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STABLE MATCHINGS, OPTIMAL ASSIGNMENTS, AND LINEAR PROGRAMMING

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Vande Vate (1989) described the polytope whose extreme points are the stable (core) matchings in the Marriage Problem. Rothblum (1989) simplified and extended this result. This paper explores a corresponding linear program, its dual and consequences of the fact that the dual solutions have an unusually direct relation to the primal solutions. This close relationship allows us to provide simple proofs both of Vande Vate and Rothblum's results and of other important results about the core of marriage markets. These proofs help explain the structure shared by the marriage problem (without sidepayments) and the assignment game (with sidepayments). The paper further explores "fractional" matchings, which may be interpreted as lotteries over possible matches or as time-sharing arrangements. We show that those fractional matchings in the Stable Marriage Polytope form a lattice with respect to a partial ordering that involves stochastic dominance. Thus, all expected utility functions corresponding to the same ordinal preferences will agree on the relevant comparisons. Finally, we provide linear programming proofs of slightly stronger versions of known incentive compatibility results.

1. Introduction. In recent years, substantial progress has been made in studying two-sided matching problems from a game theoretic perspective, both in terms of developing the basic theory (see, e.g., Roth and Sotomayor (1990), and in terms of understanding the practical problems associated with implementing matching algorithms (see, e.g., Roth (1984), Roth (1991), and Mongell and Roth (1991)). What makes two-sided matching problems different from the classical assignment problem (e.g., in which jobs are assigned to machines) is that in these problems there are people on both sides of the assignment who care about the outcome (e.g., when workers are matched with supervisors). This matters for a variety of reasons, not least of which is that if a central planner makes an assignment not to the liking of the people involved, groups of them may be able to upset the assignment by making private arrangements among themselves. This imposes new constraints on what assignments can be achieved. These are called "stability" constraints, and the nature of these constraints is the main focus of the present paper.

Furthermore, the relevant information about the quality of particular matches has to be obtained from the parties themselves. This imposes additional constraints on the problem, to take into account that the kind of information a planner collects will be influenced by the algorithm he chooses. That is, different algorithms give the parties who have the information different incentives about what to reveal. We discuss these as well, in our section on Incentives.

At a technical level, the main focus of this paper is to show how the eclectic mathematical techniques, which have been employed to study these models, can to a

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large extent be unified by linear algebraic techniques. In particular, one of the oldest puzzles arising out of the game theoretic analysis of two-sided matching concerns the fact that virtually identical conclusions about the core arise from two apparently quite different models, namely the marriage model of Gale and Shapley (1962) and the linear assignment model of Shapley and Shubik (1972). The puzzle is compounded by the fact that the arguments by which these results have so far been proved and reproved over the years remain quite different, and seem to rely on different principles. (While the marriage model and the arguments concerning it are discrete and ordinal in nature, the formulation of the assignment model and the arguments associated with it are cardinal in nature and typically involve linear programming.) As Balinski and Gale (1990, p. 274) note:

> There is, by now, a substantial literature on these problems, and one is struck by the fact that almost all results proved for the ordinal case have analogues in the cardinal case, although the techniques of proof in the two cases are in general quite different, and there is as yet no 'unified theory' which covers both.

The first task of this paper is to construct proofs of the major results for the two models based on one common set of principles. The opportunity to do this arises from recent results of Vande Vate (1989) and Rothblum (1992) showing that the matchings in the core of the marriage model can also be described by a linear programming problem whose extreme points are all integer. (We also give a new and more concise proof of this result.) We show that the dual of this linear program has a rather remarkable relationship to the primal: Each optimal solution to the primal is contained in an optimal solution to the dual. With this strong relationship we observe that many of the common results for the two models can be deduced from their linear structure via the Complementary Slackness Theorem of linear programming. We rely on existing proofs that the core is always nonempty, and do not provide a new proof of this.

One particularly important result common to the two models is that in each the set of core outcomes forms a lattice. In the assignment model the lattice extends over all solutions in the linear programming description of the core. In the marriage model the lattice results are known to hold only over the integer solutions of a corresponding linear program. We explore the "fractional" matchings or noninteger solutions of the linear program for the marriage model. These can be interpreted either as lotteries over possible matches or as time-sharing arrangements. Under either interpretation, it would appear that, in order to reach any conclusions about agents' preferences over such matchings, we would need to supplement the ordinal information about agents' preferences over possible mates with cardinal utility functions. While this would be true for arbitrary fractional matchings, we show that those which satisfy the core constraints form a lattice with respect to a partial ordering that involves stochastic dominance. Thus, all expected utility functions corresponding to the same ordinal preferences agree on the relevant comparisons. These comparisons can therefore be made without requiring further information about preferences.

The similarities between the marriage model and the assignment model arise from the fact that the stable (core) payoff vectors in the assignment model are the *solutions* to a system of linear inequalities, while the stable matchings in the marriage model are the *extreme point solutions* to a system of linear inequalities. This latter fact allows us both to give new proofs of known results for the marriage model and to extend these results to the "fractional stable matchings" which constitute the entire set of solutions to the relevant system of inequalities. The interpretation of fractional matchings as lotteries or schedules makes them of independent interest.

2. Standard linear programming terminology. Throughout the paper we rely on the following standard terminology and results about linear programming. With each linear program having the form:

$$\text{(LP)} \qquad \text{Maximize } p = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad \text{for } i = 1, 2, \ldots, m,$$

$$x_j \geqslant 0 \text{ for } j = 1, 2, \ldots, n,$$

is associated another linear program having the form:

$$\text{(DLP)} \qquad \text{Minimize } d = \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t. } \sum_{i=1}^{m} a_{ij} y_i \geqslant c_j \quad \text{for } j = 1, 2, \ldots, n,$$

$$y_i \geqslant 0 \quad \text{for } i = 1, 2, \ldots, m.$$

We refer to (LP) as the *Primal program* and to (DLP) as its *Dual*. The following relationships between these linear programs constitute a fundamental result of linear programming.

THEOREM 1 (DUALITY THEOREM). *The objective value of each feasible solution of (LP) is less than or equal to the objective value of each feasible solution of (DLP). Moreover, (LP) is feasible and has a bounded objective value if and only if the same holds for (DLP), and in this case the optimal objective values of (LP) and (DLP) are equal.*

We say that a pair of vectors $x \in \mathscr{R}^n$ and $y \in \mathscr{R}^m$ satisfies the *complementary slackness* conditions for the linear program (LP) if

$$\text{(1)} \qquad y_i\left(\sum_{j=1}^{n} a_{ij} x_j - b_i\right) = 0 \quad \text{for each } i = 1, 2, \ldots, m, \text{ and}$$

$$\text{(2)} \qquad \left(\sum_{i=1}^{m} a_{ij} y_i - c_j\right) x_j = 0 \quad \text{for each } j = 1, 2, \ldots, n.$$

Theorem 2 points out the intimate connection between optimality and the complementary slackness conditions.

THEOREM 2 (COMPLEMENTARY SLACKNESS THEOREM). *Vectors x feasible to (LP) and y feasible to (DLP) are optimal solutions to (LP) and (DLP) respectively, if and only if x and y satisfy the complementary slackness conditions for (LP).*

3. The assignment model. To set the stage, we first briefly consider the assignment model, which, following Shapley and Shubik (1972), has traditionally been studied in terms of its linear programming formulations.

The assignment game consists of two finite, disjoint sets of players, M and W (e.g., buyers and sellers), and an $|M| \times |W|$ matrix of nonnegative numbers $\{\alpha_{ij}: (i,j) \in M \times W\}$. The interpretation is that any pair of agents $(i,j) \in M \times W$ is free to form a coalition whose worth is α_{ij}, which the two agents may divide between themselves in any way. Any agent is free to remain single and receive zero, and the worth of an arbitrary coalition equals the sum of the pairwise coalitions it can form (with pairs consisting of one agent from M and one from W).

A *matching* is a one-to-one mapping μ from $M \cup W$ to itself, such that:

· $\mu(m) = w$ if and only if $\mu(w) = m$, in which case m is *matched* to w.

· If $\mu(m)$ is not in W, then $\mu(m) = m$, in which case m is *unmatched*.

· If $\mu(w)$ is not in M, then $\mu(w) = w$, in which case w is *unmatched*.

The *incidence vector* of a matching μ is a vector $x \in \{0,1\}^{|M| \times |W|}$ such that $x_{m,w} = 1$ if $\mu(m) = w$ and $x_{m,w} = 0$, otherwise. We identify each matching with its incidence vector. It is well known that a vector $x \in \mathscr{R}^{|M| \times |W|}$ is a matching if and only if it is an integer solution of the following system of linear inequalities

(3) $$\sum_{j \in W} x_{m,j} \leqslant 1 \quad \text{for each } m \in M,$$

(4) $$\sum_{i \in M} x_{i,w} \leqslant 1 \quad \text{for each } w \in W,$$

(5) $$x_{m,w} \geqslant 0 \quad \text{for each } (m,w) \in M \times W.$$

We define a *fractional matching* to be a (not necessarily integer) vector $x \in \mathscr{R}^{|M| \times |W|}$ satisfying the matching constraints (3)–(5). Birkhoff's Theorem shows that each fractional matching is a convex combination of matchings.

