Black-Scholes model

- Martingale approach
- Greeks
- American options
Black-Scholes model

- Black-Scholes model is the basic building blocks of derivatives theory.

- In 1970s, Fisher Black, Myron Scholes and Robert Merton made a major breakthrough in the pricing of stock options – they develop the Black-Scholes (or Black-Scholes-Merton) model.

- Merton and Scholes received the Nobel prize in 1997.
Martingale Approach

1. Take a stock model

2. Use the Cameron-Martin-Girsanov theorem to change it into a martingale.

3. Use the martingale representation theorem to create a replicating strategy for each claim.
Martingale Approach

Model assumptions:

– The stock price and bond price follow

\[ S_t = S_0 \exp(\mu t + \sigma W_t), \quad B_t = \exp(rt), \]

where \( r \) is the riskless interest rate, \( \sigma \) is the stock volatility and \( \mu \) is the stock drift (for simplicity, we assume that \( r, \mu \) and \( \sigma \) are deterministic).

– There are no transaction costs.

– Both instruments are freely and instantaneously tradable either long or short at the price quoted.
Martingale Approach

Three steps to replication

• Find a measure $Q$ under which $S_t$ is a martingale.

• Form the process

$$E_t = E_Q(X | \mathcal{F}_t).$$

• Find a previsible process $\phi_t$, such that

$$dE_t = \phi_t dS_t.$$
Martingale Approach – Step One

1. We first find a stochastic differential equation for $S_t$.

$$S_t = \exp(\sigma W_t + \mu t)$$

$$\implies dS_t = \sigma S_t dW_t + (\mu + \frac{1}{2} \sigma^2) S_t dt.$$ 

2. In order for $S_t$ to be a martingale, we need to remove the drift. Let $\gamma_t \equiv (\mu + \frac{1}{2} \sigma^2)/\sigma$, then based on the C-M-G theorem, there is a measure $Q$ such that $\widetilde{W}_t = W_t + \gamma t$ is a $Q$-Brownian motion. Therefore, we have

$$dS_t = \sigma S_t d\widetilde{W}_t,$$

and $S_t$ become a Q-martingale.
Martingale Approach – Step Two

For an arbitrary claim $X$ (e.g., it is $\max(S_T - K, 0)$ for an European call option),

$$E_t = E_Q(X|F_t)$$

is a $Q$-martingale.
Martingale Approach – Step Three

As $E_t$ and $S_t$ are both Q-martingales, based on the martingale representation theorem, there exists a previsible process $\phi_t$ such that

$$E_t = E_Q(X|F_t) = E_Q(X) + \int_0^t \phi_s dS_s,$$

or $dE_t = \phi_t dS_t$. 
Martingale Approach

Assume that the interest rate $r = 0$, so $B_t \equiv 1$. Our replicating strategy is to:

- hold $\phi_t$ units of stock at time $t$ and
- hold $\psi_t = (E_t - \phi_t S_t)/B_t$ units of the bond at time $t$.

The value of the portfolio at time $t$ is

$$V_t = \phi_t S_t + \psi_t B_t = E_t \implies dV_t = dE_t.$$  

The martingale representation theorem yields

$$dE_t = \phi_t dS_t.$$  

Hence, our strategy is self-financing

$$dV_t = \phi_t dS_t + \psi_t dB_t.$$
Martingale Approach

Question: What if \( r \neq 0 \)?

We consider a discounted stock \( Z_t = B_t^{-1} S_t \) and a discounted claim \( B_T^{-1} X \). Then,

\[
dZ_t = Z_t (\sigma dW_t + (\mu - r + \frac{1}{2} \sigma^2) dt).
\]

Denote \( \gamma = (\mu - r + \frac{1}{2} \sigma^2) / \sigma \), then there exists a measure \( Q \) such that \( \tilde{W}_t = W_t + \gamma t \) is a \( Q \)-Brownian motion, and

\[
dZ_t = \sigma Z_t d\tilde{W}_t.
\]

Hence \( Z_t \) is a martingale under \( Q \).
Martingale Approach

Question: What if $r \neq 0$?

— Furthermore, the process $E_t = E_Q(B_t^{-1}X|F_t)$ is also a $Q$-martingale.

— Then the martingale representation theorem says that there is a predictable $\phi_t$ such that $dE_t = \phi_t dZ_t$.

— Our replicating strategy is:
  • hold $\phi_t$ units of the stock at time $t$, and
  • hold $\psi_t = E_t - \phi_t Z_t$ units of the bond.

We can prove that $V_t = B_tE_t$ and the strategy is self-financing.

$$dV_t = \phi_t dS_t + \psi_t dB_t.$$
Martingale Approach – Summary

Suppose we have a Black-Scholes model for a continuously tradable stock and bond, which are represented by

\[ S_t = S_0 \exp(\mu t + \sigma W_t), \quad B_t = \exp(rt), \]

respectively. Then

(1) all integrable claims \( X_T \) have associated replicating strategies \((\phi_t, \psi_t)\).

