (u, v)$ be in C_μ. Define

$a_{p_i}(x, \mu) = b_{p_i}(y^{i-1}(x, \mu))$ and

$a_{\mu(p_i)}(x, \mu) = b_{\mu(p_i)}(y^{i-1}(x, \mu))$

$a_q(x, \mu) = s_q$ if q is self-matched, where

$y^{i-1}_{p_j}(x, \mu) = a_{p_j}(x, \mu)$ if p_j is in S_i, and

$y^{i-1}_{p_j}(x, \mu) = u_j$, otherwise.

$y^{i-1}_{q_j}(x, \mu) = a_{q_j}(x, \mu)$ if q_j is in $\mu(S_i)$, and

$y^{i-1}_{q_j}(x, \mu) = v_j$, otherwise.

This rebargaining function generalizes to this market the one defined in the assignment market by Rochford, who described it as follows.

> ... the components of the image vector of x are determined two at a time, and are based on the components already determined. Specifically, the components representing the incomes of the spouses of the first marriage are given to be the symmetrically bargained solution to the problem of how the output from their marriage is to be divided, where their threats are based on the partial allocation given by the vector x omitting the first two components. The next two components of the image of x, representing the incomes of the spouses in the second marriage, are likewise given to be the bargained distribution of income, where the threats are based on the partial allocation given by the components of x representing the incomes of the spouses in the third and higher-numbered marriages and the newly determined incomes of the spouses in the first marriage. The remaining components of the image of x are similarly determined.

The chief technical problem involved in establishing the required properties of the function a is to show that it does not depend on μ, i.e., to show that it is a function (only) of x. This is done in Proposition 7, which will be proved in the Appendix.

PROPOSITION 7. *If $x = (u, v)$ is in $C_{\mu'} \cap C_\mu$ then*

(a) $a_p(x, \mu) = a_p(x, \mu') = a_p(x)$, *for all p in P.*

(b) $a_q(x, \mu) = a_q(x, \mu') = a_q(x)$, *for all q in Q.*

The following proposition shows that the range of a is contained in the core. Furthermore, the image of x under a is compatible with the same matching(s) as x.

PROPOSITION 8. *If* $x=(u, v)$ *is in* C_μ *then* $a(x, \mu)$ *is in* C_μ.

Proof. We will show that for any $k=2, ..., m$, if $y^{k-1}=y^{k-1}(x, \mu)$ is in C_μ, then $y^k=y^k(x, \mu)$ is in C_μ, where we define $y^m \equiv a(x, \mu)$. Let y^{k-1} be in C_μ. If p_k is self-matched then $y^{k-1}=y^k$ and there is nothing to prove. So assume that p^k is matched under μ. Set $\mu(p_k)=q_k$. Since $f_{kk}(a_{p_k}(x, \mu))+g_{kk}(a_{q_k}(x, \mu))=0$ from the definition of a, it is enough to prove that

(a) $y^k_{p_k} \geqslant h_{kj}(y^k_{q_j})$, for all $q_j \neq q_k$

(b) $y^k_{q_k} \geqslant h^{-1}_{jk}(y^k_{p_j})$, for all $p_j \neq p_k$.

We know that $y^{k-1}_{p_k} \geqslant t_{p_k}(y^{k-1})$ and $y^{k-1}_{q_k} \geqslant t_{q_k}(y^{k-1})$, so $f_{kk}(t_{p_k}(y^{k-1}))+g_{kk}(t_{q_k}(y^{k-1})) \leqslant f_{kk}(y^{k-1}_{p_k})+g_{kk}(y^{k-1}_{q_k})=0$, where the last equality follows from the stability of y^{k-1}. The fact that the first sum above is nonpositive (i.e., that there is a nonnegative surplus for the bargainers to divide) implies, via the definition of a and the monotonicity of f_{kk} and g_{kk}, that $a_{p_k}(x, \mu) \geqslant t_{p_k}(y^{k-1})$ and $a_{q_k}(x, \mu) \geqslant t_{q_k}(y^{k-1})$. But $y^k_{p_k}=a_{p_k}(x, \mu)$, $y^k_{q_k}=a_{q_k}(x, \mu)$, and $y^{k-1}_j=y^{k-1}_j$. Now use the definition of t and get (a) and (b). With an analogous argument we can show that y^1 is in C_μ. Then y^k is in C_μ for all k, which concludes the proof.

