On the von Neumann Economic Growth Problem

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Abstract: We study the complexity of the von Neumann economic growth problem:

$$\gamma^* := \max\{\gamma \mid \exists\, y \neq 0: \quad y \geq 0, \quad (B - \gamma A)y \geq 0\},$$

where A and B are given two nonnegative and rational $m \times n$-matrices, and A has no all-zero column. Let the binary data length of A and B be L. In this paper we develop an interior-point algorithm to generate a $\bar{\gamma}$, such that $\gamma^* - 2^{-t} \leq \bar{\gamma} \leq \gamma^*$, in $O((m+n)(L + \min(m,n)t))$ iterations where each iteration solves a system of $(m+n)$ linear equations.

Key words: von Neumann's economic growth model, linear inequality, analytic center, max-potential, interior-point algorithms

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1 Introduction

Consider the von Neumann economic growth (NEG) problem:

$$\gamma^* := \max\{\gamma \mid \exists\, y \neq 0 : \quad y \geq 0, \quad (B - \gamma A)y \geq 0\},$$

where $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ are two given nonnegative and rational $m \times n$-matrices, and

(A1) A has no all-zero column.

Note that γ^* is finite based on this assumption. In fact,

$$\gamma^* \leq \max_j \frac{\sum_{i=1}^m b_{ij}}{\sum_{i=1}^m a_{ij}}.$$

A recent description of the problem can be found in Schrijver [14] and Tan and Freund [16] (also see Gale [2], Kemeny et al. [5], and Robinson [12]).

The NEG problem has an economic interpretation. Each row-index i stands for a 'good', while each column index j stands for a 'process'. Process j can convert a_{ij} units of good i, in one time period, into b_{ij} units of good i. So a process uses goods as materials or inputs, and gives goods as products or outputs. Matrix B is referred as output matrix, and A is the input matrix. Component y_j of y denotes the 'intensity' by which we let process j work. Vector By gives the amounts of outputs produced, and Ay gives the amounts of inputs consumed, during one time period. Then, γ represents the growth factor at intensity y. So $By \geq \gamma Ay$ requires that, for each good i, the amount of good i produced in period t is at least the amount of good i required in period $(t+1)$ with the growth factor γ. The NEG problem is to find the largest growth factor using an optimal intensity vector.

There is a related dual NEG problem,

$$\eta^* := \min\{\eta \mid \exists\, x \neq 0 : \quad x \geq 0, \quad (\eta A - B)^T x \geq 0\}.$$

We assume that

(A2) B has no all-zero row.

Then, η^* is also finite. But a duality overlap may exist, i.e., it is possible $\eta^* \leq \gamma^*$. However, under the following irreducibility assumption:

(A3) There is no (proper) subset S of the rows and no subset T of the columns such that $A_{ij} = 0$ for all $i \in S$ and all $j \in T$, and such that for all $i \in \{1, 2, ..., m\} \setminus S$, $B_{ij} > 0$ for some $j \in T$ (see Gale [2]), then the model is well behaved: $\eta^* = \gamma^*$. Moreover, the γ-level set,

$$\Gamma(\gamma) := \{y \in R^n : \ e^T y = 1, \quad y \geq 0, \quad (B - \gamma A)y \geq 0\}, \tag{1}$$

has a nonempty (relative) interior for $\gamma < \gamma^*$, meaning in this paper that $\{y \in R^n : \ e^T y = 1, \quad y > 0, \quad (B - \gamma A)y > 0\}$ is nonempty for $\gamma < \gamma^*$. Note that we have replaced $y \neq 0$ with $e^T y = 1$ in the NEG problem, where e is the vector of all ones. This is without loss of generality since the system is homogeneous in y.

Obviously, the underlying decision problem related to the NEG problem can be solved in polynomial time: Given two rational matrices A and B, and a rational number γ, does the linear system

$$\{e^T y = 1, \quad y \geq 0, \quad (B - \gamma A)y \geq 0\}$$

has a feasible point? With using an artificial variable vector z, the linear system can be rewritten as

$$By - \gamma z \geq 0, \quad Ay - z = 0, \quad e^T y = 1, \quad y \geq 0 \tag{2}$$

or

$$By - Az \geq 0, \quad \gamma y - z = 0, \quad e^T y = 1, \quad y \geq 0 \tag{3}$$