Consider the linear program (AP) of finding a fractional matching x, i.e., a vector $x \in \mathscr{R}^{|M| \times |W|}$ satisfying (3)–(5), which maximizes

$$\sum_{(i,j) \in M \times W} \alpha_{ij} x_{ij}.$$

Standard results show that (AP) has an integer optimal solution. Such a solution will determine a matching between buyers and sellers which will maximize total returns. We call such (integer optimal) solutions *optimal matchings*.

The dual of (AP), denoted (DAP), is the problem of finding vectors $u \in \mathscr{R}^{|M|}$ and $v \in \mathscr{R}^{|W|}$ which form a solution of

$$\text{Minimize } \sum_{i \in M} u_i + \sum_{j \in W} v_j$$

$$\text{s.t. } u_i + v_j \geqslant \alpha_{ij} \quad \text{for each } (i,j) \in M \times W,$$

$$u_i, v_j \geqslant 0.$$

Shapley and Shubik (1972) observed that (DAP) formulates the problem of finding payoff vectors in the core of the above assignment game. We call optimal solutions of (DAP) *stable payoff vectors*. The existence of optimal solutions of (AP) and the Duality Theorem (Theorem 1) show that the set of stable payoff vectors is nonempty.

A pair $(x; u, v)$ is called a *stable outcome* if x is an optimal matching and (u, v) is a stable payoff. The Complementary Slackness Theorem permits us to draw the following well known conclusions about stable outcomes (cf. Roth and Sotomayor (1990)).

LEMMA 3. *Each player with a positive payoff at a stable outcome is matched at every stable outcome.*

PROOF. If u_i (v_j) $\geqslant 0$ for an optimal solution of (DAP), then complementary slackness implies $\sum_{j \in W} x_{ij} = 1$ ($\sum_{i \in M} x_{ij} = 1$) for every optimal solution of (AP). □

LEMMA 4. *If two players i and j are matched at some stable outcome and i prefers one stable outcome to another, then j must have the opposite preferences. That is, if $x_{ij} = 1$ at some optimal assignment x, and if (u, v) and (u', v') are stable payoff vectors, then $u'_i > u_i \Leftrightarrow v'_j < v_j$ for each j.*

PROOF. $x_{ij} = 1$ implies by complementary slackness that $u_i + v_j = \alpha_{ij} = u'_i + v'_j$. So $\mu'_i > u_i \Leftrightarrow v'_j < v_j$. □

Shapley and Shubik (1972) observed that the set of stable payoff vectors is a lattice. Specifically, let $(x; u, v)$ and $(x'; u', v')$ be stable outcomes and define $\underline{u}$ and $\bar{u}$ in $\mathscr{R}^{|M|}$ and $\underline{v}$ and $\bar{v}$ in $\mathscr{R}^{|W|}$ by

$$\left.\begin{aligned} \underline{u}_i &= \min\{u_i, u'_i\}, \\ \bar{u}_i &= \max\{u_i, u'_i\}, \end{aligned}\right. \quad \text{for each } i \in M,$$

$$\left.\begin{aligned} \underline{v}_j &= \min\{v_j, v'_j\}, \\ \bar{v}_j &= \max\{v_j, v'_j\}, \end{aligned}\right. \quad \text{for each } j \in W.$$

Then $(x; \underline{u}, \bar{v})$ and $(x; \bar{u}, \underline{v})$ are also stable outcomes. (Note that each member of M weakly prefers $(\bar{u}, \underline{v})$ to (u, v) or (u', v'). Further, each member of M weakly prefers (u, v) or (u', v') to $(\underline{u}, \bar{v})$. The reverse is true for the members of W.)

We turn now to see how similar results may be obtained through the linear programming formulation of the marriage model.

4. The marriage model.

In the stable marriage problem there are again two sets of agents, the set $M = \{m_1, m_2, \dots, m_p\}$ of "men" and the set $W = \{w_1, w_2, \dots, w_q\}$ of "women". Each agent has a complete, transitive, and strict preference ordering over the agents on the other side of the market and the prospect of remaining single. We say that the pair (m, w) in $M \times W$ is *acceptable* if m and w prefer each other to remaining single. We let A denote the set of acceptable pairs.

A matching μ (of men with women) is called *individually rational* if no agent a prefers being single (i.e., being unmatched) to $\mu(a)$. A matching μ is *stable* if it is individually rational and there is no pair (m, w) in A such that both man m prefers woman w to $\mu(m)$ and woman w prefers man m to $\mu(w)$.

We write $a >_c b$ to denote that person c prefers person a to b and we write $a \geqslant_c b$ to denote that either $a = b$ or $a >_c b$. Thus, an individually rational matching μ is stable if there is no pair (m, w) in A such that both $w >_m \mu(m)$ and $m >_w \mu(w)$.

The following lemma characterizes stable matchings as integer solutions of a system of linear inequalities. Its (straightforward) proof is given in Rothblum (1992).

LEMMA 5. *A vector* $x \in \mathscr{R}^{|M| \times |W|}$ *is a stable matching if and only if it is an integer solution of* (3), (4), (5),

$$x_{m,w} = 0 \quad \text{for each } (m,w) \in (M \times W) \setminus A \tag{6}$$

$$\sum_{j >_m w} x_{m,j} + \sum_{i >_w m} x_{i,w} + x_{m,w} \geqslant 1 \quad \text{for each } (m,w) \in A, \tag{7}$$

where for the sake of brevity "$j >_m w$*" is used in the summation to denote* $\{j \in W: j >_m w\}$ *and "*$i >_w m$*" is used to denote* $\{i \in M: i >_w m\}$.

Constraints (6) are called the *individual rationality constraints*. Constraints (7) ensure that for each acceptable pair (m,w), either man m marries someone he prefers to woman w, or she marries someone she prefers to him, or they marry each other. These constraints are called the *stability constraints*.

We define a *stable fractional matching* to be a (not necessarily integer) solution of (3)–(7). Parallel to Birkhoff's Theorem, Vande Vate (1989) and Rothblum (1992) showed that each stable fractional matching is a convex combination of stable matchings. Thus, the extreme points of the polytope defined by (3)–(7) are exactly (the incidence vectors of) stable matchings. We provide a new and simpler proof of this result in §6.

Gale and Shapley (1962) established the existence of a stable matching. This result immediately implies that the set of stable fractional matchings is nonempty. It is natural to attempt and derive this conclusion from linear algebraic arguments, but the approach has not yet proven to be successful.

5. Linear programming proofs for the marriage problem. In this section we exploit the Complementary Slackness Theorem to develop short proofs for some of the important results about stable matchings and fractional matchings[1]. Just as in our discussion of the Assignment Game, these proofs exploit duality relations for a linear program.

Consider the linear program (MP) of finding a stable fractional matching x, i.e., a vector $x \in \mathscr{R}^{|M| \times |W|}$ satisfying (3)—(7), which maximizes

$$\sum_{(i,j) \in A} x_{i,j}.$$

The dual of this problem, denoted (DMP), is to find (α, β, γ) with $\alpha \in \mathscr{R}^{|M|}$, $\beta \in \mathscr{R}^{|W|}$ and $\gamma \in \mathscr{R}^{|M| \times |W|}$ to:

$$\min \sum_{i \in M} \alpha_i + \sum_{j \in W} \beta_j - \sum_{i \in M} \sum_{j \in W} \gamma_{i,j},$$

$$\text{s.t. } \alpha_m + \beta_w - \sum_{j <_m w} \gamma_{m,j} - \sum_{i \leqslant_w m} \gamma_{i,w} \geqslant 1 \quad \text{for each } (m,w) \in A,$$

$$\alpha, \beta, \gamma \geqslant 0,$$

$$\gamma_{m,w} = 0 \quad \text{if } (m,w) \notin A.$$

[1] In the model of one-to-one matching considered here, the set of stable matchings corresponds to the set of core outcomes. In models of many-to-one matching the connection is less close (stable matchings are a subset of the core), and in models of many-to-many matching, stable matchings need not be in the core (cf. Blair (1988), Roth and Sotomayor (1990)).

We first show that there is an unusually close relationship between (MP) and (DMP): Stable matchings give rise to optimal solutions not only of (MP), but also of (DMP). In fact, each stable fractional matching is an optimal solution to (MP) and gives rise to an optimal solution of (DMP). We know of no similarly rich class of linear programs whose primal and dual solutions are related in this way.

LEMMA 6. *Each stable fractional matching x is an optimal solution to (MP) and (α, β, x) is an optimal solution to (DMP) where:*

$$\alpha_m = \sum_{j \in W} x_{m,j} \quad \text{for each man } m \in M \tag{8}$$

and

$$\beta_w = \sum_{i \in M} x_{i,w} \quad \text{for each woman } w \in W. \tag{9}$$

PROOF. To see that (α, β, x) is feasible for (DMP), observe that

$$\begin{aligned}
&\alpha_m + \beta_w - \sum_{j <_m w} x_{m,j} - \sum_{i \leqslant_w m} x_{i,w} \\
&= \sum_{j \in W} x_{m,j} + \sum_{i \in M} x_{i,w} - \sum_{j <_m w} x_{m,j} - \sum_{i \leqslant_w m} x_{i,w} \\
&= \sum_{j \geqslant_m w} x_{m,j} + \sum_{i >_w m} x_{i,m} \geqslant 1
\end{aligned}$$

where the last inequality follows from the fact that x satisfies (7). To see that these two solutions are optimal, observe that

$$\begin{aligned}
&\sum_{m \in M} \alpha_m + \sum_{w \in W} \beta_w - \sum_{m \in M} \sum_{w \in W} x_{m,w} \\
&= 2 \sum_{m \in M} \sum_{w \in W} x_{m,w} - \sum_{m \in M} \sum_{w \in W} x_{m,w} \\
&= \sum_{m \in M} \sum_{w \in W} x_{m,w}
\end{aligned}$$

and so they share a common objective value. Thus, the optimality of x for (MP) and (α, β, x) for (DMP) follows from Theorem 1. □

Since each stable fractional matching gives rise to optimal solutions to (MP) and (DMP), given one such feasible solution, we can apply Theorem 2 to learn about all such solutions.

