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A NOTE ON JOB MATCHING WITH BUDGET CONSTRAINTS *

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Recent papers have obtained results for two-sided matching games in which agents may have complex preferences. This note observes that these results depend on the assumption that firms do not face budget constraints.

A number of recent papers [see Roth (1985a) for a survey] have modelled labor markets as two-sided matching games. Kelso and Crawford (1982) were the first to obtain results without assuming that firms' preferences for workers are separable. They considered a model in which firms may have more complex preferences over groups of workers, but in which the production technology is such that firms regard workers more as substitutes than as complements. They noted that, without this 'gross substitutes' assumption, the core of the market could be empty. However, although this assumption has been the object of further study [see, e.g., Roth (1985b), Blair (1984)], in what ways it can be relaxed remains an open question.

Here we note that implicit in the formulation of the model is the assumption that firms face perfect credit markets, since firms are assumed not to face budget constraints in hiring workers. This turns out to be critical to the results. It is shown here that, when technologies are identical to those in Kelso and Crawford (1982), except firms face budget constraints, the core may be empty. Furthermore, even when the core is non-empty, it may not exhibit the characteristic properties of such models such as those derived in Roth (1985b).

The model

There are m workers and n firms, indexed $i=1, \ldots, m$ and $j=1, \ldots, n$. Workers may work only at one firm but firms may hire any number of workers. The utility of worker i working at firm j for salary s_{ij} is $u^i(j; s_{ij})$, where u^i is strictly increasing and continuous in salary. The lowest salary that worker i will accept from firm j is σ_{ij}. At this salary worker i is indifferent between being unemployed and working for firm j, i.e., $u^i(j; \sigma_{ij}) \equiv u^i(0; 0)$, where $u^i(0; 0)$ is the utility of being unemployed. Let $y^j(C^j)$ be the firm's gross product when C^j is the set of workers hired by firm j. When firm j faces a vector of salaries $s^j=(s_{1j}, \ldots, s_{mj})$, which it would have to pay each worker i in order to hire him away from his next best opportunity, firm j chooses the set C of workers which maximizes its net profits. Let $M^j(s^j)$ be the set of workers (or sets, if there are more than one) that maximize net profits $\pi^j(C^j; s^j) \equiv y^j(C^j) - \sum_{i \in C^j} s_{ij}$.

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Table 1
Production technology of firms j and k.

Firm j	Firm k
$y^j(\{1\}) = 7$	$y^k(\{1\}) = 6$
$y^j(\{2\}) = 14$	$y^k(\{2\}) = 15$
$y^j(\{3\}) = 8$	$y^k(\{3\}) = 11$
$y^j(\{1, 2\}) = 21$	$y^k(\{1, 2\}) = 21$
$y^j(\{1, 3\}) = 15$	$y^k(\{1, 3\}) = 17$
$y^j(\{2, 3\}) = 22$	$y^k(\{2, 3\}) = 26$
$y^j(\{1, 2, 3\}) = 29$	$y^k(\{1, 2, 3\}) = 32$
$y^j(\{\emptyset\}) = 0$	$y^k(\{\emptyset\}) = 0$
$\sigma_{1j} = 4$	$\sigma_{1k} = 3$
$\sigma_{2j} = 7$	$\sigma_{2k} = 10$
$\sigma_{3j} = 4$	$\sigma_{3k} = 7$

It is assumed that (i) the marginal product of each worker is non-negative, that is $y^j(CU\{i\}) - y^j(C) - \sigma_{ij} \geq 0$ for all (i, j) and C, where $i \notin C$, (ii) there is 'no-free-lunch', $y^j(\emptyset) = 0$ for all j, and (iii) for all j, if $C^j \in M^j(s^j)$ and $\tilde{s}^j \geq s^j$ then there exists $\tilde{C}^j \in M^j(\tilde{s}^j)$ such that $T^j(C^j) \subseteq \tilde{C}^j$, where $T^j(C^j) \equiv \{i \mid i \in C^j \text{ and } \tilde{s}_{ij} = s_{ij}\}$. Assumption (iii), the 'gross substitutes' condition, states that a firm will continue to hire a subset $T^j(C^j) \subseteq C^j$ of desirable workers, whose salary did not rise, even when other workers' salaries increased. Thus the production technology is such that workers are not complements, in that desirable employees at a given salary remain desirable at that salary, even if the salary of some of their coworkers rises (at which point the coworkers may no longer be desirable).