Let $0 \leq \gamma^* \leq 2^t$ for some positive integer t. Then, one can use the *bisection* method to generate a $\bar{\gamma}$ such that $\gamma^* - 2^{-t} \leq \bar{\gamma} \leq \gamma^*$ in $O(t)$ bisection steps where each step solves linear feasibility problem (2) or (3) with data A, B and $\bar{\gamma}$. Let the data size of A and B be L. Since the size of $\bar{\gamma}$ is $O(t)$, the data size of the linear feasibility problem can be $O(L + \min(m, n)t)$. Thus, the overall worst-case complexity to generate a 2^{-t} approximate $\bar{\gamma}$ will be $O(t(m + n)^3(L + \min(m, n)t))$ arithmetic operations, or $O(t\sqrt{(m + n)}(L + \min(m, n)t))$ interior-point-algorithm iterations and each iteration solves a system of $(m + n)$ linear equations. (The currently best complexity of linear programming can be found in Gonzaga [3] and Renegar [11] among many others.) Therefore, the NEG problem is polynomially solvable. We are not aware of any other polynomial complexity result for this problem.

The use of interior-point algorithms to directly solve the NEG problem was considered by Tan and Freund [16], who developed a Newton-type method, based on Renegar's algorithm, for a general parametric center problem. Unfortunately, the worst-case complexity of their algorithm, when applied to the NEG problem, will exhibit exponential time bound in L and t. In this paper, we propose a similar interior-point algorithm to solve the NEG problem. The main difference between our algorithm and Renegar, Tan and Freund's is the measure of the iterative progress. They have measured the progress of γ, but we measure an "analytic volume" of the γ-level set $\Gamma(\gamma)$ of (1) as we did in a column generation algorithm [19]. We show that our algorithm generates a 2^{-t} approximate solution in $O((m+n)(L+\min(m,n)t))$ iterations where each iteration solves a system of $(m+n)$ linear equations.

2 Max-Potential of an Inequality System

Consider a bounded linear inequality system

$$\Gamma = \{y \in \mathcal{S} \subset R^n : c - Hy \geq 0\},$$

where $c \in R^m$ and $H \in R^{m\times n}$, and $\mathcal{S}$ is an affine subset of R^n. In this paper, we say Γ has a nonempty (relative) interior if and only if $\exists\ y \in \mathcal{S}$ such that $c - Hy > 0$. In this case, denote by y^a the analytic center of Γ (e.g., Megiddo [7], Renegar [11], and Sonnevend [15]). Then, define the max-potential, an "analytic volume", of Γ as the potential function value at y^a (Burke et al. [1] and Ye [19]), which is the logarithmic product of the slacks $s^a := c - Hy^a$, i.e.,

$$P(c,H) = \sum_{i=1}^{m} \log s_i^a = \sum_{i=1}^{m} \log(c_i - h_i y^a) = \max_{y\in\Gamma} \sum_{i=1}^{m} \log(c_i - h_i y). \qquad (4)$$

where h_i is the ith row of H. In fact, $P(c,H)$ is related to the logarithmic volume of the largest ellipsoid inscribing Γ in terms of the slacks of the inequality system, see Todd [17] and Ye [20]. We now apply the analytic center and the max-potential to the inequality system $\Gamma(\gamma)$ of (1) for a fixed $\gamma < \gamma^*$. Clearly, since B is a nonnegative matrix, we must have $\gamma^* \geq 0$. Without loss of generality, we further assume that $\Gamma(0)$ of (1) has a nonempty (relative) interior. This fact automatically holds if $(A2)$ is assumed.

We need to require that the system $\Gamma(\gamma)$ has a bounded and nonempty (relative) interior for all $\gamma < \gamma^*$, so that the analytic center and the max-potential are well defined for all $\gamma < \gamma^*$. As we discussed early, this is true under assumption $(A1)$ and $(A3)$. In what follows, we replace $(A3)$ by another assumption

(A4) There exists an optimal intensity vector $y^* \in \Gamma(\gamma^*)$ such that

$$(B + A)y^* > 0.$$

Lemma 1 . *Let $0 \leq \gamma < \gamma^*$. Then, the system $\Gamma(\gamma)$ under assumptions (A1), (A2) and (A4) has a bounded and nonempty (relative) interior.*

Proof. $\Gamma(\gamma)$ being bounded is obvious since $e^T y = 1$ and $y \geq 0$. Let y^* be in $\Gamma(\gamma^*)$, i.e.,

$$e^T y^* = 1, \quad y^* \geq 0, \quad (B - \gamma^* A)y^* \geq 0, \tag{5}$$

and it has

$$(B + A)y^* > 0.$$

Let

$$\delta = \frac{\gamma(1 + \gamma^*)}{1 + \gamma}.$$

Then, for $0 \leq \gamma < \gamma^*$ we must have

$$\gamma \leq \delta < \gamma^*.$$

The left inequality follows from $1 + \gamma^* > 1 + \gamma$, and the right inequality follows from $\gamma < \gamma^*$ which implies