LEMMA 7. *There is a partition of M into M_0 and M_1 and of W into W_0 and W_1 such that for each stable fractional matching x,*

$$\sum_{j \in W} x_{m,j} = \begin{cases} 0 & \text{if } m \in M_0, \\ 1 & \text{if } m \in M_1, \end{cases}$$

and

$$\sum_{i \in M} x_{i,w} = \begin{cases} 0 & \text{if } w \in W_0, \\ 1 & \text{if } w \in W_1. \end{cases}$$

PROOF. Consider a man m. If

$$\sum_{j \in W} x_{m,j} > 0$$

for some stable fractional matching x, then by Lemma 6 there is an optimal dual solution (α, β, x) with $\alpha_m > 0$. It follows From Theorem 2 that for each optimal solution x' to (MP),

$$\sum_{j \in W} x'_{m,j} = 1,$$

i.e., man m is in M_1. Otherwise,

$$\sum_{j \in W} x_{m,j} = 0$$

for each feasible solution x to (MP) and so man m is in M_0. The argument is similar for the women. □

The following known result about stable matchings (see McVitie and Wilson (1970), Roth (1985) and Theorem 2.22 in Roth and Sotomayor (1990) is the special case of Lemma 7 restricted to (the incidence vectors of) stable matchings.

COROLLARY 8. *The same set of people is matched at every stable matching.*

Note that Corollary 8 presents the analogous conclusion for the marriage model that Lemma 3 presents for the assignment model, and that we have proved both results via the Complementary Slackness Theorem.

The strong relationship between (MP) and (DMP) also leads to the following property of stable fractional matchings.

LEMMA 9. *If $x_{m,w} > 0$ for some stable fractional matching x, then for each stable fractional matching x',*

$$\sum_{j >_m w} x'_{m,j} + \sum_{i >_w m} x'_{i,w} + x'_{m,w} = 1.$$

PROOF. Suppose $x_{m,w} > 0$ for some stable fractional matching x. Then, by Lemma 6, (α, β, x) is an optimal solution to (DMP) where α and β are defined as in (8) and (9). Further, by Lemma 6, each stable fractional matching x' is an optimal solution to (MP). Thus, by Theorem 2, each stable fractional matching x' satisfies

$$\sum_{j >_m w} x'_{m,j} + \sum_{i >_w m} x'_{i,w} + x'_{m,w} = 1. \quad \square$$

Restricting Lemma 9 to the set of stable matchings gives the following two known results analogous to Lemma 4 for the assignment model.

COROLLARY 10. *If man m and woman w are matched to each other under some stable matching μ, then there is no stable matching which both man m and woman w prefer to μ.*

PROOF. This follows immediately from Lemma 9 and the observation that both man m and woman w prefer a matching x' to being matched to each other if and only if

$$\sum_{j >_m w} x'_{m,j} + \sum_{i >_m m} x'_{i,w} = 2.$$

The following lemma was originally proved by Knuth (1976).

LEMMA 11 (THE DECOMPOSITION LEMMA). *Let μ and μ' be stable matchings and define $M(\mu)$ to be the set of men who prefer μ to μ' and $M(\mu')$ to be the set of men who prefer μ' to μ. Define $W(\mu)$ and $W(\mu')$ analogously. Then μ and μ' map $M(\mu)$ onto $W(\mu')$ and $M(\mu')$ onto $W(\mu)$.*

PROOF. Let x and x' be the incidence vectors of μ and μ', respectively. Suppose $w = \mu(m)$ and $w' = \mu'(m)$. Since x and x' are stable matchings and $x_{m,w} > 0$, we have by Lemma 9 that

$$\sum_{j \geqslant_m w} x'_{m,j} + \sum_{i >_w m} x'_{i,w} = 1.$$

Now, man m is in $M(\mu)$ if and only if

$$\sum_{j \geqslant_m w} x'_{m,j} = 0$$

and woman w is in $W(\mu')$ if and only if

$$\sum_{i >_w m} x'_{i,w} = 1.$$

Thus, m is in $M(\mu)$ if and only if $w = \mu(m)$ is in $W(\mu')$. The remaining statements follow analogously. □

The next example demonstrates that the converse of Lemma 9 need not hold; i.e., even if every stable fractional matching satisfies the stability constraint for a pair (m, w) with equality, there may be no stable fractional matching x with $x_{m,w} > 0$. Note, however, that in this case the Strong Complementarity Theorem (see Schrijver (1989)) ensures that there is an optimal dual solution (α, β, γ) with $\gamma_{m,w} > 0$. Thus, the example also demonstrates that the converse of Lemma 6 need not hold. Although each stable fractional matching gives rise to an optimal dual solution, not every optimal dual solution gives rise to a stable fractional matching.

EXAMPLE 1. Consider the stable marriage problem with two men and one woman. Each player prefers marrying anyone on the opposite side of the market to remaining single and the woman prefers man 1 to man 2. The unique stable fractional matching x in this example has $x_{1,1} = 1$ and $x_{2,1} = 0$. This stable matching satisfies the stability constraint for the pair (m_2, w_1) with equality and yet $x_{2,1} = 0$. On the other hand, letting $\alpha_1 = \gamma_{1,1} = \gamma_{2,1} = 1$, $\alpha_2 = 0$ and $\beta_1 = 2$ is an optimal solution to the dual problem (DMP) with $\gamma_{2,1} > 0$. Of course, it does not correspond to a stable fractional matching.

One particularly important consequence of Lemma 11 (see Conway in Knuth (1976), Roth and Sotomayor (1990)) is that the set of stable matchings forms a lattice in the following way: Given two stable matchings μ^1 and μ^2, the mapping $\mu = \mu^1 \vee \mu^2$ that assigns to each man m his preferred choice of $\mu^1(m)$ and $\mu^2(m)$ is a stable matching. In §7 we extend this result and show that the set of all stable fractional matchings is a lattice.

6. The stable matching polytope. In this section we develop a short proof of Vande Vate (1989) and Rothblum's (1992) results showing that each stable fractional matching is a convex combination of stable matchings. Just as Birkhoff's Theorem shows that the extreme points of the feasible set of (AP) are exactly the (incidence vectors of) matchings, we show that the extreme points of the feasible set of (MP) are exactly the (incidence vectors of) stable matchings. We remind the reader of the fact that the set of stable fractional matchings is nonempty; see the end of §4.

Our proof of the characterization of extreme points of the set of stable fractional matchings relies on the next lemma. The conclusions of this lemma are stated and proved in Rothblum (1992) and we repeat them here for the sake of clarity and completeness. We will need some further definitions to state the results. Given a stable fractional matching x, define μ_x to assign each man m to his most preferred woman j among those with $x_{m,j} > 0$. If there is no woman j with $x_{m,j} > 0$, then $\mu_x(m) = m$. We show in Lemma 12 that for each stable fractional matching x, μ_x is a stable matching. Clearly then, μ_x is the stable matching that all men would agree is best if the set of admissible pairs were restricted to $A(x) = \{(m,w)\colon x_{m,w} > 0\}$. We also show that μ_x assigns to each woman her least preferred man among those admissible with respect to $A(x)$. Thus, μ_x is also the stable matching that all women would agree is worst if the set of admissible pairs were restricted to $A(x)$.

LEMMA 12. *For each stable fractional matching x, μ_x is a stable matching. Further, μ_x assigns to each woman w her least preferred man i among those with $x_{i,w} > 0$; and if there is no man i with $x_{i,w} > 0$, then $\mu_x(w) = w$.*

PROOF. We see that μ_x is a matching as follows. If μ_x is not a matching, there must be a woman w and two distinct men m and m' with $w = \mu_x(m) = \mu_x(m')$. In particular, $x_{m,w} > 0$ and $x_{m',w} > 0$. Since preferences are strict, woman w must prefer one of these two men, say man m. Since x satisfies the stability constraint (7) for the pair (m,w) and since

$$\sum_{j >_m w} x_{m,j} = 0,$$

it follows that

$$\sum_{i \geqslant_w m} x_{i,w} = 1,$$

contradicting the assertions $m' <_w m$, $x_{m',w} > 0$ and (4). Thus μ_x is, indeed, a matching.

We next show that μ_x assigns each woman w to her least preferred mate acceptable with respect to $A(x)$ as follows. If $x_{i,w} = 0$ for each man $i \in M$, then μ_x assigns no man to w and so $\mu_x(w) = w$. Otherwise, let man m be woman w's least preferred man among those with $x_{m,w} > 0$. Since $x_{m,w} > 0$, we have by Lemma 7 that

$$\sum_{i >_w m} x_{i,w} < \sum_{i \geqslant_w m} x_{i,w} = \sum_{i \in M} x_{i,w} = 1. \tag{10}$$

Further, since $x_{m,w} > 0$, we have by Lemma 9 that

$$\sum_{j >_m w} x_{m,j} + \sum_{i >_w m} x_{i,w} + x_{m,w} = 1. \tag{11}$$

Combining (10) and (11), we see that

$$\sum_{j \geqslant_m w} x_{m,j} > 0 \quad \text{and} \quad \sum_{j >_m w} x_{m,j} = 0,$$

implying that w is the most preferred woman j with $x_{m,j} > 0$. So, $\mu_x(m) = w$, proving that μ_x assigns woman w to her least preferred man m.