We next show that a is an increasing function of x. We will need the following lemma

LEMMA 1. *Let* $x=(u, v)$ *and* $x'(u', v')$ *be in* C_μ *with* $x \geqslant_P x'$. *Then* $a(x) \geqslant_P a(x')$.

Proof. Since $u_p \geqslant u'_p$ for all p in P, we have from Proposition 1 that $v_q \leqslant v'_q$ for all q in Q. Then, suppose $a_{p_i}(x) \geqslant a_{p_i}(x')$ for all p_i in S_k, where $S_k=\{p_i \in P; i<k\}$ and $k \geqslant 2$. We claim that $a_{p_k}(x) \geqslant a_{p_k}(x')$. We only need to consider the case in which p_k is matched, since otherwise the assertion clearly holds by definition of a. Let $y=y^{k-1}(x, \mu)$ and $y'=y^{k-1}(x', \mu)$. Then $y \geqslant_P y'$, so $h_{kj}(y_{q_j}) \geqslant h_{kj}(y'_{q_j})$, for all $q_j \neq \mu(p_k)$, by Proposition 1 and the monotonicity of h_{kj}. Hence $t_{p_k}(y) \geqslant t_{p_k}(y')$. Analogously we show that $t_{\mu(p_k)}(y') \geqslant t_{\mu(p_k)}(y)$. Now we can write

$$a_{p_k}(x)=f^{-1}_{k\mu(p_k)}\left[\tfrac{1}{2}\left(f_{k\mu(p_k)}(t_{p_k}(y))-g_{k\mu(p_k)}(t_{\mu(p_k)}(y))\right)\right].$$

Since the function f's and g's are strictly increasing, it follows that $a_{p_k}(x) \geqslant a_{p_k}(x')$. By a similar argument it can be shown that $a_{p_1}(x) \geqslant a_{p_1}(x')$. By induction $a_p(x) \geqslant a_p(x')$ for all p in P, which completes the proof.

PROPOSITION 9. *The function* a *is an increasing function of* x.

Proof. Let $x=(u, v)$ and $x'=(u', v')$ be in C, with $x \geqslant_P x'$. If x and x' have a common compatible matching, we can apply Lemma 1 and obtain that $a(x) \geqslant_P a(x')$. Otherwise, let x' be in C_{μ_1} and x be in C_μ. To show that $a(x) \geqslant_P a(x')$ we show that there exists a set of stable payoffs $x' = x^1 \leqslant_P x^2 \leqslant_P \cdots \leqslant_P x^k = x$ such that x^i and x^{i+1} have a common compatible matching, for all $i = 1, ..., k-1$. Lemma 1 then implies that $a(x') \leqslant_P a(x^2) \leqslant_P \cdots \leqslant_P a(x)$, which proves Proposition 9. This is equivalent to showing that there exists a set $\{u^1, u^2, ..., u^k\}$ in $\mathbf{U}$, where $u' \equiv u^1 \leqslant u^2 \leqslant \cdots \leqslant u^k \equiv u$ and such that u^i and u^{i+1} have a common compatible matching, for all $i = 1, ..., k-1$. The proof is based on the existence of an increasing and continuous path α: $[0, 1] \to \mathbf{U}$, from u' to u. Assume for the time being that such a path exists.