(2) The arbitrage price of such a claim \( X \) is given by

\[ V_t = B_t E_Q(B_T^{-1} X_T | \mathcal{F}_t) = e^{-(T-t)} E_Q(X_T | \mathcal{F}_t), \]

where \( Q \) is the martingale measure for the discounted stock \( B_t^{-1} S_t \).
Martingale Approach – European Options

The claim $X_T$ is $(S_T - K)^+ := \max(S_T - K, 0)$. Then,

$$V_0 = e^{-rT}E_Q((S_T - K)^+).$$

**Question:** How to evaluate this function?

— We first find the marginal distribution of $S_T$ under $Q$?

$$d(\log S_t) = \sigma \, d\tilde{W}_t + (r - \frac{1}{2} \sigma^2) \, dt$$

$$\implies S_t = S_0 \exp(\sigma \tilde{W}_t + (r - \frac{1}{2} \sigma^2) t)$$
Martingale Approach

Question: How to evaluate this function?

\[ V_0 = e^{-rT} E_Q((S_T - K)^+). \]

Let \( Z \sim N\left(-\frac{1}{2}\sigma^2 T, \sigma^2 T\right), \) then \( S_T = S_0 \exp(Z + rT), \) and

\[
\begin{align*}
V_0 &= e^{-rT} E\left((S_0 e^{Z + rT} - K)^+\right) \\
&= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log(S_0/K) - rT}^{\infty} (S_0 e^x - ke^{-rT}) \exp\left(-\frac{(x + \frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\right) dx \\
&= \ldots \\
&= S_0 \Phi\left(\frac{\log\frac{S_0}{K} + (r + \frac{1}{2}\sigma^2 T)}{\sigma \sqrt{T}}\right) - Ke^{-rT} \Phi\left(\frac{\log\frac{S_0}{K} + (r - \frac{1}{2}\sigma^2 T)}{\sigma \sqrt{T}}\right).
\end{align*}
\]
Martingale Approach – Other Options

1. The put option has a payoff $X_T = \max(K - S, 0)$, hence its value at time 0 is

   $$-S\Phi(-d_1) + Ke^{-rT}\Phi(-d_2).$$

2. For a binary call with $Payoff(S) = H(S - K)$, where $H$ is the Heaviside function, the value of the option at time 0 is

   $$e^{-rT}\Phi(d_2).$$

3. For a binary put with $Payoff(S) = H(K - S)$, the option has a value of

   $$e^{-rT}(1 - \Phi(d_2)).$$
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Greeks

• The Black-Scholes assumptions
  – The underlying follows an exponential Brownian motion.
  – The risk-free interest rate is a known function of time.
  – There are no dividends on the underlying.
  – Delta hedging is done continuously.
  – There are no transaction costs on the underlying.
  – There are no arbitrage opportunities.
Greeks

• Delta

1. It is the sensitivity of the option to the underlying.
   \[ \Delta = \frac{\partial V}{\partial S}. \]

2. Delta hedging means holding one of the option and short a quantity \( \Delta \) of the underlying.

3. As delta changes with time, the number of assets held must be changed (dynamic hedging).

4. For European call, \( \Delta_c = \Phi(d_1) \). For European put, \( \Delta_p = \Phi(d_1) - 1 \).
Greeks

• Gamma

1. The gamma of an option is the second derivative of the value of the option to the underlying

\[ \Gamma = \frac{\partial^2 V}{\partial S^2}. \]

2. It is the sensitivity of the delta to the underlying, so it is a measure of by how much a position must be rehedged to maintain a delta-neutral position.

3. For European call or put options,

\[ \Gamma_c = \Gamma_p = \frac{\Phi'(d_1)}{\sigma S \sqrt{T - t}}, \quad \text{where} \quad \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \]
Greeks

• **Theta**

1. It is the rate of change of the option price with time.

   \[ \Theta = \frac{\partial V}{\partial t} \]

2. For European call and put options,

   \[ \Theta_c = -\frac{\sigma S N'(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2), \]

   \[ \Theta_p = -\frac{\sigma S N'(-d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2). \]
Greeks

• Vega

1. It is the sensitivity of the option price to the volatility.

2. Vega is a derivative with respect to a parameter instead of a variable.

\[
Vega = \frac{\partial V}{\partial \sigma}
\]

\[
Vega_c = Vega_p = S \sqrt{T - t} \Phi'(d_1)
\]
Greeks

• Rho

1. It is the sensitivity of the option value to the interest rate used in the Black-Scholes formulas,

\[ \rho = \frac{\partial V}{\partial r}. \]

2. For European call and put options,

\[ \rho_c = K(T-t)e^{-r(T-t)}\Phi(d_2), \quad \rho_p = -K(T-t)e^{-r(T-t)}\Phi(-d_2). \]
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American Options

1. Recall the value of an European call at time $t$ ($t \leq T$)

$$C_t = E_Q(e^{-r(T-t)}(S_T - K)^+).$$

2. The value of an American call at time $t$ is given by

$$V_t = \max_{\tau} E_Q(e^{-r(\tau-t)}(S_{\tau} - K)^+).$$

$\tau$ is called a stopping time, and we need to find an optimal stopping time $\tau^*$ that maximizes

$$E_Q(e^{-r(\tau-t)}(S_{\tau} - K)^+).$$