Finally, we show that μ_x is stable as follows. If man m prefers woman w to his mate under μ_x, then

$$\sum_{j \geqslant_m w} x_{m,j} = 0,$$

and since x satisfies the stability constraint (7) for the pair (m, w),

$$\sum_{i >_w m} x_{i,w} = 1.$$

By the above characterization of $\mu_x(w)$, we conclude that woman w prefers her mate under μ_x to m. □

We are now ready to prove the characterization of the extreme points of the set of stable fractional matchings.

THEOREM 13. *The extreme points of* (*MP*) *are exactly the stable matchings.*

PROOF. Trivially, every integer solution of (3)—(7) is an extreme point of the set of fractional matchings. So stable matchings are extreme points of the set of stable fractional matchings. It remains to show that each extreme point of (MP) is a stable matching.

Consider a stable fractional matching x that is not a matching and let z be the incidence vector of μ_x. By Lemma 12, z is a stable matching and the assertion that x is not a matching assures that $x \neq z$. We show that x is not an extreme point of (MP) by expressing it as a convex combination of z and another stable fractional matching. Specifically, for $0 < \delta < 1$, consider the vector

$$y^\delta = \frac{x - \delta z}{1 - \delta}.$$

Since $x = \delta z + (1 - \delta) y^\delta$ for each $0 < \delta < 1$, it suffices to show that for some $0 < \delta < 1$, y^δ is a stable fractional matching.

We first observe that $z_{m,w} = 0$ whenever $x_{m,w} = 0$; hence, for sufficiently small positive δ, y^δ satisfies (5). Further, if $m \in M$ has $x_{m,j} = 0$ for all $j \in W$, then $z_{m,j} = 0$ for all $j \in W$ and $(y^\delta)_{m,j} = 0$ for all $j \in W$, assuring that y^δ satisfies (3) for all $0 < \delta < 1$. If $m \in M$ has $\sum_{j \in W} x_{m,j} > 0$, then $\mu_x(m) \neq m$ implying that $\sum_{j \in W} z_{m,j} = 1$. As x satisfies (3), it immediately follows that so do the y^δ's for all $0 < \delta < 1$. A similar argument shows that y^δ satisfies (4) for all $0 < \delta < 1$. Also, if $(m, w) \in (M \times W) \setminus A$, then we have $x_{m,w} = 0$ implying that $(y^\delta)_{m,w} = z_{m,w} = 0$ for all $0 < \delta < 1$; so for such δ, y^δ satisfies (6).

Thus, to show that x is not extreme, we need only show that for some small positive δ, y^δ satisfies the stability constraints (7). As

$$\sum_{j >_m w} y^\delta_{m,j} + \sum_{i \geqslant_w m} y^\delta_{i,w}$$

$$= \frac{1}{1 - \delta}\left[\left(\sum_{j >_m w} x_{m,j} + \sum_{i \geqslant_w m} x_{i,w}\right) - \delta\left(\sum_{j >_m w} z_{m,j} + \sum_{i \geqslant_w m} z_{i,w}\right)\right]$$

for each pair (m, w) and as x satisfies (7), it suffices to show that whenever x satisfies (7) as an equality so does z. So, assume that for $(m, w) \in A$, x satisfies (7) as an equality. As μ_x is a stable matching, z satisfies (7). Further, z satisfies (7) strictly for (m, w) if and only if m prefers $\mu_x(m)$ to w and woman w prefers $\mu_x(w)$ to m. By the definition of μ_x and by Lemma 12, the latter are equivalent to the assertions

$$\sum_{j >_m w} x_{m,j} > 0 \tag{12}$$

and

$$\sum_{i >_w m} x_{i,w} = 1, \tag{13}$$

which jointly contradict the assertion that x satisfies (7) for the pair (m, w) as an equality. This contradiction proves that whenever x satisfies (7) as an equality, so does z, thereby completing our proof. □

One consequence of Theorem 13 is that we may interpret each stable fractional matching x as arising from a lottery over the stable matchings or as a time sharing arrangement. Specifically, if $\{x^1, x^2, \ldots, x^p\}$ is the set of stable matchings, then for each stable fractional matching x there are nonnegative numbers $t_1, t_2, \ldots, t_p$ such that

$$\sum_{k=1}^{p} t_k x^k = x, \qquad \sum_{k=1}^{p} t_k = 1.$$

We may interpret t_k to be either the probability that stable matching x^k is chosen or the fraction of the time x^k applies. In particular, x can be interpreted as a *schedule* of matchings over a period lasting one time-unit; in this period, the stable matching x^k is in force for a time-interval of length t_k (e.g., all agents would be matched according to x^1 for a time-interval of length t_1 and then according to x^2 for a time-interval of length t_2, etc.).

The following example shows that the structure of a stable fractional matching, i.e., how it is achieved by lottery or time sharing, may be as important as the simple totals $x_{i,j}$.

EXAMPLE 2. Consider the stable marriage problem involving three men and three women. The preferences of the players are:

Man					Woman				
m_1:	w_1	w_2	w_3	m_1	w_1:	m_3	m_2	m_1	w_1
m_2:	w_2	w_3	w_1	m_2	w_2:	m_1	m_3	m_2	w_2
m_3:	w_3	w_1	w_2	m_3	w_3:	m_2	m_1	m_3	w_3

The stable matchings in this example are:

$$x^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad x^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad x^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The stable fractional matching

$$x = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

which assigns each man to each woman for an equal amount of time, can be expressed as the schedule of stable matchings in which each stable matching is in force for one-third the time. It can, however, also be expressed as the schedule in which each of the following unstable matchings is in force for one-third the time:

$$y^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad y^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad y^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The situation in which the stable fractional matching x is achieved via the matchings y^1, y^2 and y^3 would not be stable, since there are pairs whose members have a mutual interest in changing partners. Theorem 12 shows that each stable fractional matching can, however, be achieved in a way that is stable at each point in time.

In the next section we extend the lattice on the set of stable matchings to a lattice on the set of stable fractional matchings.

7. The lattice of stable fractional matchings. An important fact about stable matchings is that the set of stable matchings forms a lattice under the partial order of the men's common preferences, and this lattice is dual to the lattice formed under the partial order of the women's common preferences. This result is mirrored in the Assignment Game where the set of stable payoff vectors forms a lattice under the partial order of agents' common preferences on one side of the market that is dual to the lattice formed under the partial order of the agents' preferences on the other side of the market. This latter result is derived from the duality relations associated with the linear programming formulation of the model. In this section, we extend the lattice on the set of stable matchings to a lattice on the set of all stable fractional matchings. As in the Assignment Game, our derivation is based on duality relations associated with the linear programming formulation. We begin by defining an appropriate partial order over the set of stable fractional matchings.

We say that a fractional matching x *weakly dominates* a fractional matching y *in man m's opinion*, denoted $x \succcurlyeq_m y$, if

$$\sum_{j \geqslant_m w} x_{m,j} \geqslant \sum_{j \geqslant_m w} y_{m,j}$$

for each $w \in W$. We further say that x *strongly dominates* y *in man m's opinion*, denoted $x \succ_m y$, if the above inequalities hold with at least one strict inequality for some woman w. Weak domination and strong domination in a woman's opinion are defined analogously. If we interpret $x_{m,w}$ as the probability that m is matched to w, then $x \succcurlyeq_m y$ if $x_{m,\cdot}$ stochastically dominates $y_{m,\cdot}$. Note that in this case we can assert that man m prefers x to y, even though we know only man m's ordinal preferences $\geqslant_m$ and not his expected utility function, since all expected utility functions having the same ordinal preferences agree on comparisons of stochastically dominated lotteries. Of course, if x is the incidence vector of a matching μ and y is the incidence vector of a matching μ', then $x \succcurlyeq_m y$ if and only if $\mu(m) \geqslant_m \mu'(m)$.

We define a partial order $\succcurlyeq_M$ on the set of fractional matchings where $x \succcurlyeq_M y$ if $x \succcurlyeq_m y$ for each man $m \in M$. That is, $x \succcurlyeq_M y$ means that x weakly dominates y in every man's opinion.