Set $u' \equiv u^1$ in $\mathbf{U}_{\mu_1}$. Since α is continuous and increasing and $\mathbf{U}_{\mu_1}$ is a compact lattice it follows that $\{\alpha(t); \alpha(t) \in \mathbf{U}_{\mu_1}, t \in [0, 1]\}$ is a compact lattice. Hence $\sup\{\alpha(t); \alpha(t) \in \mathbf{U}_{\mu_1}, t \in [0, 1]\} \equiv u^2 \equiv \alpha(t_2)$ is in $\mathbf{U}_{\mu_1}$, where $t_2 = \sup \alpha^{-1}(u^2)$. Then $u^1 \leqslant u^2$. Now we will show that u^2 is in $\mathbf{U}_{\mu_2}$ for some $\mu_2 \neq \mu_1$. Let $t_n \to t_2$, when $n \to \infty$; $1 \geqslant t_n > t_2$ for all n. Then $\alpha(t_n) > \alpha(t_2)$, for all n, by the definition of t_2 and the monotonicity of α. So $\alpha(t_n)$ is in $\bigcup_{i \neq 1} \mathbf{U}_{\mu_i}$ which is closed since it is a finite union of closed sets, so $\alpha(t_2)$ is in $\bigcup_{i \neq 1} \mathbf{U}_{\mu_i}$, since $\alpha(t_n) \to \alpha(t_2)$. Then $u^2 = \alpha(t_2)$ is in $\mathbf{U}_{\mu_2}$, for some $\mu_2 \neq \mu_1$. If $\mu_2 \neq \mu$, take $\sup\{\alpha(t); \alpha(t) \in \mathbf{U}_{\mu_2}, t \in [0, 1]\} = u^3$. By a similar argument to the one used above we can show that $u^3 = \alpha(t^3)$ is in $\mathbf{U}_{\mu_2}$, where $t_3 = \sup \alpha^{-1}(u^3)$ and $u^3 \geqslant u^2 \geqslant u^1$. Furthermore, if $\mu^3 \neq \mu$, u^3 is in $\mathbf{U}_{\mu_3}$ for some $\mu_3 \notin \{\mu_1, \mu_2\}$. Since we have a finite number of matchings and $u = \alpha(1)$ is in $\mathbf{U}_{\mu_k}$, for some k, in a finite number of steps we will reach u. So we will get a set $u^1, ..., u^k$ such that $u^1 \leqslant u^2 \leqslant \cdots \leqslant u^k$ as required.

It remains to show the existence of α. Let γ: $[0, 1] \to [\underline{u}, \bar{u}]$, where $\gamma(t) = tu + (1-t)u'$. Then γ is continuous, and, since $u > u'$, γ is increasing. Define α: $[0, 1] \to \mathbf{U}$ by $\alpha(t) = \underline{u}(\gamma(t))$, where $\underline{u}(r) = \inf\{u \in \mathbf{U}; u \geqslant r\}$, for all r in $[\underline{u}, \bar{u}]$. From Propositions 4 and 5 we have that α has the desired properties, which concludes the proof.

It remains to prove that SPB is the set of fixpoints of a.

PROPOSITION 10. *x is in SPB if and only if $a(x) = x$.*

Proof. Let $x = (u, v) \in C_\mu$. If x is in SPB, suppose that $a_{p_i}(x) = u_i$ for all $i < k$. Since $a(x)$ is in C_μ (Proposition 8) and h is one-to-one, it follows that $a_{q_i}(x) = v_i$ for all q_i in $\mu(S_k)$. Then $y^{k-1}(x) = x$ and $a_{p_k}(x) = b_{p_k}(x) = u_k$. By a similar argument we show that $a_{p_1}(x) = u_1$. Then $a_{p_i}(x) = u_i$ for all p_i in P and consequently $a_{q_i}(x) = v_i$ for all q_i in $\mu(P)$. Since $a_{q_i}(x) = s_{q_i}$ if q_i is unmatched, $a(x) = x$. In the other direction, if $a(x) = x$, $y^{i-1}(x) = x$ for all $i = 1, ..., m$, so the result follows directly from the definition of a.

Our last result is that the rebargaining function a converges to the P-optimal SPB when it starts from the P-optimal stable outcome, and to the Q-optimal SPB when it starts from the Q-optimal stable outcome.

