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RISK AVERSION AND SOLUTIONS TO NASH's BARGAINING PROBLEM

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1. INTRODUCTION

Starting with Nash [1950], axiomatic models of bargaining have by and large made only indirect use of the theory of rational individual choice under uncertainty, in spite of the fact that a bargaining problem is usually defined in terms of the expected utility functions of the bargainers. The only property of an individual's utility function of which Nash makes explicit use is that it is uniquely defined only up to order-preserving linear transformations. Here we consider the effect on various solutions to the bargaining problem of an individual's aversion to risk, as expressed in his utility function.

Following Nash, we will consider two-player bargaining games defined by a pair (S,d), where d is a point in the plane, and S is a compact convex subset of the plane which contains d and at least one point x such that $x > d$. The interpretation is that S is the set of feasible expected utility payoffs to the players, any one of which can be achieved if it is agreed to by both players. If no such agreement is reached, then the disagreement point d is the result.

We will consider games which arise from bargaining over the set L of all lotteries defined on some convex, compact set of certain alternatives $C \subseteq R^n$, by individuals with concave utility functions u_1 and u_2. The feasible set of utility payoffs is the convex set

(1) $$S = \{(x_1,x_2) \mid x_1 = u_1(\ell) \text{ and } x_2 = u_2(\ell) \text{ for some } \ell \text{ in } L\},$$

and the disagreement point d is

(2) $$d = (u_1(\bar{c}),\ u_2(\bar{c}))$$

where $\bar{c} \in C$ is the alternative which results in the case of disagreement.

Denote the (strong) Pareto optimal subset of S by $P(S)$, and note that each point of $P(S)$ is of the form $(x_1, x_2) = (u_1(c), u_2(c))$ for some $c \in C$. (This follows from the concavity of u_1 and u_2 and the convexity of C.) Let $\underline{x}_1$ and $\overline{x}_1$ denote the minimum and maximum values of x_1 on the set $P(S)$. Then there exists a monotonically decreasing concave function ϕ defined on the interval $[\underline{x}_1,\overline{x}_2]$ such that $(x_1,x_2) \varepsilon P(S)$ if and only if $x_2 = \phi(x_1)$. That is, the Pareto optimal set $P(S)$ consists of points of the form $(x_1,\phi(x_1))$.

We can now consider the effect of replacing player i in a bargaining game (S,d) with a more risk averse player. Since our results will be independent of i, we can take $i = 2$ in what follows, without loss of generality.

Let (S,d) be defined as in (1) and (2), with $u_2 = w$. Let $\hat{w}$ be a utility function which is more risk averse than w, i.e. $\hat{w}(c) = k(w(c))$ for all c in C, where k is an increasing, concave function (c.f. Arrow [1965], Pratt [1964], or Kihlstrom and Mirman [1974]). Consider the game $(\hat{S},\hat{d})$ derived from (S,d) by replacing individual w with the more risk averse individual $\hat{w}$. Any outcome $c \varepsilon C$ which is Pareto optimal in (S,d) is also Pareto optimal in $(\hat{S},\hat{d})$, so $P(\hat{S})$ consists of points of the form $(x_1,\hat{\phi}(x_1))$, where $\hat{\phi}(x_1) = k(\phi(x_1))$. We can now proceed to study the effect of such a change on the predictions made about the outcome of bargaining by alternative models of the bargaining process.

2. NASH's SOLUTION

Nash proposed that bargaining between rational players be modelled by means of a function called a solution, which selects a feasible outcome for every bargaining game. That is, if we denote the class of all bargaining games by B, a solution is the function $f: B \to R^2$ such that f(S,d) is an element of S. Nash further proposed that a solution should possess the properties of Pareto optimality, symmetry, independence of irrelevant alternatives, and independence of equivalent utility representations, all of which have been amply described elsewhere (c.f. Nash [1950], Luce and Raiffa [1957], Harsanyi [1977] and Roth [1979]). Note that only the last of these properties, stated below, deals at all with the cardinal properties of utility functions.

> Independence of equivalent utility representations:
> If (S,d) and $(\hat{S},\hat{d})$ are bargaining games such that $\hat{S} = \{(a_1x_1 + b_1,\ a_2x_2 + b_2)|(x_1,x_2) \varepsilon S\}$ and $\hat{d} = (a_1d_1 + b_1,\ a_2d_2 + b_2)$ where a_1, a_2, b_1 and b_2 are numbers such that $a_1 > 0$ and $a_2 > 0$, then $f(\hat{S},\hat{d}) = (a_1f_1(S,d) + b_1,\ a_2f_2(S,d) + b_2)$.