Given two stable fractional matchings x and y, define $x \vee y$ to be the $|M| \times |W|$ matrix given by

$$(14)\quad (x \vee y)_{m,w} = \max\left\{ \sum_{j \geqslant_m w} x_{m,j}, \sum_{j \geqslant_m w} y_{m,j} \right\} - \max\left\{ \sum_{j >_m w} x_{m,j}, \sum_{j >_m w} y_{m,j} \right\}$$

and define $x \wedge y$ to be the $|M| \times |W|$ matrix given by:

$$(15)\quad (x \wedge y)_{m,w} = \min\left\{ \sum_{j \geqslant_m w} x_{m,j}, \sum_{j \geqslant_m w} y_{m,j} \right\} - \min\left\{ \sum_{j >_m w} x_{m,j}, \sum_{j >_m w} y_{m,j} \right\}$$

Note that $x \vee y$ is the unique $|M| \times |W|$ matrix satisfying

$$(16)\quad \sum_{k \geqslant_m w} (x \vee y)_{m,k}$$

$$= \sum_{k \geqslant_m w} \left[\max\left\{ \sum_{j \geqslant_m w} x_{m,j}, \sum_{j \geqslant_m k} y_{m,j} \right\} - \max\left\{ \sum_{j >_m k} x_{m,j}, \sum_{j >_m k} y_{m,j} \right\} \right]$$

$$= \sum_{k \geqslant_m w} \left[\max\left\{ \sum_{j \geqslant_m k} x_{m,j}, \sum_{j \geqslant_m k} y_{m,j} \right\} \right]$$

$$- \sum_{k >_m w} \left[\max\left\{ \sum_{j \geqslant_m k} x_{m,j}, \sum_{j \geqslant_m k} y_{m,j} \right\} \right]$$

$$= \max\left\{ \sum_{j \geqslant_m w} x_{m,j}, \sum_{j \geqslant_m w} y_{m,j} \right\}$$

for each $(m, w) \in M \times W$ and $x \wedge y$ is the unique $|M| \times |W|$ matrix satisfying

$$(17)\quad \sum_{i \geqslant_w m} (x \wedge y)_{i,w} = \min\left\{ \sum_{j \geqslant_m w} x_{m,j}, \sum_{j \geqslant_m w} y_{m,j} \right\}$$

for each $(m, w) \in M \times W$.

The next lemma interprets $x \vee y$ and $x \wedge y$ for given stable fractional matchings x and y in terms of the women's preferences.

LEMMA 14. *If x and y are stable fractional matchings, then*

$$(18)\quad (x \vee y)_{m,w} = \min\left\{ \sum_{i \geqslant_w m} x_{i,w}, \sum_{i \geqslant_w m} y_{i,w} \right\} - \min\left\{ \sum_{i >_w m} x_{i,w}, \sum_{i >_w m} y_{i,w} \right\}$$

and

$$(19)\quad (x \wedge y)_{m,w} = \max\left\{ \sum_{i \geqslant_w m} x_{i,w}, \sum_{i \geqslant_w m} y_{i,w} \right\} - \max\left\{ \sum_{i >_w m} x_{i,w}, \sum_{i >_w m} y_{i,w} \right\}.$$

PROOF. If $x_{m,w} = y_{m,w} = 0$, then $(x \vee y)_{m,w} = (x \wedge y)_{m,w} = 0$ and (18) is clearly true. If either $x_{m,w} > 0$ or $y_{m,w} > 0$, then by Lemma 9,

$$\sum_{j \geqslant_m w} x_{m,j} = 1 - \sum_{i >_w m} x_{i,w},$$

$$\sum_{j >_m w} x_{m,j} = 1 - \sum_{i \geqslant_w m} x_{i,w},$$

$$\sum_{j \geqslant_m w} y_{m,j} = 1 - \sum_{i >_w m} y_{i,w},$$

$$\sum_{j >_m w} y_{m,j} = 1 - \sum_{i \geqslant_w m} y_{i,w}.$$

Thus,

$$\max\left\{\sum_{j \geqslant_m w} x_{m,j}, \sum_{j \geqslant_m w} y_{m,j}\right\} = 1 - \min\left\{\sum_{i >_w m} x_{i,w}, \sum_{i >_w m} y_{i,w}\right\}$$

and

$$\max\left\{\sum_{j >_m w} x_{m,j}, \sum_{j >_m w} y_{m,j}\right\} = 1 - \min\left\{\sum_{i \geqslant_w m} x_{i,w}, \sum_{i \geqslant_w m} y_{i,w}\right\},$$

from which (18) follows immediately. Similar arguments prove (19). □

To demonstrate that the set of stable fractional matchings forms a lattice under the partial order $\succcurlyeq_M$ it suffices to show that if x and y are stable fractional matchings, then so are $x \vee y$ and $x \wedge y$. We first show that they are fractional matchings which immediately implies that $x \vee y$ is the unique least upper bound and $x \wedge y$ is the unique greatest lower bound of x and y under the partial order $\succcurlyeq_M$ (defined on the set of fractional matchings).

LEMMA 15. *Let x and y be stable fractional matchings. Then $x \vee y$ and $x \wedge y$ are fractional matchings.*

PROOF. It is immediate from (14) that $(x \vee y) \geqslant 0$, i.e., $x \vee y$ satisfies (5). Next, consider a man $m \in M$ and let w be the least preferred acceptable woman in man m's eyes. By Lemma 7,

$$\sum_{j \geqslant_m w} x_{m,j} = \sum_{j \in W} x_{m,j} = \sum_{j \in W} y_{m,j} = \sum_{j \geqslant_m w} y_{m,j}$$

and so

$$\sum_{j \in W} (x \vee y)_{m,j} = \sum_{j \geqslant_m w} (x \vee y)_{m,j} = \max\left\{\sum_{j \geqslant_m w} x_{m,j}, \sum_{j \geqslant_m w} y_{m,j}\right\}$$
$$= \sum_{j \in W} x_{m,j} \leqslant 1.$$

Thus, $x \vee y$ satisfies (3). A similar argument using the representation of $x \vee y$ given in Lemma 14 shows that $x \vee y$ satisfies (4) and so $x \vee y$ is a fractional matching. Similar arguments apply for $x \wedge y$. □

Lemmas 15 and 14 show that if x and y are stable fractional matchings, then $x \vee y$ is the unique fractional matching satisfying

$$(20) \qquad \sum_{i \geqslant_w m} (x \vee y)_{i,w} = \min\left\{ \sum_{i \geqslant_w m} x_{i,w}, \sum_{i \geqslant_w m} y_{i,w} \right\}$$

and $x \wedge y$ is the unique fractional matching satisfying

$$(21) \qquad \sum_{i \geqslant_w m} (x \wedge y)_{i,w} = \max\left\{ \sum_{j \geqslant_m w} x_{m,j}, \sum_{j \geqslant_m w} y_{m,j} \right\}$$

for each $(m, w) \in A$. Thus, while $x \vee y$ is the unique, *least upper* bound of x and y under the partial order $\succcurlyeq_M$ of the men's common preferences, it is the unique *greatest lower* bound under the analogous partial order $\succcurlyeq_W$ of the women's common preference.

The next result shows that if x and y are stable fractional matchings, then so are $x \vee y$ and $x \wedge y$; hence, the set of stable fractional matchings forms a lattice under the partial order $\succcurlyeq_M$.

LEMMA 16. *If x and y are stable fractional matchings, then $x \vee y$ and $x \wedge y$ are also stable fractional matchings.*

PROOF. By Lemma 15, $x \vee y$ and $x \wedge y$ are fractional matchings. Also, as x and y are individually rational, (14) and (15) imply that so are $x \vee y$ and $x \wedge y$. Thus, we need only show that they also satisfy the stability constraints (7).

Consider a pair $(m, w) \in A$ and assume, without loss of generality, that

$$\sum_{i \geqslant_w m} x_{i,w} \geqslant \sum_{i \geqslant_w m} y_{i,w}.$$

Then, by (16) and (20),

$$\begin{aligned}
&\sum_{j >_m w} (x \vee y)_{m,j} + \sum_{i \geqslant_w m} (x \vee y)_{i,w} \\
&\quad = \max\left\{ \sum_{j >_m w} x_{m,j}, \sum_{j >_m w} y_{m,j} \right\} + \min\left\{ \sum_{i \geqslant_w m} x_{i,w}, \sum_{i \geqslant_w m} y_{i,w} \right\} \\
&\quad \geqslant \sum_{j >_m w} y_{m,j} + \sum_{i \geqslant_w m} y_{i,w} \geqslant 1.
\end{aligned}$$

Thus, $x \vee y$ satisfies the stability constraint (7). Similar arguments show that so does $x \wedge y$. □

We have shown the following theorem stating that the set of stable fractional matchings enjoys a lattice structure analogous to the lattice structure on the core of the Assignment Game.

THEOREM 17. *The set of stable fractional matchings forms a distributive lattice under the partial order $\succcurlyeq_M$ of the men's common preferences. Moreover, this lattice is the dual of the lattice under the partial order $\succcurlyeq_W$ of the women's common preferences.*

PROOF. Let x and y be stable fractional matchings. Then by Lemma 16, $x \vee y$ and $x \wedge y$ are also stable fractional matchings and by (16) and (17); they are the least upper bound and greatest lower bound, respectively, of x and y. Further, the distributivity of this lattice follows from (14), (15), (16) and (17). Finally, the fact that the dual of this lattice is obtained by considering the partial order derived from the preferences of the women follows from Lemma 14. □

Stable fractional matchings can be viewed as arising from a lottery over the stable matchings or as a time sharing arrangement; see the discussion following the proof of Theorem 12. Also, recall Example 2 of §5 which demonstrates the fact that it is important how a stable fractional matching is achieved. The following lemmas show that these arrangements can be made in a manner consistent with the partial order $\succcurlyeq_M$ and hence further illustrate the fact that comparisons between stable fractional matchings can be made without further information about preferences.