THEOREM 4. *Let $\bar{x}$ and $\underline{x}$ be the P-optimal and Q-optimal stable outcomes, respectively. Then $a^n(\bar{x}) \to \sup SPB$ and $a^n(\underline{x}) \to \inf SPB$, when $n \to \infty$.*

We will need to show that a is continuous.

PROPOSITION 11. *The function a is a continuous function of x.*

Proof. It is enough to see that $a|_{C_\mu}: C_\mu \to C_\mu$ is continuous, for all $\mu \in \Sigma$, since C is the union of a finite number of closed sets C_μ. Suppose that for all p_i in S_k and q_j in $\mu(S_k)$ the functions $a_{p_i}(x)$ and $a_{q_j}(x)$ are continuous in x, where $k \in \{2, ..., n\}$. Then $y^{k-1}(x)$ is continuous in x. Now observe that the functions $t_{p_k}(y^{k-1}(x))$ and $t_{\mu(p_k)}(y^{k-1}(x))$ are continuous for they are maxima of continuous functions. From the continuity of the functions f, g, f^{-1}, and g^{-1} it follows that $a_{p_k}(x)$ and $a_{\mu(p_k)}(x)$ are continuous. By a similar argument we obtain the result for p_1 and $\mu(P_1)$, so $a_{p_i}(x)$ and $a_{q_j}(x)$ are continuous for all p_i in P and q_j in $\mu(P)$. Since $a_{q_j}(x) = s_j$ if q_j is unmatched, this concludes the proof.

Proof of Theorem 4. Since a is increasing and $\bar{x} \geqslant_P a(\bar{x})$, it follows that $\bar{x} \geqslant_P a^n(\bar{x}) \geqslant_P a^{n+1}(\bar{x}) \geqslant_P \underline{x}$ for all n. That is $\{a^n_p(\bar{x})\}_n(\{a^n_q(\bar{x})\}_n)$ is a decreasing (increasing) and bounded sequence of real numbers, for all p in P (q in Q). So $a^n_p(\bar{x}) \to w_p$ and $a^n_q(\bar{x}) \to w_q$ when $n \to \infty$, for some numbers w_p and w_q, for all (p,q) in $P \times Q$. Then $a^n \to w$ when $n \to \infty$, and w is in C, since C is closed. On the other hand, $w = \lim a(a^n(\bar{x})) = a(w)$, by the continuity of a, so w is in SPB. Now suppose that z is in SPB. Then $z \leqslant_P \bar{x}$. So $w = \lim a^n(\bar{x}) \geqslant_P \lim a^n(z) = z$. Hence $w = \sup$ SPB. The other part of the theorem follows dually.

4. DISCUSSION

We have shown how to capitalize on the lattice structure of the core of a two-sided matching market, to draw conclusions about certain fixed points within the core. On a technical level, the advantage of using Tarski's theorem rather than a topological result like Brouwer's fixed point theorem to prove existence is that it permitted us also to draw conclusions about the structure of the set of fixed points.[2] In economic terms, this structure permitted us to make welfare comparisons between different outcomes, e.g., to observe that there is a P-optimal SPB that is simultaneously the best for all the P-agents and the worst for all the Q-agents. Thus the set SPB of

[2] Although the mathematical tools used in game theory and mathematical economics are much more often topological rather than algebraic in nature, this paper is by no means the first use of such tools in game theory. Tarski's theorem itself has been used to study the equilibria of noncooperative games in [32] and [33], and algebraic fixed point theorems closely related to Tarski's theorem ([20, 5]) were used in cooperative game theory in [21].

fixed points, which is a subset of the core, exhibits the same polarization of interests that characterizes the core. Again on a technical level, this suggests that many of the properties of the core of such a market are intimately connected to its lattice structure.[3]

Since this lattice structure appears to be characteristic of two-sided matching markets, the techniques used in this paper can be applied to any such market. What is needed is to construct an order-preserving function a from the core to itself, whose fixed points are of interest.[4] If the function a is also continuous, convergence results of the kind obtained in Theorem 4 will also apply.

