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WEAK VERSUS STRONG DOMINATION IN A MARKET WITH INDIVISIBLE GOODS

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The core of a market in indivisible goods can be defined in terms of strong domination or weak domination. The core defined by strong domination is always non-empty, but may contain points which are unstable in a dynamic sense. However, it is shown that there are always stable points in the core, and a characterization is obtained. The core defined by weak domination is always non-empty when there is no indifference, and has no instability problems. In this case, the core coincides with the unique competitive allocation.

1. Introduction

In a recent paper, Shapley and Scarf (1974) consider a market with indivisible goods as a game without side payments. They define the core of this market in the usual way, as the set of allocations which are not strongly dominated, and prove that it is always non-empty. However, they show by an example that this result depends on the core being defined in terms of strong rather than weak domination; if the core is defined by weak domination, then there are markets for which the core is empty. The purpose of this paper is to point out several other implications of the differences between strong and weak domination in this type of market game.

The first consequence of using strong instead of weak domination is that it is possible for a point in the core of a market to be unstable in the following sense: an allocation x can be in the core of a given market, but not in the core of the market in which x itself is the initial endowment. In a sense, x would be stable only until it was realized. There will always exist stable allocations in the core, however, and we will characterize the set of stable allocations in terms of prices.

Also, the relation between the core and the set of competitive allocations depends on whether strong or weak domination is used to define the core. When the core is defined by strong domination, it always contains the set of competitive allocations (which is itself non-empty, and can contain several allocations). The core can be strictly larger than the set of competitive allocations.

When the core is defined by weak domination, it can be empty. In particular the core need not contain the non-empty set of competitive allocations. However, it can be shown that if no trader is indifferent between any of the goods on the market, then the core defined by weak domination is always non-empty, and is in fact precisely equal to a *unique* competitive allocation.

If some traders are indifferent between some of the goods on the market, we observe another anomalous effect. It is possible for all traders to be indifferent between two allocations x and y and yet x is competitive and y is not. Further, we can have x competitive, y not competitive but have y weakly Pareto superior to x.

2. Model

We consider markets with n traders, each of whom owns one indivisible good. (Shapley and Scarf suggest a market in houses as an appropriate example.) The traders each have purely ordinal preferences over the goods, and no trader has any use for more than one item.

We will denote the initial endowment of the market by $w = (w_1, \ldots, w_n)$ where w_i is the good brought to the market by the ith trader. For a given set of preferences we will denote by $M(w)$ the market with endowment vector w. Denote the ith trader's preference relation by R_i, where $w_j R_i w_k$ means trader i likes the item w_j at least as well as w_k. If $w_j R_i w_k$ but $\sim w_k R_i w_j$, we say trader i *strictly prefers* w_j to w_k, and denote this by $w_j P_i w_k$. If $w_j R_i w_k$ and $w_k R_i w_k$, we say that trader i is *indifferent* between w_j and w_k and write this as $w_j I_i w_k$.

We define an *allocation* to be any permutation of the initial endowment w. Thus the set of allocations represents the set of all possible trades which result in each trader having possession of exactly one item. A general allocation will sometimes be denoted as a vector $x = (x_1, \ldots, x_n)$, where it is understood that the x_i can be mapped by some one-to-one mapping into the corresponding w_j.

Let N denote the set of all traders, and let S be a subset of N. We say that an allocation x (strongly) *dominates* an allocation y if there is some coalition S such that

(i) $\{x_i \mid i \in S\} = \{w_i \mid i \in S\}$,

(ii) $x_i P_i y_i$ for all $i \in S$.

The first condition says that the coalition S is *effective* for the allocation x, and the second condition says that every member of S strictly prefers x to y. Thus x (strongly) dominates y if, by trading among themselves, a coalition S could arrive at a reallocation x which is *strictly* preferred by each member of S.

We define *weak domination* by relaxing condition (ii) to read (iia) $x_i R_i y_i$ for all $i \in S$, and $x_i P_i y_i$ for some $i \in S$.

For whichever type of domination is under consideration, we define the *core* of the market to be the set of undominated allocations. Shapely and Scarf show that the core defined by strong domination is always non-empty, but the core defined by weak domination may be empty. (By definition, the core defined by weak domination is contained in the core defined by strong domination).