Nash showed that there is a unique solution which possesses Properties 1-4. It is the solution F defined by $F(S,d) = x$ such that $x > d$ and $(x_1 - d_1)(x_2 - d_2) > (y_1 - d_1)(y_2 - d_2)$ for all y in S such that $v \neq x$ and $y > d$. We can state the following well-known alternative characterization of Nash's solution. (For simplicity, we state the following lemma for the case that ϕ is differentiable.)

Lemma 1: $F(S,d) = (x_1,\phi(x_1))$ <u>is the point such that</u> $(\phi(x_1)-d_2)/(x_1-d_1) = -\phi'(x_1)$.

We can now state the following results which strengthens a result of Kannai [1977].

<u>Theorem 1: The utility which Nash's solution assigns to a player increases as his opponent becomes more risk averse. That is,</u> $F_1(\hat{S},\hat{d}) \geq F_1(S,d)$, <u>where</u> $(\hat{S},\hat{d})$ <u>is obtained from</u> (S,d) <u>by replacing player 2 with a more risk averse player.</u>

Note that there is no ambiguity about the meaning of the comparison of $F_1(S,d)$ and $F_1(\hat{S},\hat{d})$ made in the theorem, since both quantities are payoffs defined by the same utility function of the same individual (over the same set of events).

<u>Proof</u>: Since F is independent of equivalent utility representations, it will be sufficient to prove the lemma for the case when $d = \hat{d} = \overline{0}$, where $\overline{0}$ denotes the origin (i.e. $\overline{0} = (0,0)$). So let $z = F(S,\overline{0})$ and $\hat{z} = F(\hat{S},\overline{0})$; we want to show that $\hat{z}_1 \geq z_1$. Since Nash's solution selects the point in S which maximizes the geometric average of the gains, it will be sufficient to show that the geometric average $A(y_1) = k(\phi(y_1))y_1$ has a positive first derivative at z_1. But

$$A'(z_1) = k'(\phi(z_1))\phi'(z_1)z_1 + k(\phi(z_1)) ,$$

and by Lemma 1, $\phi'(z_1)z_1 = -\phi(z_1)$, so

$$\begin{aligned} A'(z_1) &= -k'(\phi(z_1))\phi(z_1) + k(\phi(z_1)) \\ &= -k'(z_2)z_2 + k(z_2) = z_2[-k'(z_2) + \frac{k(z_2)}{z_2}] . \end{aligned}$$

The concavity of the function k implies that $(k(z_2)/z_2) \geq k'(z_2)$, while the individual rationality of Nash's solution implies $z_2 \geq 0$, so $A'(z_1) \geq 0$, as required. ||||

In Roth [1978], it was shown that Nash's solution could be interpreted as the utility function for a certain kind of individual, reflecting his preferences for bargaining in different games. Interpreted in this way, Theorem 2 states that such a player prefers to bargain against the more risk averse of any pair of possible opponents.

3. RISK POSTURE AND OTHER SOLUTIONS

Another solution for two-person bargaining games, axiomatized by Kalai and Smorodinsky [1975], responds to changes in risk posture in qualitatively the same way as Nash's solution. For any game (S,d), let the <u>ideal point</u> $I(S,d)=x^I=(X_1^I,x_2^I)$ be defined by $x_1^I = \max\{x_1|x \in S$ and $x \geq d\}$ and $x_2^I = \max\{x_2|x \in S$ and $x \geq d\}$, and let G be the solution such that $G(S,d)$ selects the maximal feasible point on the line joining d to X^I. That is, $G(S,d) = x$ is the Pareto optimal point in S such that $(x_1 - d_1)/(x_2 - d_2) = (x_1^I - d_1)/(x_2^I - d_2)$.

The solution G shares with Nash's solution the properties of Pareto optimality, symmetry, and independence of equivalent utility representations. It also shares with Nash's solution a sensitivity to changes in risk posture, which permits us to state the following parallel to Theorem 1.

Theorem 2: The utility which the solution G assigns to a player increases as his opponent becomes more risk averse. That is, $G_1(\hat{S},\hat{d}) \geq G_1(S,d)$, where $(\hat{S},\hat{d})$ is obtained from (S,d) by replacing player 2 with a more risk averse player.