In terms of practical importance, these results suggest some avenues by which to address the issues of equity and distribution that arise in markets of this kind by virtue of the polarization of interests that exists in the core. For example, it is shown in [24] that the labor market for residency positions in American teaching hospitals is administered in such a way as to result in the hospital-optimal stable outcome. At the same time, there are concerns in the medical community about the distribution of certain kinds of resident physicians to certain kinds of hospitals (see, e.g., [29]). It seems plausible that particular concerns of this type could be formulated in terms of a "redistribution function" whose fixed points would be core outcomes with respect to which no further redistribution would achieve the desired goals. What the kind of results presented here suggest is that, while such redistribution may move the outcome to an interior point of the core, some polarization of interests among the fixed points of the redistribution process will remain, unless there is a unique fixed point. In this sense, the polarization of interests that exists in the core of two-sided matching markets may be even more fundamental than previous results have suggested.

I. Appendix: Example 1 and Proofs of Propositions 7 and 5

I.1. *Example* 1

The following example shows that not every stable matching need be compatible with every stable outcome.

[3] However, not all of the distinctive properties of such markets will be explainable in terms of the lattice structure of the core. For example, core outcomes in a wide variety of matching markets involving one-to-one matching have been shown to share some striking incentive properties (see, e.g., [10, 23, 8, 17, 13, 9]), but these have been shown [28] not to generalize to the case of many-to-one matching, even though the lattice structure persists in these more general models.

[4] Obviously such a function a may arise in ways entirely removed from the kind of bargaining considerations that led to the particular function considered in this paper. Conversely, notions of bargaining equilibrium need not be confined to the core: see e.g., [7] for discussion of such a formulation.

Let the agents be $P=\{p_1, p_2\}$ and $Q=\{q_1, q_2\}$, with "reservation prices" $r_1=r_2=0$, and $s_1=s_2=-4\pi$. The compensation functions are

$$f_{11}(u)=u-\sin(u), \qquad g_{22}(v)=v+\sin(v)$$

$$f_{pq}(u)=u \qquad \text{if} \quad (p, q)\neq(1, 1)$$

$$g_{pq}(v)=v \qquad \text{if} \quad (p, q)\neq(2, 2).$$

There are only two possible matchings. These are μ and μ' given by $\mu(p_1)=q_1$, $\mu(p_2)=q_2$; and $\mu'(p_1)=q_2$, $\mu'(p_2)=q_1$.

The payoffs

$$x=(u, v)=(\pi/2,\ \pi/2+1,\ 1-\pi/2,\ -\pi/2),$$

and

$$x'=(u', v')=(3\pi/2,\ 1+3\pi/2,\ -1-3\pi/2,\ -3\pi/2)$$

are stable with matchings μ and μ', respectively. However μ is not compatible with x', and μ' is not compatible with x.

I.2. *Proof of Proposition 7*

The proof of Proposition 7 will be accomplished in parts, by proving Lemmas 2, 3, and 4 below.

We will assume that $x=(u, v)$ is in $C_{\mu'}\cap C_{\mu}$. Let $z\equiv a(x, \mu)$, $z'\equiv a(x, \mu')$, $P^0\equiv\{p\in P;\ z_p=z'_p\}$, and $Q^0\equiv\{q\in Q;\ z_q=z'_q\}$. We need to show that $P^0=P$ and $Q^0=Q$, i.e., that every agent is indifferent between z and z'.

LEMMA 2. *Let* $P'=\{p\in P^0; z_p=u_p\}$ *and* $Q'=\{q\in Q^0; z_q=v_q\}$. *Then for all p in* P',

(a) $z_{\mu(p)}=z'_{\mu(p)}=v_{\mu(p)}$ *if p is matched by* μ.