When no confusion will result, we will refer to the core defined by strong domination simply as the core, and to strong domination simply as domination.

3. Shapley–Scarf example

Let $N = \{1, 2, 3\}$ and $w = (w_1, w_2, w_3)$. The preferences of the three players are as follows:

(1) $\qquad w_3 P_1 w_2 P_1 w_1,$

(2) $\qquad w_1 P_2 w_2 P_2 w_3,$

(3) $\qquad w_2 P_3 w_3 P_3 w_1.$

The allocation $x = (w_3, w_1, w_2)$ is clearly in the core of the market $M(w)$, since it assigns to each trader his most preferred good. In fact, x is a competitive allocation in $M(w)$ supported by prices $(1, 1, 1)$.

Let us now look at $y = (w_2, w_1, w_3)$, which gives only trader 2 his most preferred good. It is straightforward to show that y cannot be competitive, but y is in the core of $M(w)$, since it is undominated by any other allocation. In particular, the allocation x fails to dominate y, because the coalition of traders $\{1, 3\}$ which strictly prefers x to y is not effective for x (i.e., they cannot accomplish x without trader 2), while the traders in the grand coalition $\{1, 2, 3\}$ do not all strictly prefer x to y (since trader 2 gets w_1 in both allocations).

Nevertheless, the allocation y cannot be considered stable. For, suppose that the market $M(w)$ should result in the allocation y, i.e., suppose that the only trade should be the bilateral one between traders 1 and 2. As soon as the traders take possession of their new goods, a new market comes into being: the market $M(y)$. And in this market, y *is* dominated by x, since the coalition $\{1, 3\}$, which strictly prefers x to y, is effective for x. Simply stated, once the endowment of the market becomes $y = (w_2, w_1, w_3)$, the coalition $\{1, 3\}$ is effective for the mutually profitable bilateral trade which results in the allocation x.

The difficulty in the previous example arises from the fact that the allocation y is not in the core of the market $M(y)$. We therefore define an allocation x to be *stable* if and only if it is in the core of $M(x)$.

To see the relation between stability and prices, we define a price vector π to be any non-zero vector of non-negative numbers $\pi = (\pi_1, \ldots, \pi_n)$, and we say that a pair (π, x) where π is a price vector and x is an allocation is an *efficiency equilibrium* if, for every trader $i \in N$, $x_j P_i x_i$ implies $\pi_j > \pi_i$. Intuitively, if (π, x) is an efficiency equilibrium, then the allocation x gives to each trader i the best good he could purchase at the prices π, were he to sell his own assignment x_i at the price π_i. We say that an allocation x is efficient if there exists a price vector π such that (π, x) is an efficiency equilibrium.[1] We will call an efficiency equilibrium a competitive equilibrium if for each trader x the price of his final good equals the price of his initial good. An allocation x is called competitive if there exists a price π such that (π, x) is a competitive equilibrium.

Theorem 1. An allocation is stable if and only if it is efficient.

Proof. Let x be an efficient allocation. Then there exists a price vector π such that for all $i \in N$, $x_j P_i x_i$ implies $\pi_j > \pi_i$. Suppose now that x were not stable. Then x is not in the core of $M(x)$, so there is an allocation y which dominates x, via some coalition $S \subseteq N$ in the market $M(x)$. This means $\{y_i \mid i \in S\} = \{x_i \mid i \in S\}$ and $y_i P_i x_i$ for all $i \in S$. Thus each trader $i \in S$ strictly prefers some good $y_i = x_j$, which implies $\pi_j > \pi_i$. But since the good x_j must belong to some player $j \in S$, we can construct a 'cycle' $\pi_j > \pi_i > \ldots > \pi_k > \pi_j$, where i, j, k are all members of S. This is plainly an absurdity, so we see that x must be stable if it is efficient.

Now let x be a stable outcome. Then no coalition of traders exists which, by trading among its members, could allocate to each a good which they strictly prefer to that which they receive at the allocation x. Thus there must be some trader i who likes the good x_i at least as well as all other goods in the market (otherwise a profitable trade could be arranged among a coalition of players $S = \{i_1, i_2, \ldots, i_s = i_0\}$ such that trader i_p likes the good belonging to i_{p+1} at least as well as any good in the market). Renumbering if necessary, let $i = 1$. Thus trader 1 has no desire to trade with any of the remaining traders.