LEMMA 18. *Let x and y be stable fractional matchings and suppose that $x \succcurlyeq_M y$. Then there are nonnegative numbers $t_1, t_2, \ldots, t_s$ and stable matchings $x^1, x^2, \ldots, x^s, y^1, y^2, \ldots, y^s$ such that*

$$\sum_{k=1}^{s} t_k = 1, \qquad \sum_{k=1}^{s} t_k x^k = x, \qquad \sum_{k=1}^{s} t_k y^k = y \quad \textit{and}$$

$$x^k \succcurlyeq_M y^k \quad \textit{for each } k \in [1, \ldots, s].$$

PROOF. Let x^1 be the incidence vector of μ_x and let y^1 be the incidence vector of μ_y. Then the assertion $x \succcurlyeq_M y$ immediately implies that $x^1 \succcurlyeq_m y^1$. Let t_1^x be the largest value of t such that $x'(t) = [x - tx^1]/(1 - t)$ is a stable fractional matching and let t_1^y be the largest value of t such that $y'(t) = [y - ty^1]/(1 - t)$ is a stable fractional matching. Let $t_1 = \min\{t_1^x, t_1^y\}$.

We will next show that $x^1(t_1) \succcurlyeq_M y^1(t_1)$. Let $(m, w) \in A$. We consider two cases. First, assume that $w \leqslant_m \mu_y(m)$. In this case

$$\sum_{j \geqslant_m w} x_{m,j} - (1 - t_1) \sum_{j \geqslant_m w} x'(t_1)_{m,j}$$

$$= t_1 = \sum_{j \geqslant_m w} y_{m,j} - (1 - t_1) \sum_{j \geqslant_m w} y'(t_1)_{m,j}.$$

As the assertion $x \succcurlyeq_m y$ implies that

$$\sum_{j \geqslant_m w} x_{m,j} \geqslant \sum_{j \geqslant_m w} y_{m,j},$$

we conclude that

$$\sum_{j \geqslant_m w} x'(t_1)_{m,j} \geqslant \sum_{j \geqslant_m w} y'(t_1)_{m,j}. \tag{22}$$

Of course, this inequality is trivial if $w >_m \mu_M(m)$, for in this case the right-hand side of (22) is zero. So, indeed, $x'(t_1) \succcurlyeq_M y'(t_1)$. The conclusion of our lemma now follows by induction, replacing x and y with $x'(t_1)$ and $y'(t_1)$. □

Lemma 19. *Let x and y be stable fractional matchings. Then there are nonnegative numbers* $t_1, t_2, \ldots, t_s$ *and stable matchings* $x^1, x^2, \ldots, x^s, y^1, y^2, \ldots, y^s$ *such that*

$$\sum_{k=1}^{s} t_k = 1, \qquad \sum_{k=1}^{s} t_k x^k = x, \qquad \sum_{k=1}^{s} t_k y^k = y \quad \textit{and}$$

$$\sum_{k=1}^{s} t_k (x^k \vee y^k) = x \vee y.$$

Proof. Let x^1 be the incidence vector of μ_x, y^1 the incidence vector or μ_y. Then $x^1 \vee y^1$ is the incidence vector of $\mu_{x \vee y}$ and the conclusion of our lemma follows by an inductive argument as was the case for the proof of Lemma 18. □

8. Strong stability. If we view $x_{m,w}$ as the fraction of time man m and woman w are assigned to each other, a stable fractional matching x may assign both m and w a portion of time with people they like less than each other. These two players have an incentive to increase the time they spend with each other at the expense of those they like less. (Of course if x is obtained by a time-sharing schedule

$$x = \sum t_k x^k$$

in which each matching x^k is stable, then there will be no point in time at which two players would *simultaneously* prefer to be matched to each other.) We say that a stable fractional matching x is *strongly stable* if no two people both spend time with people they like less than each other, i.e., if x satisfies the *strong stability condition*

$$\left[1 - \sum_{j \geqslant_m w} x_{m,j}\right] \cdot \left[1 - \sum_{i \geqslant_w m} x_{i,w}\right] = 0 \tag{23}$$

for each pair $(m, w) \in A$.

Clearly every strongly stable fractional matching is stable and the incidence vector of any stable matching is strongly stable. The following example demonstrates a strongly stable fractional matching that is not a matching and also shows that not all stable fractional matchings are strongly stable.

Example 3. Consider the stable marriage problem of Example 2. It is easily seen that the fractional matching

$$x = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is stable; but, it is not strongly stable since

$$\sum_{j \geqslant_{m_1} w_2} x_{1,j} = \sum_{i \geqslant_{w_2} m_1} x_{i,2} = \frac{1}{2}$$

implying that

$$\left[1 - \sum_{j \geqslant_{m_1} w_2} x_{1,j}\right] \cdot \left[1 - \sum_{i \geqslant_{w_2} m_1} x_{i,2}\right] = \frac{1}{4} \neq 0.$$

We also observe that the stable fractional matching

$$x = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

is strongly stable.

The following lemma and its corollary give necessary conditions for a stable fractional matching to be strongly stable.

LEMMA 20. *A stable fractional matching x is strongly stable only if*

(a). *if there are two women w_1 and w_2 such that $w_1 <_m w_2$ and x_{m,w_1} and x_{m,w_2} are positive, then there is no woman $w_1 <_m w <_m w_2$ such that $y_{m,w} > 0$ in a stable fractional matching y, and*

(b). *if there are two men m_1 and m_2 such that $m_1 <_w m_2$ and $x_{m_1,w}$ and $x_{m_2,w}$ are positive, then there is no man $m_1 <_w m <_w m_2$ such that $y_{m,w} > 0$ in a stable fractional matching y.*

PROOF. Suppose x is a stable fractional matching and for some man m there are $w_1 <_m w_2$ such that x_{m,w_1} and x_{m,w_2} are positive. If there is a woman w with $w_1 <_m w <_m w_2$ and $y_{m,w} > 0$ for some stable fractional matching y, then, by Lemma 9,

$$1 - \sum_{j \geqslant_m w} x_{m,j} = \sum_{j <_m w} x_{m,j} \geqslant x_{m,w_1} > 0$$

and,

$$1 - \sum_{i \geqslant_w m} x_{i,w} = \sum_{j >_m w} x_{m,j} \geqslant x_{m,w_2} > 0,$$

so x is not strongly stable. Similar arguments show the necessity of (b). □

COROLLARY 21. *A stable fractional matching is strongly stable only if*

(a). *for each man $m \in M$, $x_{m,j} > 0$ for at most two women $j \in W$, and*

(b). *for each woman $w \in W$, $x_{i,w} > 0$ for at most two men $i \in M$.*

We next provide sufficient conditions for strong stability. To describe these conditions, we note that, as a lattice, the set of stable fractional matchings contains a greatest element with respect to the partial order $\succcurlyeq_M$. Further, as for each stable fractional matching x, the stable matching μ_x defined in §6 has $\mu_x \succcurlyeq_m x$, we have that the above greatest element is a stable matching which we denote μ_M. Symmetric arguments show that the greatest element in the set of stable fractional matchings with respect to the partial order $\succcurlyeq_W$ is also a (stable) matching and we denote it by μ_W. Finally, the results of Theorem 16 show that μ_M and μ_W are the least stable fractional matchings with respect to the partial orders $\succcurlyeq_W$ and $\succcurlyeq_M$, respectively.

LEMMA 22. *A stable fractional matching x is strongly stable if either of the following holds*:

(a). *For each man $m \in M$ and collection w_1, w_2 and w of women such that $w_1 <_m w <_m w_2$ and x_{m,w_2} are positive, $\mu_M(w) >_w m$.*

(b). *For each woman $w \in W$ and collection m_1, m_2 and m of men such that $m_1 <_w m <_w m_2$ and $x_{m_2,w}$ are positive, $\mu_W(m) >_m w$.*

PROOF. To establish the sufficiency of (a), consider a pair $(m,w) \in A$ and recall the partitions of M and W obtained in Lemma 7. If $m \in M_0$ then the stability of x implies that

$$\sum_{i >_w m} x_{i,w} = 1$$

and if $w \in W_0$ then the stability of x implies that

$$\sum_{j >_m w} x_{m,j} = 1.$$

In either case, it is immediate that x satisfies the strong stability condition for (m,w).

Suppose then that $m \in M_1$, $w \in W_1$ and

$$\sum_{j \geqslant_m w} x_{m,j} < 1.$$

Then there is a woman $w_1 <_m w$ with $x_{m,w_1} > 0$. Now, if $x_{m,j} = 0$ for each woman j for which $j >_m w$, then

$$\sum_{j >_m w} x_{m,j} = 0$$

and the stability constraint (7) for the pair (m,w) implies that

$$\sum_{i \geqslant_w m} x_{i,w} = 1;$$

so, x satisfies the stability condition for the pair (m,w). Alternatively, if there is some woman w_2 with $x_{m,w_2} > 0$ and $w <_m w_2$, then (a) implies that $m <_w \mu_M(w)$. In particular, if z is the incidence vector of μ_M we have from the fact that $x \succcurlyeq_W z$ that

$$\sum_{i \geqslant_w m} x_{i,w} \geqslant \sum_{i \geqslant_w m} z_{i,w} \geqslant \sum_{i \geqslant_w \mu_M(w)} z_{i,w} = 1,$$

and again x satisfies the stability constraint for the pair (m,w).