$$\gamma(1 + \gamma^*) < \gamma^*(1 + \gamma)$$

which further implies

$$\gamma^* > \frac{\gamma(1 + \gamma^*)}{1 + \gamma} = \delta.$$

Since $(B+A)y^* > 0$, we have

$$(\gamma^* - \delta)(B+A)y^* > 0. \tag{6}$$

Adding two inequalities in (5) and (6), we have

$$((1+\gamma^* - \delta)B - \delta A)y^* > 0,$$

or

$$(B - \frac{\delta}{1+\gamma^*-\delta}A)y^* > 0,$$

which implies

$$(B - \gamma A)y^* > 0,$$

since

$$\gamma = \frac{\delta}{1+\gamma^*-\delta}.$$

Therefore, there is an $0 < \epsilon < 1$ such that

$$\bar{y} = (1-\epsilon)y^* + \epsilon e/n > 0$$

and

$$(B - \gamma A)\bar{y} > 0.$$

That is, $\bar{y}$ is in the (relative) interior of $\Gamma(\gamma)$.

$\square$

Note that the above lemma may not be true in general. For example, let

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for this problem $\gamma^* = 2$. However, for any $1 < \gamma < 2$, it must be true that $(y_1 = 1,\ y_2 = 0)$ is on the boundary of $\Gamma(0)$.

It is easy to show that $(A3)$ implies $(A4)$, see Theorem 2 of Robinson [12]. However, $(A4)$ is weaker than $(A3)$. Consider

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This system is reducible but $\Gamma(\gamma)$ has nonempty (relative) interior for all $\gamma < \gamma^* = 1$, and it satisfies $(A4)$.

There is also an economic interpretation for assumption $(A4)$: if $\Gamma(\gamma)$ has an empty (relative) interior for some $\gamma < \gamma^*$, then at every optimal intensity vector y^*

$$(B + A)y^* \not> 0,$$

which implies that for some good i

$$b_i y^* = a_i y^* = 0$$

at every optimal intensity vector y^*, where b_i and a_i are the ith row of the matrices B and A, respectively. Thus, the ith good is irrelevant, as it is neither produced nor consumed, and so can be removed from further consideration. Therefore, we can set up a reduced NEG problem (both row and column dimensions may be reduced) like

$$\max\{\gamma \mid \exists\, y \neq 0 : \quad y \geq 0, \quad (B_2 - \gamma A_2)y \geq 0, \quad (b_i + a_i)y = 0\}$$

where B_2 and A_2 are the remaining matrices of B and A after deleting b_i and a_i, respectively. Now the reduced γ-level set will have a nonempty (relative) interior for any $\gamma < \gamma^*$. We will discuss more for this case in the last section.

Based on Lemma 1, for every $\gamma < \gamma^*$ we can define the max-potential of $\Gamma(\gamma)$ of (1) as

$$P(\gamma) = \max_{y \in \Gamma(\gamma)} \left(\sum_{i=1}^{m} \log(By - \gamma Ay)_i + \sum_{i=1}^{n} \log y_i \right).$$

In the following, we frequently use the notation $s := By - \gamma Ay$.

We now prove two more lemmas to indicate that the max-potential of $\Gamma(\gamma)$ is an effective *measure* for the γ-level set of the NEG problem.

Lemma 2 . *Let $\gamma^0 < \gamma^1 < \gamma^*$. Then,*

$$P(\gamma^0) > P(\gamma^1).$$

Proof. Let y^1 be the analytic center of $\Gamma(\gamma^1)$. Then, since

$$(B - \gamma^0 A)y^1 = (B - \gamma^1 A)y^1 + (\gamma^1 - \gamma^0)Ay^1$$

and $(\gamma^1 - \gamma^0)Ay^1 \geq 0$, we must have

$$(B - \gamma^0 A)y^1 \geq (B - \gamma^1 A)y^1 > 0.$$

This shows that y^1 is in the (relative) interior of $\Gamma(\gamma^0)$. Moreover, $(\gamma^1 - \gamma^0)Ay^1 \neq 0$, since A has no all-zero columns and $y^1 > 0$. Thus,

$$\begin{aligned}
P(\gamma^0) &\geq \sum_{i=1}^{m} \log(By^1 - \gamma^0 Ay^1)_i + \sum_{i=1}^{n} \log y_i^1 \\
&= \sum_{i=1}^{m} \log(By^1 - \gamma^1 Ay^1 + (\gamma^1 - \gamma^0)Ay^1)_i + \sum_{i=1}^{n} \log y_i^1 \\
&> \sum_{i=1}^{m} \log(By^1 - \gamma^1 Ay^1)_i + \sum_{i=1}^{n} \log y_i^1 \\
&= P(\gamma^1).
\end{aligned}$$