(b) $z_{\mu'(p)}=z'_{\mu'(p)}=v_{\mu'(p)}$ *if p is matched by* μ'.

and for all q in Q',

(c) $z_{\mu(q)}=z'_{\mu(q)}=u_{\mu(q)}$ *if q is matched by* μ.

(d) $z_{\mu'(q)}=z'_{\mu'(q)}=u_{\mu(q)}$ *if q is matched by* μ'.

Proof. We will prove (a) and (b). The proof of (c) and (d) is analogous. From the stability of x, z and z' (Proposition 8) we have that

$$v_{\mu(p)}=h^{-1}_{p\mu(p)}(u_p)=h^{-1}_{p\mu(p)}(z_p)=z_{\mu(p)}$$

$$v_{\mu'(p)}=h^{-1}_{p\mu'(p)}(u_p)=h^{-1}_{p\mu'(p)}(z'_p)=z'_{\mu'(p)}$$

The result now follows from Proposition 1.

Note that Corollary 1 implies that any agent who is unmatched at either μ or μ' is indifferent between z and z'.

LEMMA 3. *If* $\mu(p) \neq \mu'(p)$ *then*

(a) $z_p = z'_p = u_p$

(b) $z_{\mu(p)} = z'_{\mu(p)} = v_{\mu(p)}$ *if* p *is matched by* μ.

(c) $z_{\mu'(p)} = z'_{\mu'(p)} = v_{\mu'(p)}$ *if* p *is matched by* μ'.

Proof. Let p_i be such that $\mu(p_i) \neq \mu'(p_i)$. If p_i is self-matched under μ or μ' the result follows directly from Corollary 1. Then we will assume that p_i is matched under μ and μ'. Now, construct a graph whose vertices are $P \cup Q$. We say that p and q are connected by an arc if either $\mu(p) = q$ or $\mu'(p) = q$. Let $\tilde{P} \cup \tilde{Q}$ be all vertices which can be reached by a path starting from p_i. Since p_i is matched under μ and μ', $\tilde{P}$ is nonempty and $|\tilde{Q}| \geqslant 2$. Suppose Lemma 3 is true for all p_j in $\tilde{P}$ such that $j < k$ and p_k is in $\tilde{P}$. We will show that it is true for p_k. In fact, if p_k is self-matched under μ or μ' we do not have anything to prove, by Corollary 1. So suppose p_k is matched under μ and μ'. We have that $\mu(p_k) \neq \mu'(p_k)$, since otherwise $(p_k, \mu(p_k))$ would be an isolated component of the graph, which contradicts the fact that $|\tilde{Q}| \geqslant 2$ and $\tilde{P} \cup \tilde{Q}$ is connected to p_i. So let $\mu(p_k) = q_i$ and $\mu'(p_k) = q_j$ for some $q_i \neq q_j$. By Lemma 2 we need only consider the case where there are some p_l and p_m such that $\mu'(q_i) = p_m$ and $\mu(q_j) = p_l$. Then p_l and p_m are in $\tilde{P}$, $p_l \neq p_k$ and $p_m \neq p_{k'}$ since $|\tilde{Q}| \geqslant 2$. There are three cases:

Case 1. $l < k$.

Then, $z_{\mu'(p_k)} = z'_{\mu'(p_k)} = v_{\mu'(p_k)}$, by assumption, since $\mu(p_l) = q_j = \mu'(p_k) \neq \mu'(p_l)$. Then, q_j is in Q' by the inductive hypothesis, and from Lemma 2 it follows that $z_{p_k} = z'_{p_k} = u_k$. But then p_k is in P' and from Lemma 2, again, we get the assertion (b).

Case 2. $m < k$.

Then $z_{\mu(p_k)} = z'_{\mu(p_k)} = v_{\mu(p_k)}$ by assumption, since $\mu'(p_m) = q_i = \mu(p_k) \neq \mu(p_m)$. The argument follows analogously to Case 1.

Case 3. $l > k$ and $m > k$.