Finally, the sufficiency of (b) follows from the sufficiency of (a) by exchanging the roles of M and W. □

9. Incentives. Recall that the set of stable matchings has a greatest element, μ_M that all men agree is best, and a least element, μ_W that all men agree is worst. In particular, Gale and Shapley's well-known deferred acceptance algorithm can be used to compute these matchings; see Roth and Sotomayor (1990) for details. In the current section we use this algorithm to sketch simple linear programming proofs of incentive compatibility results for the stable marriage problem (cf. Dubins and Freedman (1981) and Roth (1982)). Specifically, we use linear programming and the Gale-Shapley algorithm to show that the men-optimal stable matching μ_M gives each man the best outcome not precluded by the preferences of the others. That is, the men-optimal stable matching gives each man m the same outcome as his most preferred outcome in the set of "quasi-stable" matchings defined to be those outcomes in which blocking pairs are not excluded, but each blocking pair must include man m. The linear programming formulation is well suited to make this connection clear because the m-quasi-stable matchings can be studied by removing all the stability constraints involving man m. We will show that if a man misrepresents his preferences, every resulting stable (fractional) matching would be (weakly) dominated by the men-optimal stable matching under his original preferences. Thus, no man can do better than to state his true preferences. Beyond highlighting the connections between incentive compatibility, the deferred acceptance algorithm and the polyhedral structure of the stable marriage problem, these proofs illustrate a general technique for demonstrating incentive compatibility.

We consider a linear programming formulation of the problem in which the individual rationality and stability constraints involving one man, say man 1, are relaxed. We will see that it is still impossible for him to be matched, for any positive fraction of time, with a woman he prefers to $\mu_M(1)$.

Consider the problem of finding the best 1-quasi-stable matching, i.e., the fractional matching satisfying the individual rationality and stability constraints of the other players, which maximizes the fraction of time man 1 spends with women he prefers to $\mu_M(1)$. To include all man 1's possible misrepresentations, the set of admissible pairs in this problem must be expanded to include those pairs $(1, w)$ for which woman w prefers man 1 to remaining single, but man 1 prefers remaining single to marrying woman w. We denote the set by A_1. The problem we then consider, denoted (MP1), is:

$$\max \sum_{j >_1 \mu_M(1)} x_{1,j}.$$

$$\text{s.t.} \sum_{j \in W} x_{m,j} \leqslant 1 \quad \text{for each } m \in M, \tag{24}$$

$$\sum_{i \in M} x_{i,w} \leqslant 1 \quad \text{for each } w \in W, \tag{25}$$

$$\sum_{j >_m w} x_{m,j} + \sum_{i \geqslant_w m} x_{i,w} \geqslant 1 \quad \text{for each } (m, w) \in A \text{ with } m \neq 1, \tag{26}$$

$$x_{m,w} \geqslant 0 \quad \text{for each } (m, w) \in A_1, \tag{27}$$

$$x_{m,w} = 0 \quad \text{for each } (m, w) \in (M \times W) \setminus A_1. \tag{28}$$

The constraints defining (MP1) are analogous to the constraints of (MP) except that A_1 replaces A and the stability constraints that consider the preferences of man 1 have been dropped.

We show that the optimal objective value of (MP1) is zero by considering the dual program (DMP1):

$$\min \sum_{i \in M} \alpha_i + \sum_{j \in W} \beta_j - \sum_{(i,j) \in A} \gamma_{i,j}$$

$$\alpha_m + \beta_w - \sum_{j <_m w} \gamma_{m,j} - \sum_{i \leqslant_w m} \gamma_{i,w} \geqslant 0 \quad \text{for each } (m, w) \in A,\ m \neq 1, \tag{29}$$

$$\alpha_1 + \beta_w - \sum_{j <_1 w} \gamma_{1,j} - \sum_{i <_w 1} \gamma_{i,w} \geqslant 0 \quad \text{for } (1, w) \in A_1,\ w \leqslant_1 \mu_M(1), \tag{30}$$

$$\alpha_1 + \beta_w - \sum_{j <_1 w} \gamma_{1,j} - \sum_{i <_w 1} \gamma_{i,w} \geqslant 1 \quad \text{for } (1, w) \in A_1,\ w >_1 \mu_M(1), \tag{31}$$

$$\gamma_{m,w} = 0 \quad \text{if } m = 1 \text{ or } (m, w) \in (M \times W) \setminus A, \tag{32}$$

$$\alpha, \beta, \gamma \geqslant 0. \tag{33}$$

The next lemma shows that feasible solutions to relaxations of (DMP1) are generated in the course of the deferred acceptance algorithm (which we do not describe here; see Gale and Shapley (1962) or Roth and Sotomayor (1990)).

LEMMA 23. *Let x be the incidence vector of a matching obtained in the course of the deferred acceptance algorithm when man* 1 *is single. Then* (α, β, γ) *defined by*

(34) $$\alpha_m = 0 \quad \textit{for each } m \in M,$$

(35) $$\beta_w = \sum_{i \in M} x_{i,w} \quad \textit{for each } w \in W,$$

(36) $$\gamma_{m,w} = x_{m,w} \quad \textit{for each } (m,w) \in M \times W,$$

satisfies every constraint of (*DMP*1) *except possibly those constraints* (31) *for women w to whom man* 1 *has not yet proposed. Further, the dual objective value associated with* (α, β, γ) *is zero.*

PROOF. By the definition of (α, β, γ) given in (34), (35) and (36),

(37) $$\alpha_m + \beta_w - \sum_{j <_m w} \gamma_{m,w} - \sum_{i \leqslant_w m} \gamma_{i,w} = \sum_{i >_w m} x_{iw} - \sum_{j <_m w} x_{m,j}.$$

Now, suppose that man m is currently matched and he prefers a woman w to the woman currently accepting his proposal. The deferred acceptance algorithm ensures that, in this case, man m has already proposed to w and that she has rejected him for someone she prefers. As each woman only rejects a proposal for one she prefers, it follows that woman w prefers her current mate to man m. Thus, in this case

$$\sum_{i >_w m} x_{i,w} = 1 \quad \text{and} \quad \sum_{j <_m w} x_{m,j} = 1,$$

and hence the right-hand side of (37) is nonnegative. Further, if man m is single or if he is currently matched to w or to a woman he prefers to w, then

$$\sum_{j <_m w} x_{m,j} = 0$$

and the right-hand side of (37) is trivially nonnegative. So (29) and (30) hold. Next, assume that man 1 has already proposed to w. Then the above arguments show that $\sum_{i >_w m} x_{i,w} = 1$. As man 1 is single, $\sum_{j >_m w} x_{m,j} = 0$ and (31) holds for the pair $(1, w)$. Also, trivially, (α, β, γ) satisfies (32) and (33). So, (α, β, γ) satisfies the desired constraints. The fact that the objective value of (α, β, γ) is zero is immediate. □

THEOREM 24. *The optimal objective value of* (*MP*1) *is zero.*

PROOF. Every incidence vector of a stable matching is feasible for (MP1) and such vectors are known to exist. Further, the objective value of each feasible solution of (MP1) is nonnegative. Thus, by the (Duality) Theorem 1 it suffices to construct a feasible solution of (DMP1) with (dual) objective value zero.

Now if man 1 is never rejected, $\mu_M(1)$ is his most preferred woman and the theorem is immediate. Otherwise, consider the matching obtained when, in the course of the deferred acceptance algorithm, man 1 receives his last rejection. In this matching, man 1 is single and has been rejected by all women he prefers to $\mu_M(1)$. By Lemma 23 the corresponding solution (α, β, γ) defined by (34)—(36) is feasible for (DMP1) and has objective value 0. □

COROLLARY 25. *No man m can, by misrepresenting his preferences, be matched to a woman he prefers to $\mu_M(m)$.*

We next modify the above results to consider incentive compatibility with respect to groups consisting of more than a single man. This strengthens a result of Damange, Gale and Sotomayor in the case of no indifference (see Roth and Sotomayor (1990, Theorem 4.11)).

THEOREM 26. *No coalition of men and women which includes at least one man can, by falsifying its members' preferences, obtain a (fractional) matching which strongly dominates the men-optimal stable matching in the opinion of each of its members.*

PROOF. We only sketch the main ideas of the proof which relies on the theory of linear inequalities. Consider a coalition C including at least one man and assume for the sake of presentation that each member of C is matched in the man-optimal stable matching. If some man in C receives no rejection, then his mate under μ_M is his favorite woman and so he cannot be matched to someone he prefers and the theorem follows. So, assume each man in C receives at least one rejection.

Let $A_C = A \cup \{(m,w): m \in M \cap C$ and $w \in W \cap C$; or $m \in M \cap C, w \in W \setminus C$ and $m >_w w$; or $m \in M \setminus C$, $w \in W \cap C$ and $w >_m m\}$ and consider the linear system:

$$(38) \qquad \sum_{j \in W} x_{m,j} \leqslant 1 \quad \text{for each } m \in M \setminus C,$$

$$(39) \qquad \sum_{j \in W} x_{m,j} = 1 \quad \text{for each } m \in M \cap C,$$

$$(40) \qquad x_{m,w} = 0 \quad \text{for each } m \in M \cap C \text{ and } w <_m \mu_M(m)$$

$$(41) \qquad \sum_{j >_m \mu_M(m)} x_{m,j} > 0 \quad \text{for each } m \in M \cap C,$$

$$(42) \qquad \sum_{i \in M} x_{i,w} \leqslant 1 \quad \text{for each } w \in W \setminus C,$$

$$(43) \qquad \sum_{i \in M} x_{i,w} = 1 \quad \text{for each } w \in W \cap C,$$

$$(44) \qquad x_{m,w} = 0 \quad \text{for each } w \in W \cap C \text{ and } m <_w \mu_M(m),$$

$$(45) \qquad \sum_{i >_w \mu_M(w)} x_{i,w} > 0 \quad \text{for each } w \in W \cap C,$$

$$(46) \qquad \sum_{j >_m w} x_{m,j} + \sum_{i \geqslant_w m} x_{i,w} \geqslant 1 \quad \text{for } (m,w) \in A,\ m \in M \setminus C \text{ and } w \in W \setminus C,$$

$$(47) \qquad x_{m,w} \geqslant 0 \quad \text{for each } (m,w) \in A_C,$$

$$(48) \qquad x_{m,w} = 0 \quad \text{for each } (m,w) \in (M \times W) \setminus A_C.$$

Constraints (39)–(41) and (43)–(45) ensure that any feasible solution x strictly dominates μ_M in the opinion of the male and female members of C, respectively. We establish the theorem by demonstrating the infeasibility of (38)–(48).

