1 Semi-Lagrangian Advection (cont.)

In the previous lecture we described semi-Lagrangian advection (method of characteristics) for solving the advection equation

$$\rho_t + \vec{V} \cdot \nabla \rho = 0$$

where the numerical scheme was given by

$$\rho^{n+1}(\vec{x}) = \rho^n(\vec{x} - \vec{V} \triangle t)$$

As we noted previously the point $\vec{x} - \vec{V} \triangle t$ is generally not a grid point. To get a value for $\rho$ at that point, we use linear interpolation in 1D (bilinear interpolation in 2D, or trilinear interpolation in 3D). Let us look at the interpolation in detail in 1D. Assume that the point $x_j - u \triangle t$ lies between grid points $x_i$ and $x_{i+1}$, and that its distance from grid point $x_i$ is $s$. 

![Diagram showing interpolation between grid points](image-url)
Therefore its distance from grid point \( x_{i+1} \) is \( \Delta x - s \). Then linear interpolation gives

\[
\rho_{j}^{n+1} = \rho_{i}^{n} + \frac{s}{\Delta x} (\rho_{i+1}^{n} - \rho_{i}^{n}) \\
= \frac{\Delta x \rho_{i}^{n} + s \rho_{i+1}^{n} - s \rho_{i}^{n}}{\Delta x} \\
= \frac{(\Delta x - s) \rho_{i}^{n} + s \rho_{i+1}^{n}}{\Delta x}
\]

or

\[
\rho_{j}^{n+1} = \left(1 - \frac{s}{\Delta x}\right) \rho_{i}^{n} + \frac{s}{\Delta x} \rho_{i+1}^{n}
\]

Since \( 1 - \frac{s}{\Delta x} \leq 1 \) and \( \frac{s}{\Delta x} \leq 1 \), we have that

\[
\min(\rho_{i}^{n}, \rho_{i+1}^{n}) \leq \rho_{j}^{n+1} \leq \max(\rho_{i}^{n}, \rho_{i+1}^{n})
\]

This fact is important for unconditional stability. For example, it immediately implies that the method is unconditionally stable in the max norm.

### 1.1 Semi-Lagrangian Velocity Advection

Recall that for incompressible flow, we have the momentum equation

\[
\vec{v}_{t} + \vec{V} \cdot \nabla \vec{v} + \frac{\nabla p}{\rho} = \vec{g}
\]

which in 2D is

\[
\begin{align*}
\frac{u}{\Delta t} + \vec{V} \cdot \nabla u + \frac{p_{x}}{\rho} & = 0 \\
\frac{v}{\Delta t} + \vec{V} \cdot \nabla v + \frac{p_{y}}{\rho} & = g
\end{align*}
\]

Using the projection method the steps in the numerical solution are

1. \[
\begin{align*}
\frac{u^{n+1} - u^{n}}{\Delta t} + \vec{V}^{n} \cdot \nabla u^{n} & = 0 \\
\frac{v^{n+1} - v^{n}}{\Delta t} + \vec{V}^{n} \cdot \nabla v^{n} & = g
\end{align*}
\]

2. \[
\nabla \cdot \left( \frac{1}{\rho} \nabla p \right) = \frac{\nabla \cdot \vec{V}^{*}}{\Delta t}
\]

3. \[
\begin{align*}
\frac{u^{n+1} - u^{*}}{\Delta t} + \frac{p_{x}}{\rho} & = 0 \\
\frac{v^{n+1} - v^{*}}{\Delta t} + \frac{p_{y}}{\rho} & = 0
\end{align*}
\]
Only step 1 has a CFL condition for the hyperbolic terms. The other two steps do not have a time step restriction for stability. Therefore, if we use the semi-Lagrangian method for the velocity advection in step 1 we can eliminate the time step restriction. For \( u^* \), the method is

\[
u_j^* = u^n \left( \mathbf{x}_j - \mathbf{V}_j^n \Delta t \right)
\]

For \( v^* \) we must also account for gravity, so we have

\[
v_j^* = v^n \left( \mathbf{x}_j - \mathbf{V}_j^n \Delta t \right)
v_j^* + = \Delta t g
\]

where we are computing

1) \( v_t + \mathbf{V} \cdot \nabla v = 0 \)
2) \( v_t = g \)

This is a Godunov splitting, which is first order accurate.

## 2 Vorticity

Here we describe a method to counteract the numerical dissipation that damps out many interesting features in the flow.

Taking the curl of the momentum equation (1) gives

\[
\mathbf{\omega}_t + \mathbf{V} \cdot \nabla \mathbf{\omega} + \ldots = 0
\]

where

\[
\mathbf{\omega} = \nabla \times \mathbf{V}.
\]

In 2D,

\[
\mathbf{\omega} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
u & v & 0
\end{vmatrix} = \left( -\frac{\partial}{\partial z} v, \frac{\partial}{\partial z} u, \frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u \right)
\]

Since

\[
\frac{\partial}{\partial z} u = \frac{\partial}{\partial z} v = 0
\]

we have

\[
\mathbf{\omega} = (0, 0, v_x - u_y)
\]

So this is particularly nice in 2D as we get one scalar equation for \( \omega \) (in 3D we still get a 3-vector). Since \( \omega \) will be either positive or negative, the vorticity vector \( \mathbf{\omega} \) is pointing either into or out of the \( x - y \) plane. Vorticity can be thought of as a paddle wheel which is trying to spin the flow. The direction of the spinning depends on the sign of \( \omega \).

Some points of interest regarding vorticity are
• Vorticity is conserved.
• Vorticity stays confined in high Reynolds number flows.

Here we discuss a simple turbulence model due to Steinhoff. First we compute vorticity location vectors

$$\vec{N} = \frac{\nabla |\vec{\omega}|}{|\nabla |\vec{\omega}||}.$$ 

Then we compute the paddle wheel force as

$$\vec{F} = \vec{N} \times \vec{\omega}.$$ 

Steinhoff’s idea was to add a forcing term to the momentum equations

$$\vec{V}_t + \vec{V} \cdot \nabla \vec{V} + \frac{\nabla p}{\rho} = \vec{g} + \epsilon \Delta x \vec{F}.$$ 

It is interesting to note that if you linearize the forcing term, it looks like $-\Delta \vec{V}$. 

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