Since no mutually profitable trade is possible among the remaining traders, there must be one among them who likes his good at least as well as all of those in the market except possibly that of trader 1. Renumber if necessary so that this trader is number 2.

In this manner we can order all of the traders so that k likes his good at least as well as that of any higher numbered trader.[2] If we let π be any price vector in which $\pi_i > \pi_{i+1}$, then (π, x) is an efficiency equilibrium, and thus x is efficient. ■

The construction of the coalitions which trade with each other in the above theorem is the method of 'top trading cycles'. It is due to David Gale and discussed in Shapley and Scarf (1974), where it is used to demonstrate the existence of a competitive allocation. Since a competitive allocation is efficient, we have as a corollary to the above theorem:

[1]This terminology is borrowed from Shitovitz (1973).

[2]This ordering constitutes a 'top trading cycle' in which each cycle consists of exactly one player.

Corollary. There exists at least one stable allocation in the core of every market $M(w)$.

It is also worth noting that the existence of unstable allocations in the core is a phenomenon that results directly from the indivisibility of goods in the market. In a market with divisible goods (and with continuous and insatiable preferences), every allocation (commodity bundle) in the core is stable.

To see this, consider an unstable allocation x. The fact that x is not in the core of $M(x)$ means that there is some allocation y which dominates x in the market $M(x)$. Letting y be individually rational, we may assume without loss of generality that each trader (weakly) prefers y_i to x_i. However, in a market with a divisible commodity, this implies the existence of another allocation, y^1, such that every trader *strictly* prefers y^1 to x. The allocation y^1 is produced from y by means of an infinitesimal transfer of the divisible good from traders who strictly preferred y to x to traders who were indifferent between y and x.

Since every trader strictly prefers the allocation y^1 to x, x is not in the core of any market $M(w)$ (since the coalition of all traders is effective regardless of the initial endowment). So every allocation in the core of a market with divisible goods is stable.

Note also that since every unstable allocation can be *weakly* dominated (via the coalition of all traders), no allocation in the core defined by weak domination is unstable.

It is straightforward to show that any competitive allocation can be thought of as resulting from the method of top trading cycles described in Theorem 1, and hence must be contained in the core defined by strong domination [see Shapley and Scarf (1974, p. 18)]. Using the fact that a competitive allocation must come from top trading cycles, we get the following lemma:

Lemma 1. If no trader is indifferent between any goods, then a competitive allocation weakly dominates any other allocation.

Proof. If x is any competitive allocation, we saw above that we can think of x as being arrived at via trading among top trading cycles $S_1, S_2, \ldots, S_p$. Let y be any allocation. If $y_i \neq x_i$, $\forall i \in S_1$, x dominates y via the coalition S_1 since S_1 is effective for x and x gives each member of S_1 its most preferred good. If $y_i = x_i$, $\forall i \in S_1$, but $y_i \neq x_i$, $\forall i \in S$, x dominates y via $S_1 U S_2$, since $S_1 U S_2$ is effective for x and $x_i R_i y_i$, $\forall i \in S_1 U S_2$, and $x_i P_i y_i$ for some $i \in S_1 U S_2$ (again since both x and y give S_1 its most preferred good and x gave S_2 its most preferred of what was left). Proceeding in this manner we see that x weakly dominates all other allocations. ■

Theorem 2. If no trader is indifferent between any goods, then the core defined by weak *domination is always non-empty, and contains exactly one allocation. This allocation is the* unique *competitive allocation.*

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Proof. We know that under the conditions above a competitive allocation weakly dominates every other allocation, (competitive or not). Thus we need only show that *no* allocation weakly dominates a competitive allocation.

Let x be a competitive allocation. Associated with x are coalitions $S_1, \ldots, S_p$ (top trading cycles) which are effective for x, and prices $\pi_1, \ldots, \pi_N$ which are constant for each S_j and such that $i < j$ implies $\pi_i \geqq \pi_j$.

Suppose y weakly dominates x via some coalition T. Then $\{y_i \mid i \in T\} = \{w_i \mid i \in T\}$, $y_i R_i x_i$ for all $i \in T$, and $y_i P_i x_i$ for at least one $i \in T$.