□

Lemma 3 . *Let $\gamma^* \leq 2^t$ for some positive integer t, and let γ be a (real) number such that $\gamma \leq \gamma^* - 2^{-t}$. Then, the max-potential of the system $\Gamma(\gamma)$ is greater than $-O((m+n)(L + \min(m,n)t))$, i.e.,*

$$P(\gamma) \geq -O((m+n)(L + \min(m,n)t)).$$

Proof. Obviously, γ^* is in the interval $(k/2^t, (k+1)/2^t]$ for some integer $k \leq 2^{2t}$. Let $\bar{\gamma} = k/2^t$. Then $\bar{\gamma}$ is a rational number and its size is $O(t)$. Since $\bar{\gamma}$ is strictly less than γ^*, the system $\Gamma(\bar{\gamma})$ of (1) has a bounded and nonempty (relative) interior. Note that the size of B, A and $\bar{\gamma}I$ is $O(L + \min(m,n)t)$ in the linear systems (2) or (3). Thus, there must be an interior-point $\bar{y} \in \Gamma(\bar{\gamma})$ such that

$$\bar{y} \geq 2^{-O(L+\min(m,n)t)}e \quad \text{and} \quad (B - \bar{\gamma}A)\bar{y} \geq 2^{-O(L+\min(m,n)t)}e.$$

The point $\bar{y}$ can be generated as the convex combination of $(m+n)$ *basic feasible* points $\bar{y}^i$'s in $\Gamma(\bar{\gamma})$, where the ith slack of $\Gamma(\bar{\gamma})$ is greater than $2^{-O(L+\min(m,n)t)}$ at $\bar{y}^i$ (see Schrijver [14]), for example, let

$$\bar{y} = \sum_{i=1}^{n+m} \frac{1}{m+n} \bar{y}^i.$$

Thus, the max-potential of $\Gamma(\bar{\gamma})$

$$P(\bar{\gamma}) \geq -O((m+n)(L+\min(m,n)t)).$$

Furthermore, since $\gamma \leq \bar{\gamma}$, from Lemma 2 we have

$$P(\gamma) \geq P(\bar{\gamma}) \geq -O((m+n)(L+\min(m,n)t)).$$

$\square$

Lemma 3 indicates that if we have

$$P(\gamma) \leq -O((m+n)(L+\min(m,n)t)),$$

then $\gamma^* - 2^{-t} \leq \gamma \leq \gamma^*$. In what follows, we develop an algorithm to reduce the max-potential of $\Gamma(\gamma)$ as γ is increased.

3 Technical Results

Most of technical results presented in this section are related to those in Burke et al. [1] and Ye [19]. Let $0 \leq \gamma^0 < \gamma^*$. Then, the analytic center y^0 of $\Gamma(\gamma^0)$, or simply Γ^0, satisfies the following conditions.

$$\begin{pmatrix} X^0 s^0 \\ Z^0 y^0 \end{pmatrix} = \begin{pmatrix} e \\ e \end{pmatrix}, \tag{7}$$

where the diagonal matrices $X = \mathrm{Diag}(x)$ and $Z = \mathrm{Diag}(z)$, (s^0, y^0) is feasible with $\gamma = \gamma^0$ for the system

$$s - (B - \gamma A)y = 0, \quad e^T y = 1, \quad (s, y) \geq 0, \tag{8}$$

and (x^0, z^0) is feasible with $\gamma = \gamma^0$ for the system

$$(B - \gamma A)^T x + z = (m+n)e, \quad (x, z) \geq 0. \tag{9}$$

Now let y^1 be the analytic center of $\Gamma^1 = \Gamma(\gamma^1)$ with $\gamma^1 = \gamma^0 + \Delta\gamma$. For a suitable choice of $\Delta\gamma$, we show that $\gamma^0 < \gamma^1 < \gamma^*$ and

$$P(\gamma^1) \leq P(\gamma^0) - \Omega(1).$$

We first prove a lemma on how to select $\Delta\gamma$. In what follows, $\|.\|$ denotes the standard ℓ_2 norm.