The theorem of the alternative for linear systems with weak and strict inequalities (see Schrijver (1989)) states that the system (38)–(48) has a feasible solution if and only if there is no $\alpha \in \mathscr{R}^{|M|}$, $\beta \in \mathscr{R}^{|W|}$, $\gamma \in \mathscr{R}^{|M| \times |W|}$ and $\delta \in \mathscr{R}^{|M \cup W|}$ satisfying:

$$(49) \qquad \sum_{i \in M} \alpha_i + \sum_{j \in W} \beta_j - \sum_{(i,j) \in A} \gamma_{i,j} \leqslant 0$$

$$(50) \qquad \alpha_m + \beta_w - \sum_{j <_m w} \gamma_{m,j} - \sum_{i \leqslant_w m} \gamma_{i,w} \geqslant \begin{cases} 0 & \text{if } (m,w) \in A_C,\ m \in M \setminus C, \\ & \quad w \in W \setminus C \\ \delta_m & \text{if } (m,w) \in A_C,\ m \in M \cap C, \\ & \quad w >_m \mu_M(m),\ w \in W \setminus C, \\ \delta_w & \text{if } (m,w) \in A_C,\ w \in W \cap C, \\ & \quad m >_w \mu_M(w),\ m \in M \setminus C, \\ \delta_m + \delta_w & \text{if } (m,w) \in A_C, \\ & \quad m \in C,\ w >_m \mu_M(m), \\ & \quad w \in C,\ m >_w \mu_M(w), \\ 0 & \text{if } (m,w) \in A_c,\ \mu_M(m) = w. \end{cases}$$

$$(51) \qquad \gamma_{m,w} = 0 \quad \text{if } m \in C, \text{or } w \in C \text{ or } (m,w) \in (M \times W) \setminus A,$$

$$(52) \qquad \gamma \geqslant 0,$$

$$(53) \qquad \alpha_m \geqslant 0 \quad \text{if } m \in M \setminus C,$$

$$(54) \qquad \beta_w \geqslant 0 \quad \text{if } w \in W \setminus C,$$

$$(55) \qquad \delta_k = 0 \quad \text{if } k \in (M \cup W) \setminus C,$$

$$(56) \qquad \delta \geqslant 0$$

$$(57) \qquad \sum_{i \in M} \alpha_i + \sum_{j \in W} \beta_j - \sum_{(i,j) \in A_C} \gamma_{i,j} \neq 0 \quad \text{or} \quad \delta \neq 0.$$

(We note that the first and last cases in (50) can easily be made mutually exclusive.)

Let μ be the matching obtained in the course of the deferred acceptance algorithm when the last rejection of a man in the coalition C occurs. In the Appendix, we demonstrate that (38)–(48) is infeasible by showing that the following is a feasible solution to (49)–(57). Let

$$\alpha_m = \begin{cases} -1 & \text{if } m \in C,\ \mu(m) \neq m, \\ 0 & \text{otherwise;} \end{cases}$$

$$\beta_w = \begin{cases} 1 & \text{if } \mu(w) \in M \text{ and } w \in W \setminus C \text{ or if } w \in W \cap C \text{ and } \mu(w) \in M \cap C, \\ 0 & \text{otherwise;} \end{cases} \quad \square$$

$$\gamma_{m,w} = \begin{cases} 1 & \text{if } \mu(m) = w,\ m \in M \setminus C \text{ and } w \in W \setminus C, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_k = \begin{cases} 1 & \text{if } k \in M \cap C \text{ and } \mu(k) = k, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

We note that in the proof of Theorem 24, the constructed β and δ satisfy $\beta_w \geqslant 0$ and $\delta_w = 0$ for each $w \in W \cap C$. Consequently, our proof establishes infeasibility of a relaxed version of (38)–(48) where constraints (45) are dropped and constraints (43) are weakened by replacing the equality by an inequality "$\leqslant$". Similarly, if $M \cap C$ consists of one man m, $\delta_m = 0$ and our proof establishes the infeasibility of a relaxed version of (38)–(48) where (41) is dropped. In particular, the last observation shows how the arguments of the proof of Theorem 24 can be used to establish Corollary 23.

10. Conclusions. The marriage model has been studied with eclectic tools of discrete mathematics since its introduction by Gale and Shapley (1962). Concurrently, the assignment model introduced by Shapley and Shubik (1972) has been studied with the algebraic tools of linear programming. As noted in the introduction, the similarities between the results obtained by the two models (and their generalizations) has been an enduring puzzle. Here we have shown how linear programming can be brought to bear on the marriage model. In doing so, we hope both to have shed some light on the similarities between the two models and to have shown how this alternative set of tools can be used to extend what is known about matching.

Appendix.

DETAILED PROOF OF THEOREM 24. First, one may easily verify from their definitions that α, β, γ and δ satisfy (53)–(56). Also, since

$$\sum_{i \in M} \alpha_i + \sum_{j \in W} \beta_j \quad \text{and} \quad \sum_{(i,j) \in A} \gamma_{i,j}$$

are the number of matched pairs in μ not involving a member of C, (49) is satisfied with equality. Since $\delta_m = 1$ for the man $m \in C$ who was last rejected, (57) is satisfied.

We show that α, β, γ and δ satisfy (50) by considering the five cases separately.

Case 1. $(m, w) \in A_C$, $m \notin C$ and $w \notin C$.

In this case, we see that α, β, γ and δ satisfy (50) exactly as in the proof of Lemma 21.

Case 2. $(m, w) \in A_C$, $m \in C$, $w \notin C$ and $w >_m \mu_M(m)$.

In this case, $\alpha_m - \delta_m = -1$ and $\gamma_{m,j} = 0$ for each $j \in W$. Thus,

$$\alpha_m - \delta_m + \beta_w - \sum_{j <_w m} \gamma_{m,j} - \sum_{i \leqslant_w m} \gamma_{i,w} = -1 + \beta_w - \sum_{i <_w m} \gamma_{i,m}. \tag{58}$$

Now, since $w >_m \mu_M(m)$ and m receives no further rejections under the deferred acceptance algorithm, woman w must have already rejected man m. It follows that $\mu(w) \in M$ and $\mu(w) >_w m$ and so the right-hand side of (58) is zero.

Case 3. $(m, w) \in A_C$, $m \notin C$, $w \in C$ and $m >_w \mu_M(w)$.

In this case, since $m >_w \mu_M(w) \geqslant_w \mu(w)$, woman w has not yet rejected man m and so either $\mu(m) = m$ or $\mu(m) >_m w$. In either case,

$$\begin{aligned} \alpha_m + \beta_w - \delta_w - \sum_{j <_w m} \gamma_{m,j} - \sum_{i \leqslant_w m} \gamma_{i,w} \\ = \beta_w - \sum_{j <_w m} \gamma_{m,j} = \beta_w \geqslant 0. \end{aligned} \tag{59}$$

Case 4. $(m, w) \in A_C$, $w >_m \mu_M(m)$ and $m >_w \mu_M(w)$.

This case is precluded by the stability of μ_M.

Case 5. $(m, w) \in A_C$ and $\mu_M(m) = w$.

We consider first the case in which man m is single under μ, i.e., $\mu_M(m) = m$. In this case,

$$\alpha_m + \beta_w - \sum_{j <_w m} \gamma_{m,j} - \sum_{i \leqslant_w m} \gamma_{i,w} = \beta_w - \sum_{i <_w m} \gamma_{i,m}. \tag{60}$$

Now, if $w \in C$, then $\gamma_{i,w} = 0$ for each $i \in M$ and the right-hand side of (60) is trivially nonnegative. If $w \notin C$, then

$$\beta_w - \sum_{i <_w m} \gamma_{i,w} \geqslant \sum_{i \geqslant_w m} \gamma_{i,m} \geqslant 0$$

and so the right-hand side of (60) is again nonnegative.

We finally consider the case in which man m is matched under μ, i.e., $\mu(m) = \mu_M(m) \neq m$. In this case, if $m \in C$ then

$$\alpha_m + \beta_w - \sum_{j <_w m} \gamma_{m,j} - \sum_{i \leqslant_w m} \gamma_{i,w} = \alpha_m + \beta_w - \gamma_{m,w} = 0,$$

and if $m \notin C$, then we need only consider the case in which $w \in C$ (since we considered $m \notin C$ and $w \notin C$ in Case 1). Since $w = \mu_M(m) \leqslant_m \mu(m)$, $\alpha_m = 0$ and $\gamma_{i,w} = 0$ for each $i \in M$ from which (50) follows. □

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