Let j be the smallest integer such that $S_j \cap T \neq \phi$. Since x is competitive, if $y_i P_i x_i$, for $i \in S_j \cap T$, then y_i must have sold at a higher price than x_i. But this implies that y_i must have been traded in some S_k for $k < j$, and hence must have been the initial endowment of some member m of S_k. But T is effective for y, so $m \in T$. This contradicts our assumption that j was the smallest integer such that $S_j \cap T \neq \phi$.

Thus it is false that $y_i P_i x_i$ for $i \in S_j \cap T$, and since there is no indifference, and $y_i R_i x_i$ for all $i \in T$, it must be that $y_i = x_i$ for all $i \in S_j \cap T$.

If we assume the $S_1, \ldots, S_p$ are *minimal* cycles, then it follows that $S_j \subset T$, and $T - S_j$ is effective for y.

Continuing in the same manner, we see that for all $i \in T$, $x_i = y_i$, contradicting the assertion that y dominates x. ■

If we had merely wanted to show that there was a unique competitive allocation, we would have done so by using the fact that every competitive equilibrium can be generated using top trading cycles.

These results partially clarify a question Shapley and Scarf raise: Can there be non-competitive points in the core defined by strong domination which are not weakly dominated by a competitive allocation? If there is no indifference, Lemma 1 says that a competitive allocation weakly dominates everything else. Thus if there exists a market of this type with allocations in the core defined by strong domination which are not weakly dominated by a competitive allocation, there must be indifference in some peoples preferences.

The effect of not ruling out indifferences in traders' preferences can be shown by the following examples:

Example 1. A market in which allocations x and y are completely indifferent for all traders and x is competitive but y is not.

Let there be four traders with the following preferences over the four goods,

$$P_1: \quad w_2 I_1 w_4 P_1 w_1 P_1 w_3,$$

$$P_2: \quad w_1 P_2 w_3 P_2 w_2 P_2 w_4,$$

$$P_3: \quad w_1 P_3 w_4 I_3 w_2 P_3 w_3,$$

$$P_4: \quad w_1 P_4 w_3 P_4 w_4 P_4 w_2.$$

Then it is easy to verify that $x = (w_2, w_1, w_4, w_3)$ is competitive where $\pi_1 = \pi_2 > \pi_3 = \pi_4$ and that all traders are indifferent between x and $y = (w_4, w_1, w_2, w_3)$. But y cannot be competitive since $\pi_1 = \pi_2 = \pi_3 = \pi_4$ in any price system that makes y possible. But trader 4 would not choose w_3 but rather w_1 is this situation. Note that there is another competitive allocation $z = (w_4, w_3, w_2, w_1)$ supported by prices $\pi_1 = \pi_4 > \pi_3 = \pi_2$ which weakly dominates both x and y via the coalition $\{1, 4\}$. Perhaps even stranger is the second example.

Example 2. A market in which x is competitive, y is not competitive and y is weakly Pareto superior to x.

Let there again be four traders with the following preferences, which are the same as the previous example except for trader 3,

$$P_1:\quad w_2 I_1 w_4 P_1 w_1 P_1 w_3,$$

$$P_2:\quad w_1 P_2 w_3 P_2 w_2 P_2 w_4,$$

$$P_3:\quad w_1 I_3 w_2 P_3 w_4 P_3 w_3,$$

$$P_4:\quad w_1 P_4 w_3 P_4 w_4 P_4 w_2.$$

Again $x = (w_2, w_1, w_4, w_3)$ is competitive at prices $\pi_1 = \pi_2 > \pi_3 = \pi_4$ and $y = (w_4, w_1, w_2, w_3)$ cannot be competitive. But now we see that traders 1, 2, and 4 are indifferent between x and y, but that trader 3 *prefers* y to x.

Again, however, $z = (w_4, w_3, w_2, w_1)$ is competitive at prices $\pi_1 = \pi_4 > \pi_3 = \pi_2$ and z weakly dominates both x and y via $\{1, 4\}$.

That a competitive allocation with indivisible commodities may not be Pareto optimal is not new; Emmerson (1972) has shown an example of this phenomenon. But in Emmerson's example the Pareto optimal allocation which dominates the competitive allocation is itself competitive, whereas it is not in our example.

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