Lemma 4 . *Let*

$$\Delta\gamma = \beta \min\{\frac{1}{\|X^0AY^0e\|}, \frac{1}{\|Y^0A^TX^0e\|}\}.$$

for some constant $0 < \beta < 1$ *and let* $\gamma^1 = \gamma^0 + \Delta\gamma$. *Furthermore, let*

$$\bar{s}^0 = s^0 - \Delta\gamma A y^0 \quad \textit{and} \quad \bar{y}^0 = y^0,$$

and

$$\bar{x}^0 = x^0 \quad \textit{and} \quad \bar{z}^0 = z^0 + \Delta\gamma A^T x^0.$$

Then, $(\bar{s}^0, \bar{y}^0)$ *and* $(\bar{x}^0, \bar{z}^0)$ *are feasible for systems (8) and (9) with* $\gamma = \gamma^1$*, respectively. Moreover,*

$$\left\| \begin{pmatrix} \bar{X}^0\bar{s}^0 - e \\ \bar{Z}^0\bar{y}^0 - e \end{pmatrix} \right\| \leq \sqrt{2}\beta,$$

and

$$(\bar{s}^0, \bar{y}^0) > 0 \quad \textit{and} \quad (\bar{x}^0, \bar{z}^0) > 0.$$

Proof. The two equations in systems (8) and (9) for $\gamma = \gamma^1$ can easily be verified. The inequality for the norm can be proved from relation (7) and the choice of $\Delta\gamma$.

$$\begin{aligned} \|\bar{X}^0\bar{s}^0 - e\|^2 &= \|X^0(s^0 - \Delta\gamma A y^0) - e\|^2 \\ &= \|\Delta\gamma X^0 A Y^0 e\|^2 \\ &\leq \beta^2 \end{aligned}$$

Similarly,

$$\begin{aligned} \|\bar{Z}^0\bar{y}^0 - e\|^2 &= \|Y^0(z^0 + \Delta\gamma A^T x^0) - e\|^2 \\ &= \|\Delta\gamma Y^0 A^T X^0 e\|^2 \\ &\leq \beta^2 \end{aligned}$$

These relations also imply that

$$\bar{s}^0 > 0 \quad \text{and} \quad \bar{z}^0 > 0,$$

since

$$\bar{x}^0 = x^0 > 0 \quad \text{and} \quad \bar{y}^0 = y^0 > 0.$$

$\square$

In the next section, we show that $(\bar{x}^0, \bar{z}^0)$ and $(\bar{s}^0, \bar{y}^0)$ can be used as an initial pair to generate the new center pair (x^1, z^1) and (s^1, y^1) where y^1 is the analytic center for the system $\Gamma^1 = \Gamma(\gamma^1)$. In fact, we will show that we are able to generate the new center pair using Newton's method with a quadratic convergence rate. We will also show that the new center is relatively close to the old center. More precisely, the following relations hold

$$\|(Y^1)^{-1}Y^0\| \leq O(1) \quad \text{and} \quad \|(Y^0)^{-1}Y^1\| \leq O(1). \tag{10}$$

We now state a technical fact.

Proposition 5 . *Let $H = \{h_{ij}\}$ be a nonnegative $m \times n$-matrix. Then*

$$\|He\| \leq e^T He,$$

and

$$\|H^T e\| \leq e^T H^T e = e^T He$$

Proof. This directly from the relation between ℓ_1 and ℓ_2 norms:

$$\|He\| = \|He\|_2 \leq \|He\|_1 = e^T He.$$

The second inequality can be proved similarly.

$\square$

We now present a max-potential reduction theorem.

Theorem 6 . *Let $\Delta\gamma$ and γ^1 be selected as in Lemma 4. Assume that relation (10) holds. Then,*

$$\gamma^0 < \gamma^1 < \gamma^*,$$

and

$$P(\gamma^1) \le P(\gamma^0) - \Omega(1).$$

Proof. $\gamma^1 > \gamma^0$ because $\Delta\gamma > 0$ and $\gamma^1 < \gamma^*$ because $\Gamma^1 = \Gamma(\gamma^1)$ has a nonempty (relative) interior. Now we prove the potential-reduction inequality. First, note that the old center pair (s^0, y^0) and (x^0, z^0) is feasible for systems (8) and (9) with $\gamma = \gamma^0$, and the new center pair (s^1, y^1) and (x^1, z^1) is feasible for systems (8) and (9) with $\gamma = \gamma^1$. Thus,

$$\begin{aligned}
(x^0)^T s^1 + (z^0)^T y^1 &= (x^0)^T (B - \gamma^1 A) y^1 + (z^0)^T y^1 \\
&= (y^1)^T ((B - \gamma^0 A)^T x^0 + z^0) - \Delta\gamma (x^0)^T A y^1 \\
&= (m+n) e^T y^1 - \Delta\gamma (x^0)^T A y^1 \\
&= (m+n) - \Delta\gamma (x^0)^T A y^1.
\end{aligned}$$