Proof: Let $(\hat{S},\hat{d})$ be derived from (S,d) by replacing the utility function w of player 2 with the more risk averse function $\hat{w}$ such that $\hat{w}(c) = k(w(c))$ for all c in the underlying set of sure alternatives C. Since the solution G is independent of equivalent utility representations, we can choose any normalization for $\hat{w}$, and hence for k. So let $k(d_2) = d_2$ and $k(x_2^I) = x_2^I$. (This is equivalent to letting $\hat{w}(\bar{c}) = w(\bar{c}) = d_2$ and $\hat{w}(m_2) = w(m_2) = x_2^I$, where $\bar{c}$ is the disagreement outcome and m_2 the outcome which yields player 2 his maximum payoff in the Pareto set of the set $S^+ \equiv \{x \in S | x \geq d$. Let m_1 be the lottery which yields player 1 his maximum payoff in $P(S^+)$. (Then $u(m_1) = x_1^I$).

Note that $w(m_2) \geq w(m_1) \geq w(\bar{c})$, and so there exists some number α between 0 and 1 such that $w(m_1) = \alpha w(\bar{c}) + (1-\alpha)w(m_2)$. Consequently, the concavity of k implies that

$$\hat{w}(m_1) \geq w(m_1) \quad ,$$

since

$$\hat{w}(m_1) = k(w(m_1)) \geq \alpha k(w(\bar{c})) + (1-\alpha)k(w(m_2)) = w(m_1) \; .$$

Consequently $\hat{\phi}(x_1^I) \geq \phi(x_1^I)$, and so the fact that $\hat{\phi}(u(m_2)) = \phi(u(m_2))$ implies that $\hat{\phi}(x) > \phi(x)$ for all $x \in [u(m_2),x_1^I]$, since $\hat{\phi}$ is a concave transformation of ϕ. That is, every point in the Pareto set of S^+ is less than or equal to some point in the Pareto set of $\hat{S}^+$. Since $I(S^+,d) = I(\hat{S}^+,\hat{d})$ and $d = \hat{d}$, it therefore follows that $G(\hat{S},\hat{d}) = G(\hat{S}^+,\hat{d}) \geq G(S^+,d) = G(S,d)$, and, in particular, $G_1(\hat{S},\hat{d}) \geq G_1(S,d)$, as required. ||||

A third solution to the Nash bargaining problem has recently been proposed by Perles and Maschler [1980]. This solution is called the super-additive solution and is obtained by replacing Nash's independence of irrelevant alternatives axiom with super-additivity and continuity axioms.

The super-additive solution, to be denoted by H, has been characterized by the following equation

$$\int_0^{H_1} \sqrt{-\phi'(x_1)}\, dx_1 = \int_{H_1}^{\bar{x}_1} \sqrt{-\phi'(x_1)}\, dx_1 \quad , \tag{3}$$

where $H_1 = H_1(S,(\overline{0}))$. Perles-Maschler restrict their solution to bargaining problems in which the disagreement point d is the origin and the set S is comprehensive in the nonnegative quadrant. For the simplicity, we will further restrict ourselves to the case in which the weak and strong Pareto sets coincide; i.e. $\underline{x}_1 = \phi(\overline{x}_1) = 0$.

The following theorem is an analog of Theorems 1 and 2 for H.

Theorem 3: <u>The utility which H assigns to a player increases as his opponent becomes more risk averse. That is, $H_1(\hat{S},\overline{0}) \geq H_1(S,\overline{0})$, where $(\hat{S},\overline{0})$ is obtained from $(S,\overline{0})$ by replacing player 2 with a more risk averse player.</u>

Proof: As in the proof of Theorem 2, we denote by k the concave transformation of w which yields $\hat{w}$. Here k is normalized so that $k(0) = 0$ and $k(\overline{x}_1) = \overline{x}_1$. We now use the above characterization of the super-additive solution.