An *individually rational allocation* is an assignment of workers to firms with a salary schedule such that, if f: $(1, \ldots, m) \to (1, \ldots, n)$ is the function representing the assignments and $C^j \equiv \{i \mid j = f(i)\}$ then (i) $s_{if(i)} \geq \sigma_{if(i)}$, and (ii) $\pi^j(C^j; s^j) \equiv y^j(C^j) - \Sigma_{i \in C^j} s_{ij} \geq 0$.

Kelso and Crawford (1982) consider a model in which salary can only be paid in discrete units. A *(discrete) strict core allocation* is an individually rational allocation $(f; s_{1f(1)}, \ldots, s_{mf(m)})$ such that there does not exist any set of firms and workers (j, C) with salaries $\tilde{s}^j \equiv (\tilde{s}_{1j}, \ldots, \tilde{s}_{mj})$ that satisfy (i) $u^i(j; \tilde{s}_{ij}) \geq u^i[f(i); s_{if(i)}]$ for all $i \in C$, and (ii) $\pi^j(C; \tilde{s}^j) \geq \pi^j(C^j; s^j)$ with strict inequality holding for at least one worker or firm.

Kelso and Crawford (1982) proved the non-emptiness of the core and the existence of a firm optimal core allocation, that is, a core allocation which all firms weakly prefer to any other allocation.

Consider a market which consists of firms j and k. (See table 1.) The production technology satisfies all the conditions specified in the model. Now impose a budget constraint on firms j and k respectively; $\Sigma s_{ij} \leq B^j = 4.40$ for $i \in C^j$ and $\Sigma s_{ik} \leq B^k = 10.75$ for $i \in C^k$. A feasible allocation must meet the requirement that the sum of the salaries of workers employed by a firm does not exceed the budget constraint. The budget-constrained production technology [1] for firm j is $y^j(\{1\}) = 7$ and $y^j(\{3\}) = 8$ and for firm k is $y^k(\{1\}) = 6$, $y^k(\{2\}) = 15$, $y^k(\{3\}) = 11$, and $y^k(\{1, 3\}) = 17$. The gross substitutes condition no longer holds for firm k. Consider two salary vectors $s^k \equiv (s_{1k}, s_{2k}, s_{3k}) = (3, 10, 7)$ and $\tilde{s}^k \equiv (\tilde{s}_{1k}, \tilde{s}_{2k}, \tilde{s}_{3k}) = (3.80, 10, 7)$. While the unique preferred set of workers for firm k is $\{1, 3\}$ at both salary vectors s^k and $\tilde{s}^k$, the firm cannot afford this set of workers at the salary vector $\tilde{s}^k$. The firm has to choose the set of workers $\{2\}$ at the salary vector $\tilde{s}^k$.

It is straightforward to verify that the core in this market is empty. [2] However, non-emptiness is

[1] Note that under these production technologies, the productivities of the workers are separable.

Table 2
Production technology of firms m and p.

Firm m	Firm p
$y^m(\{1\}) = 7$	$y^p(\{1\}) = 6$
$y^m(\{2\}) = 7.5$	$y^p(\{2\}) = 15$
$y^m(\{3\}) = 14$	$y^p(\{3\}) = 11$
$y^m(\{1, 2\}) = 14.5$	$y^p(\{1, 2\}) = 21$
$y^m(\{1, 3\}) = 21$	$y^p(\{1, 3\}) = 17$
$y^m(\{2, 3\}) = 21.5$	$y^p(\{2, 3\}) = 26$.
$y^m(\{1, 2, 3\}) = 28.5$	$y^p(\{1, 2, 3\}) = 32$
$y^m(\{\emptyset\}) = 0$	$y^p(\{\emptyset\}) = 0$
$\sigma_{1m} = 4$	$\sigma_{1p} = 1.5$
$\sigma_{2m} = 4$	$\sigma_{2p} = 8$
$\sigma_{3m} = 7$	$\sigma_{3p} = 6$

not all that is lost. Even when a core allocation does exist, firm optimality [see Kelso and Crawford (1982), Roth (1985b)] and the property that, at the firm optimal outcome, firms are not harmed by additional workers entering the market, is also lost [see Kelso and Crawford (1982)].