Thus, from relation (7)

$$\begin{aligned}
\frac{\exp(P(\Gamma^1))}{\exp(P(\Gamma^0))} &= \prod_{i=1}^m \frac{s_i^1}{s_i^0} \prod_{i=1}^n \frac{y_i^1}{y_i^0} \\
&= \prod_{i=1}^m (x_i^0 s_i^1) \prod_{i=1}^n (z_i^0 y_i^1) \\
&\le \left(\frac{(x^0)^T s^1 + (z^0)^T y^1}{m+n} \right)^{m+n} \\
&= \left(1 - \frac{\Delta\gamma (x^0)^T A y^1}{m+n} \right)^{m+n} \\
&\le \exp(-\Delta\gamma (x^0)^T A y^1).
\end{aligned}$$

Moreover, since $(x^0)^T A y^1 > 0$, due to Lemma 5 and relation (10), we have

$$\begin{aligned}
\Delta\gamma (x^0)^T A y^1 &= \Delta\gamma e^T X^0 A Y^1 e \\
&= \beta \min\{\frac{1}{\|X^0 A y^0\|}, \frac{1}{\|Y^0 A^T x^0\|}\} e^T X^0 A Y^1 e \\
&\ge \beta \frac{e^T X^0 A Y^1 e}{e^T X^0 A Y^0 e} \\
&\ge \Omega(1).
\end{aligned}$$

The last inequality follows from $X^0 A \geq 0$ and

$$\begin{aligned} e^T X^0 A Y^1 e &= \sum_i \sum_j a_{ij} x_i^0 y_j^1 \\ &= \sum_i \sum_j a_{ij} x_i^0 y_j^0 \frac{y_j^1}{y_j^0} \\ &\geq \sum_i \sum_j a_{ij} x_i^0 y_j^0 \Omega(1) \\ &= \Omega(1) e^T X^0 A Y^0 e. \end{aligned}$$

□

4 Conceptual Algorithm Using Perfect Centers

We now present a conceptual algorithm, which uses the perfect center, to illustrate the basic idea of our approach.

Algorithm 1.

Step 0. Let $\gamma^0 = 0$. Then generate the analytic center y^0 of $\Gamma^0 = \Gamma(\gamma^0)$. Set $k := 0$.

Step 1. Let

$$\Delta\gamma^k = \beta \min\{\frac{1}{\|X^k A Y^k e\|}, \frac{1}{\|Y^k A^T X^k e\|}\}$$

for some constant $0 < \beta < 1$ and let $\gamma^{k+1} = \gamma^k + \Delta\gamma^k$. Then use Newton's method, which is described below, to generate the analytic center y^{k+1} of $\Gamma^{k+1} = \Gamma(\gamma^{k+1})$.

Step 2. If $P(\gamma^{k+1}) > -O((m+n)(L + \min(m,n)t))$, then set $k := k+1$ and return to Step 1.

□

We now discuss how to use Newton's method to accomplish Step 1 of Algorithm 1. For simplicity, let $k = 0$. We wish to generate the new center pair (s^1, y^1) and (x^1, z^1) that satisfies systems (8) and (9) with $\gamma = \gamma^1$, and satisfies

$$\begin{pmatrix} X^1 s^1 \\ Z^1 y^1 \end{pmatrix} = \begin{pmatrix} e \\ e \end{pmatrix}.$$

We use a Newton procedure, which was originally used by Kojima et al. [6] and Monteiro and Adler [9].

Newton Procedure.

Step 0. Let $(\bar{x}^0, \bar{z}^0)$ and $(\bar{s}^0, \bar{y}^0)$ be defined as in Lemma 4. Set $l = 0$.

Step 1. Solve for $(\Delta s, \Delta y)$ and $(\Delta x, \Delta z)$:

$$\begin{array}{rcl} \bar{S}^l \Delta x + \bar{X}^l \Delta s &=& e - \bar{X}^l \bar{s}^l, \\ \bar{Y}^l \Delta z + \bar{Z}^l \Delta y &=& e - \bar{Y}^l \bar{z}^l, \\ (B - \gamma^1 A)^T \Delta x + \Delta z &=& 0, \\ \Delta s - (B - \gamma^1 A) \Delta y &=& 0. \end{array} \tag{11}$$

Let

$$\bar{x}^{l+1} = \bar{x}^l + \Delta x \quad \text{and} \quad \bar{z}^{l+1} = \bar{z}^l + \Delta z,$$

and

$$\bar{s}^{l+1} = \bar{s}^l + \Delta s \quad \text{and} \quad \bar{y}^{l+1} = \bar{y}^l + \Delta y.$$

Step 2. Set $l := l + 1$ and return to Step 1.

$\square$

We have the following quadratic convergence result.