For the bargaining problem $(\hat{S},\overline{0})$, the equality becomes

$$\int_0^{\hat{H}_1} \sqrt{k'(\phi(x_1))}\,\sqrt{-\phi'(x_1)}\,dx_1 = \int_{\hat{H}_1}^{\overline{x}_1} \sqrt{k'(\phi(x_1))}\,\sqrt{-\phi'(x_1)}\,dx_1,$$

where $\hat{H}_1 = H_1(\hat{S},0)$. In order to prove that $\hat{H}_1 \geq H_1$, it suffices to show that

$$\int_0^{H_1} \sqrt{k'(\phi(x_1))}\,\sqrt{-\phi'(x_1)}\,dx_1 \leq \int_{H_1}^{\overline{x}_1} \sqrt{k'(\phi(x_1))}\,\sqrt{-\phi'(x_1)}\,dx_1 . \tag{4}$$

First note that, because ϕ is a decreasing function and k is concave,

$$x_1 \leq (\geq) \quad H_1$$

implies

$$\sqrt{k'(\phi(x_1))} \leq (\geq) \quad \sqrt{k'(\phi(\hat{H}_1))} . \tag{5}$$

We now multiply both sides of (3) by the constant $\sqrt{k'(\phi(H_1))}$ to obtain

$$\int_0^{H_1} \sqrt{k'(\phi(H_1))}\,\sqrt{-\phi'(x_1)}\,dx_1 = \int_{H_1}^{\overline{x}_1} \sqrt{k'(\phi(H_1))}\,\sqrt{-\phi'(x_1)}\,dx_1 . \tag{6}$$

Finally, we replace the constant $\sqrt{k'(\phi(H_1))}$ by $\sqrt{k'(\phi(x_1))}$ in the integrals in (6) and use the inequalities (5) to yield (4). ||||

Note that if k is strictly concave, all of the inequalities become strict.

Thus F, G and H are all solutions which possess an intuitively plausible sensitivity to changes in risk posture, and we can investigate this property in the more general context of an arbitrary solution f. Specifically, we will consider solutions f which possess the following property.

Risk sensitivity: If a bargaining game (S,d) is transformed into a game $(\hat{S},\hat{d})$ by replacing player 2, say, with a more risk averse player, then $f_1(\hat{S},\hat{d}) \geq f_1(S,d)$.

Any risk sensitive solution f models a bargaining process in which it is advantageous to have a highly risk-averse opponent. A surprising consequence of this property is that a solution which is both risk sensitive and Pareto optimal must also be independent of equivalent utility representations.

Theorem 4: If f is a solution which is both Pareto optimal and risk sensitive, then f is independent of equivalent utility representations.

Proof: If $(\hat{S},\hat{d})$ is derived from (S,d) by subjecting player 2's utility function to a concave transformation, then (S,d) can be derived from $(\hat{S},\hat{d})$ by subjecting player 2's utility to a convex transformation. So risk sensitivity implies that convex transformations of one player's utility lower the other player's payoff, just as concave transformations raise it, and so linear transformations of one player's utility leave the other player's utility unchanged. Thus if (S,d) and $(\hat{S},\hat{d})$ are related by a linear transformation k of player 2's utility, and if $f(S,d) = (x_1,x_2)$ and $f(\hat{S},\hat{d}) = (y_1,y_2)$, then $y_1 = x_1$. This, of course, is half of independence of equivalent utility representations: the other half is that a linear transformation of a player's utility function should change his own payoff by the same transformation. But this follows from the Pareto optimality of f, since $x_2 = \phi(x_1)$ and $y_2 = \hat{\phi}(y_1) = k(\phi(y_1)) = k(\phi(x_1)) = k(x_2)$. ||||

This theorem is somewhat counterintuitive, since it deduces the linear invariance of solutions from the risk sensitivity property, which is specifically concerned with the nonlinearity of utility functions due to risk aversion. One explanation may be that the intuitive plausibility of the risk sensitivity property derives in part from the feeling that the outcome of bargaining may turn out not to be Pareto optimal. In particular, a disagreement may occur, and the fear of this hleps cause a highly risk-averse player to settle for an unfavorable agreement.

Note that an immediate corollary of Theorem 4 is that risk sensitivity can replace independence of equivalent utility representations in a characterization of Nash's solution.

Theorem 4 suggests several approaches to studying risk-sensitive solutions different from Nash's. One of these is to study risk sensitive solutions which are also Pareto optimal. The theorem shows that such solutions must also be independent of equivalent utility representations, so they must differ from Nash's solution by not being independent of irrelevant alternatives (or by not being symmetric). Theorems 2 and 3 show that G is a solution of this kind as is H.

Another approach will be to explore risk-sensitive solutions which need not always yield Pareto optimal outcomes. Solutions of this sort may be able to provide more descriptive models of bargaining in which there is a non-zero probability of ending in disagreement.

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