Consider a problem with three workers and two firms, m and p, where $B^m = B^p = 12$. (See table 2.) A core allocation, f, yields the following profits for firm m and p, respectively;

$$\pi^m(\{3\};\, s^m) = 7 \quad \text{where} \quad s^m = (s_{1m} > 5.01,\ s_{2m} > 5.01,\ s_{3m} = 7),$$
$$\pi^p(\{1, 2\};\, s^p) = 9.48 \quad \text{where} \quad s^p = (s_{1p} = 2.51,\ s_{2p} = 9.01,\ s_{3p} > 6).$$

Another stable allocation, $\tilde{f}$, yields the following profits:

$$\pi^m(\{3\};\, \tilde{s}^m) = 5.99 \quad \text{where} \quad \tilde{s}^m = (\tilde{s}_{1m} > 4,\ \tilde{s}_{2m} > 4,\ \tilde{s}_{3m} = 8.01),$$
$$\pi^p(\{1, 2\};\, \tilde{s}^p) = 11.50 \quad \text{where} \quad \tilde{s}^p = (\tilde{s}_{1p} = 1.5,\ \tilde{s}_{2p} = 8,\ \tilde{s}_{3p} > 7.01).$$

Firm m prefers allocation f to allocation $\tilde{f}$ whereas firm p prefers allocation $\tilde{f}$. The unique feasible allocation f at which firm m receives a profit of 7 *and* firm p receives a profit at 11.50 is unstable. [Note that in order for firm p to receive a profit of 11.50, $\underline{C}^p = \{1, 2\}$.] Hence, there is no firm optimal allocation.

When the budget constraints for the two firms are changed to $B^m = 9$ and $B^p = 8$, it can be shown that the presence of additional workers may harm the firm.

Consider the above market without worker 1, where now $B^m = 9$ and $B^p = 8$. The firm optimal allocation yields a profit of 7 each for firms m and p,

$$\pi^m(\{3\};\, s^m) = 7 \quad \text{where} \quad s^m = (s_{2m} > 4,\ s_{3m} = 7), \quad \text{and}$$
$$\pi^p(\{2\};\, s^p) = 7 \quad \text{where} \quad s^p = (s_{2p} = 8,\ s_{3p} > 6).$$

The inclusion of worker 1 yields a firm optimal allocation with profits of 6.50 for firm m and 7 for

[2] One approach is to observe that there exists no stable outcome in which worker 2 is employed and none in which he is unemployed.

firm p,

$$\pi^m(\{3\};\ \hat{s}^m) = 6.50 \quad \text{where} \quad \hat{s}^m = (\hat{s}_{1m} > 4,\ \hat{s}_{2m} > 4,\ \hat{s}_{3m} = 7.5),$$
$$\pi^p(\{2\};\ \hat{s}^p) = 7 \quad \text{where} \quad \hat{s}^p = (\hat{s}_{1p} > 1.5,\ \hat{s}_{2p} = 8,\ \hat{s}_{3p} > 6.5).$$

The addition of worker 1 resulted in a decrease on the profit of firm m from 7 to 6.50. Hence, the inclusion of worker 1 into the market harmed firm m.

References

Blair, Charles, 1984, Stable matching with multiple partners and lattice structure, Mimeo. (University of Illinois, Urbana, IL).

Kelso, A.S., Jr. and V.P. Crawford, 1982, Job matching, coalition formation, and gross substitutes, Econometrica 50, 1483–1504.

Roth, Alvin E., 1985a, Common and conflicting interests in two-sided matching markets, European Economic Review 27, 75–96.

Roth, Alvin E., 1985b, Conflict and coincidence of interest in job matching: Some new results and open questions, Mathematics of Operations Research 10, 379–389.