Lemma 7 . *In the Newton procedure, let*

$$\left\| \begin{pmatrix} \bar{X}^l \bar{s}^l - e \\ \bar{Z}^l \bar{y}^l - e \end{pmatrix} \right\| \le \beta^l < 1.$$

Then,

$$\left\| \begin{pmatrix} \bar{X}^{l+1} \bar{s}^{l+1} - e \\ \bar{Z}^{l+1} \bar{y}^{l+1} - e \end{pmatrix} \right\| \le \frac{\sqrt{2}}{4} \frac{(\beta^l)^2}{1 - \beta^l},$$

and

$$\|(\bar{Y}^l)^{-1} (\bar{y}^{l+1} - \bar{y}^l)\| \le \frac{\beta^l}{1 - \beta^l}.$$

Proof. This is a standard lemma in the primal-dual interior-point algorithms. A proof can be found in Mizuno et al. [8]. Also see Kojima et al. [6] and Roos and Vial [13].

$\square$

For a suitable choice of β, for example $\beta = \sqrt{2}/3$, one can show that relation (10) holds (e.g., see [3] and [13]), i.e.,

$$\|(Y^1)^{-1}Y^0\| \leq O(1) \quad \text{and} \quad \|(Y^0)^{-1}Y^1\| \leq O(1),$$

where y^1 is the analytic center of $\Gamma^1 = \Gamma(\gamma^1)$. Furthermore, one can verify that the initial max-potential of $\Gamma^0 = \Gamma(0)$ is bounded above by $O(mL)$, because

$$0 < y^0 \leq e \quad \text{and} \quad 0 < s^0 = By^0 \leq 2^L e.$$

Therefore, based on Lemma 3 and Theorem 6 we have the following iteration-complexity result: Algorithm 1, with a suitable choice of β, terminates in $k = O((m+n)(L+\min(m,n)t))$ steps, and the resulting γ^k satisfies $\gamma^* - 2^{-t} \leq \gamma^k < \gamma^*$.

5 Algorithm Using Approximate Centers

In this section, we show that the use of the perfect center in Algorithm 1 is unnecessary. Similar issues have been discussed by Gonzaga [3], Renegar [11] and Sonnevend [15] in a path-following algorithm for linear programming. We now use an approximate center instead of the perfect center in Algorithm 1. We first prove the following lemma to show that the potential function value at a δ-approximate center suffices to terminate the algorithm.

Lemma 8 . *Let $\Gamma(\gamma^k)$ be bounded and have nonempty (relative) interior, and let an interior-point pair (x^k, z^k) and (s^k, y^k) satisfy systems (8) and (9) with $\gamma = \gamma^k$, and*

$$\left\| \begin{pmatrix} X^k s^k - e \\ Z^k y^k - e \end{pmatrix} \right\| \leq \delta < 1.$$

Then,

$$P(\gamma^k) \geq \phi(s^k, y^k) := (\sum_{i=1}^{m} \log s_i^k + \sum_{i=1}^{m} \log y_i^k) \geq P(\gamma^k) - \frac{\delta^2}{2(1-\delta)}.$$

Proof. The proof can be found in Lemma 2 of Ye [19], where we used a standard technical result (e.g., see Gonzaga [3]): Given $c \in R^n$, $c > 0$, $e^T c = n$ and $\|c - e\| \le \delta < 1$. Then,

$$\sum_{j=1}^{n} \log c_j \ge -\frac{\delta^2}{2(1-\delta)}. \tag{12}$$

$\square$

The above lemma indicates that the potential function value at a δ-approximate center y^k of $\Gamma^k = \Gamma(\gamma^k)$ differs from the max-potential by a constant when constant $0 < \delta < 1$. Therefore, the potential function value at the approximate center y^k of the kth iteration of Algorithm 1 can be used to terminate the algorithm.

Termination Criterion.

$$\phi(s^k, y^k) = (\sum_{i=1}^{m} \log s_i^k + \sum_{i=1}^{m} \log y_i^k) \le -O((m+n)(L + \min(m,n)t)).$$

We now analyze the complexity of using an approximate center in Algorithm 1. Again for simplicity, let $k = 0$. The following lemma is an analogue to Lemma 4.

Lemma 9 . *Let positive constants δ and β satisfy $\delta + \sqrt{2}\beta < 1$. Let (x^0, z^0) and (s^0, y^0) be an approximate center pair for $\Gamma(\gamma^0)$, where (s^0, y^0) and (x^0, z^0) are feasible for systems (8) and (9) with $\gamma = \gamma^0$, respectively, and*

$$\left\| \begin{pmatrix} X^0 s^0 - e \\ Z^0 y^0 - e \end{pmatrix} \right\| \le \delta. \tag{13}$$