Let $y = y^{k-1}(x, \mu)$. Since $\mu(p_l) = \mu'(p_k) \notin \mu(S_k)$ and $p_m \notin S_{k'}$ we have that $y_{\mu'(p_k)} = v_{\mu'(p_k)}$ and $y_{p_m} = u_m$. Then, from the definition of t we have that

$$u_k \geqslant t_{p_k}(y) \geqslant h_{k\mu'(p_k)}(v_{\mu'(p_k)}) = u_k$$

$$v_{\mu(p_k)} \geqslant t_{\mu(p_k)}(y) \geqslant h^{-1}_{m\mu(p_k)}(u_m) = v_{\mu(p_k)}$$

since $\mu'(p_m) = q_i = \mu(p_k)$. So $t_{p_k}(y) = u_k$ and $t_{\mu(p_k)}(y) = v_{\mu(p_k)}$. From this it follows that $z_{p_k} = u_k$ and $z_{\mu(p_k)} = v_{\mu(p_k)}$.

Now, let $y' = y^{k-1}(x, \mu')$. By a similar argument we have that $z'_{p_k} = u_k$ and $z'_{\mu'(p_k)} = v_{\mu'(p_k)}$, which completes the proof that Lemma 3 is true for p_k. Analogously we prove that it is true for p_{i_1}, where i_1 is the minimum index for the p's in $\tilde{P}$. Then by induction we get that Lemma 3 is true for all p in $\tilde{P}$, and in particular for p_i.

LEMMA 4. *If* $\mu(p) = \mu'(p)$ *then*

(a) $z_p = z'_p$

(b) $z_{\mu(p)} = z'_{\mu(p)}$.

Proof. Let $P' = \{p \in P;\ \ \mu(p) = \mu'(p)\} = \{p_{i_1}, p_{i_2}, ..., p_{i_r}\}$, where $i_1 < i_2 \cdots < i_r$. Define $S'_i = \{p_j \in P'; j < i\}$. We claim that Lemma 4 is true for all p in $S'_{i_{k+1}}$, for all $k = 1, ..., r$. In fact, if we suppose it is true for all p in S_{i_k}, then, using Lemma 3 we can write:

$$y^{i_k - 1}_{p_i}(x, \mu) = y^{i_k - 1}_{p_i}(x, \mu') = z_{p_i}, \qquad \text{if } p_i \text{ is in } S'_{i_k}\text{; and} = u_i, \text{ otherwise}$$

$$y^{i_k - 1}_{q_i}(x, \mu) = y^{i_k - 1}_{q_i}(x, \mu') = z_{q_j}, \qquad \text{if } q_j \text{ is in } \mu(S'_{i_k})\text{; and} = v_{q_j}, \quad \text{otherwise.}$$

Then $y^{i_k - 1}(x, \mu) = y^{i_k - 1}(x, \mu')$. Now use the definition of a and get the assertions (a) and (b) for all p in $S_{i_{k+1}}$.

By an analogous argument we can see that (a) and (b) hold for p_{i_1}. Then the result follows by induction. This completes the proof of Lemma 4, and of Proposition 7.

I.3. *Proof of Proposition* 5

We will need Lemmas 5 and 6. Lemma 5 will be only stated; its proof can be found in Demange and Gale [9].

LEMMA 5. *If* $|Q| \leqslant |P|$ *then there exists some* p *in* P *such that* $\underline{u}_p(r) = r_p$.

Lemma 6 and Proposition 5 are due to D. Gale (personal communication). In order to simplify our notation, we will write the vector inequality $u < u'$ to signify that $u_p \leqslant u'_p$ for all p in P and $u_{p'} < u'_{p'}$ for some p' in P.

LEMMA 6. *Let* $u < u'$ *in* U *and let* $P^1 = \{p; u'_p > u_p\}$. *Choose* $u < \tilde{r} < u'$ *so that* $u_p < \tilde{r}_p < u'_p$ *for all* p *in* P^1. *Then there is some* $(\tilde{u}, \tilde{v})$ *stable for* $M(\tilde{r}) \equiv M(P, Q, f, g, \tilde{r}, s)$ *such that* $\tilde{u}_p = \tilde{r}_p$ *for some* p *in* P^1.

