Appendix B: Interchanging Limits and Expectation

In this appendix, we discuss conditions that guarantee that the interchange of limit and expectation is valid. To see that some conditions are necessary, suppose that $N = (N(t) : t \geq 0)$ is a Poisson process having unit rate. Note that if $t$ is not a jump epoch of $N(\cdot)$, $N(t + h) = N(t)$ for $h$ sufficiently small and positive. It follows that

$$
\lim_{h \downarrow 0} \frac{N(t + h) - N(t)}{h} = 0 \quad \text{a.s.}
$$

On the other hand,

$$
\lim_{h \downarrow 0} E \left( \frac{N(t + h) - N(t)}{h} \right) = 1.
$$

So, the limit interchange fails.

One of the contributions of measure-theoretic probability is a set of sufficient conditions guaranteeing the validity of the limit interchange.

The Bounded Convergence Theorem (BCT): Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$ and that there exists $c \in \mathbb{R}$ such that

$$
|X_n(\omega)| \leq c < \infty
$$

for $n \geq 0$ and $\omega \in \Omega$. Then, $EX_n \to EX_\infty$ as $n \to \infty$.

The Dominated Convergence Theorem (DCT): Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$ and that there exists a rv $Y$ (called the “dominating rv”) such that $EY < \infty$ and

$$
|X_n(\omega)| \leq Y(\omega)
$$

for $n \geq 0$ and $\omega \in \Omega$. Then, $EX_n \to EX_\infty$ as $n \to \infty$.

The Monotone Convergence Theorem (MCT): Suppose that $(X_n : n \geq 0)$ is a sequence of non-negative rv’s for which $X_n(\omega) \leq X_{n+1}(\omega)$ for $n \geq 0$ and $\omega \in \Omega$. Then,

$$
X_\infty = \lim_{n \to \infty} X_n
$$

exists (and is possibly $\infty$-valued) and $EX_n \nearrow EX_\infty$ as $n \to \infty$.

Fatou’s Lemma: If $(X_n : n \geq 0)$ is a sequence of non-negative rv’s, then

$$
E \liminf_{n \to \infty} X_n \leq \liminf_{n \to \infty} EX_n.
$$

When the $X_n$’s are non-negative, a necessary and sufficient condition guaranteeing validity of the interchange can be identified.
Definition B.1 A sequence \((X_n : n \geq 0)\) is said to be *uniformly integrable* if, for each \(\epsilon > 0\), there exists \(c = c(\epsilon)\) such that
\[
\limsup_{n \to \infty} E|X_n|I(|X_n| \geq c) < \epsilon.
\]

Suppose that \((X_n : n \geq 0)\) is a sequence of non-negative rv’s for which \(X_n \to X_\infty\) a.s. as \(n \to \infty\), where \(EX_\infty < \infty\). Then, \(EX_n \to EX_\infty\) as \(n \to \infty\) if and only if \((X_n : n \geq 0)\) is uniformly integrable.

If \(X_n \Rightarrow X_\infty\) as \(n \to \infty\), we can often reduce the limit interchange problem to one involving almost sure convergence by invoking the following result.

Exercise B.1 Suppose that \(X_n \Rightarrow X_\infty\) as \(n \to \infty\). Prove that there exists a probability space supporting rv’s \((X'_n : 1 \leq n \leq \infty)\) satisfying:

- \(X_n \overset{D}{=} X'_n\) for \(1 \leq n \leq \infty\)
- \(X'_n \to X'_\infty\) a.s. as \(n \to \infty\)

(Hint: Consider \(X'_n = F_{X_n}^{-1}(U)\), where \(U\) is uniform on \([0, 1]\), where \(F_{X_n}^{-1}(x) = \sup\{z : X_n(z) \leq x\}\).)

Finally, note that the MCT implies that if the \(X_n\)’s are non-negative rv’s then
\[
E \sum_{n=1}^\infty X_n = \sum_{n=1}^\infty EX_n.
\]

This is one version of *Fubini’s theorem*. Other variants:

- If \((X_n : n \geq 1)\) is a sequence of rv’s for which
\[
E \sum_{n=1}^\infty |X_n| = \sum_{n=1}^\infty E|X_n| < \infty,
\]
then
\[
E \sum_{n=1}^\infty X_n = \sum_{n=1}^\infty EX_n.
\]

- If \((X(t) : t \geq 0)\) is a non-negative process, then
\[
E \int_0^\infty X(t)dt = \int_0^\infty EX(t)dt.
\]

- If \((X(t) : t \geq 0)\) is a process for which
\[
E \int_0^\infty |X(t)|dt = \int_0^\infty E|X(t)|dt < \infty,
\]
then
\[
E \int_0^\infty X(t)dt = \int_0^\infty EX(t)dt.
\]