Proof. Let μ and μ' be compatible matchings for u and u', respectively. By Proposition 1, $\mu(P^1) = \mu'(P^1) = Q^2$. Let $(\underline{u}'', \bar{v}'')$ be the Q-optimal payoff for $M(P^1, Q^2, f, g, \tilde{r}, s)$. Now define $(\tilde{u}, \tilde{v})$ to agree with $(\underline{u}'', \bar{v}'')$ on P^1 and

Q^2, and with (u, v) on $P-P^1$ and $Q-Q^2$, and let $\tilde{\mu}$ be the corresponding matching. By Lemma 5 $\underline{u}''_p=\tilde{r}_p$ for some p in P^1. It remains to show that $(\tilde{u}, \tilde{v})$ is stable for $M(\tilde{r})$. Consider a possible blocking pair (p, q). If p is in P^1 and q is in $Q-Q^2$, then $\tilde{u}_p \geqslant \tilde{r}_p > u_p$, so if (p, q) blocks $(\underline{u}, \tilde{v})$ it would also block (u, v), contradicting stability of (u, v). Now for q in Q^2, $\tilde{v}_q=\bar{v}''_q \geqslant v'_q$, since (u', v') is stable in $M(P^1, Q^2; \tilde{r})$. Hence if p is in $P-P^1$, q is in Q^2, and (p, q) blocks $(\tilde{u}, \tilde{v})$, it would also block (u', v'), contradicting the stability of (u', v').

Proof of Proposition 5. If $\underline{u}(r)$ were discontinuous at r, since **U** is compact, there would be a sequence $r^n \rightarrow r$, when $n \rightarrow \infty$, r^n in $[\underline{u}, \bar{u}]$, such that $\underline{u}(r^n) \rightarrow u' \neq \underline{u}(r)$. Since $\underline{u}(r^n) \geqslant r^n$, it follows that $u' \geqslant r$, so $u' > \underline{u}(r)$, since u' is stable. Let P^1 and $\tilde{r}$ be as in Lemma 6: i.e., $\underline{u}_p(r) < \tilde{r}_p < u'_p$ for all p in P^1, and $\underline{u}_p(r)=\tilde{r}_p=u'_p$ for all p in $P-P^1$. Then, for n sufficiently large we must have:

(a) $r^n < \tilde{r}$, since $r^n \rightarrow r$ when $n \rightarrow \infty$, and $r \leqslant \underline{u}(r) < \tilde{r}$; and

(b) $\underline{u}_p(r^n) > \tilde{r}_p$, for p in P^1, since $\underline{u}(r^n) \rightarrow u'$ when $n \rightarrow \infty$, and $u'_p > \tilde{r}_p$.

But from Lemma 6 there is a stable payoff $(\tilde{u}, \tilde{v})$ for $M(\tilde{r})$ with $\tilde{u}_p=\tilde{r}_{p'}$ for some p in P^1. Every agent in P^1 is matched under $\tilde{\mu}$ so if p' is self-matched under $\tilde{\mu}$ it follows that p' is in $P-P^1$, so it is self-matched under μ. By Corollary 1 this means that $\underline{u}_{p'}=\bar{u}_{p'}=r_{p'}$ which implies that $r^n_{p'}=r_{p'}=\tilde{r}_{p'}$ for all n, since r^n is in $[\underline{u}, \bar{u}]$. Then $\tilde{u}_{p'}=r^n_{p'}$. Since, from (a), $r^n < \tilde{r}$, we can conclude that $(\tilde{u}, \tilde{v})$ is stable for $M(P, Q; r^n)$, so $\tilde{r}_p=\tilde{u}_p \geqslant \underline{u}_p\, t^n)$, by definition of $\underline{u}(r^n)$. But this contradicts (b) above, and completes the proof.

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