Let $\Delta\gamma$, γ^1, $(\bar{s}^0, \bar{y}^0)$ and $(\bar{x}^0, \bar{z}^0)$ be selected as in Lemma 4. Then, $(\bar{s}^0, \bar{y}^0)$ and $(\bar{x}^0, \bar{z}^0)$ are feasible for systems (8) and (9) with $\gamma = \gamma^1$, respectively. Moreover,

$$\left\| \begin{pmatrix} \bar{X}^0 \bar{s}^0 - e \\ \bar{Z}^0 \bar{y}^0 - e \end{pmatrix} \right\| \le \delta + \sqrt{2}\beta < 1,$$

and

$$(\bar{s}^0, \bar{y}^0) > 0 \quad \textit{and} \quad (\bar{x}^0, \bar{z}^0) > 0.$$

Proof. The two equations in systems (8) and (9) with $\gamma = \gamma^1$ can again be verified. Furthermore,

$$\left\| \begin{pmatrix} \bar{X}^0\bar{s}^0 - e \\ \bar{Z}^0\bar{y}^0 - e \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} X^0 s^0 - e \\ Z^0 y^0 - e \end{pmatrix} \right\| + \left\| \begin{pmatrix} \Delta\gamma X^0 A y^0 \\ \Delta\gamma Y^0 A^T x^0 \end{pmatrix} \right\| \leq \delta + \left\| \begin{pmatrix} \Delta\gamma X^0 A y^0 \\ \Delta\gamma Y^0 A^T x^0 \end{pmatrix} \right\|.$$

Thus, the result follows from the proof of Lemma 4.

□

Now, using $(\bar{x}^0, \bar{z}^0)$ and $(\bar{s}^0, \bar{y}^0)$ as the initial pair, we apply the Newton procedure described in the previous section to generate a new approximate center pair (s^1, y^1) and (x^1, z^1) for $\Gamma(\gamma^1)$. This time, we terminate the procedure in *one* Newton step, then assign

$$(s^1, y^1) = (\bar{s}^1, \bar{y}^1) \quad \text{and} \quad (x^1, z^1) = (\bar{x}^1, \bar{z}^1).$$

Note that Lemmas 7 and 9 indicate that

$$\left\| \begin{pmatrix} X^1 s^1 - e \\ Z^1 y^1 - e \end{pmatrix} \right\| \leq \frac{\sqrt{2}}{4} \frac{(\delta + \sqrt{2}\beta)^2}{1 - \delta - \sqrt{2}\beta}, \tag{14}$$

and

$$\|(Y^0)^{-1} Y^1\| \leq \frac{\delta + \sqrt{2}\beta}{1 - \delta - \sqrt{2}\beta}. \tag{15}$$

Next, for suitable constants δ and β, for example,

$$\delta = 1/12 \quad \text{and} \quad \beta = 1/4\sqrt{2}, \tag{16}$$

we show that the potential value at y^1 for $\Gamma^1 = \Gamma(\gamma^1)$ is reduced by a constant from the value at y^0 for $\Gamma^0 = \Gamma(\gamma^0)$.

Theorem 10 . *Let δ and β be chosen as in (16), and (x^0, y^0, s^0, z^0) satisfy (13). Let*

$$\Delta\gamma = \beta \min\{\frac{1}{\|X^0 A Y^0 e\|}, \frac{1}{\|Y^0 A^T X^0 e\|}\}$$

and $\gamma^1 = \gamma^0 + \Delta\gamma$. Let (s^1, y^1) and (x^1, y^1) be generated in one Newton step of the Newton procedure. Then,

$$\gamma^0 < \gamma^1 < \gamma^*,$$

$$\left\| \begin{pmatrix} X^1 s^1 - e \\ Z^1 y^1 - e \end{pmatrix} \right\| \le \delta,$$

and

$$\phi(s^1, y^1) \le \phi(s^0, y^0) - \Omega(1).$$

Proof. $\gamma^1 > \gamma^0$ because $\Delta\gamma > 0$ and $\gamma^1 < \gamma^*$ because $\Gamma^1 = \Gamma(\gamma^1)$ has a nonempty (relative) interior. From inequality (14) we have

$$\left\| \begin{pmatrix} X^1 s^1 - e \\ Z^1 y^1 - e \end{pmatrix} \right\| \le \frac{\sqrt{2}}{4}\frac{1}{6} < \delta.$$

Now we prove the potential reduction inequality. Again, similar to the proof of Theorem 6, we have

$$\begin{aligned} (x^0)^T s^1 + (z^0)^T y^1 &= (x^0)^T (B - \gamma^1 A) y^1 + (z^0)^T y^1 \\ &= (y^1)^T ((B - \gamma^0 A)^T x^0 + z^0) - \Delta\gamma (x^0)^T A y^1 \\ &= (m+n) e^T y^1 - \Delta\gamma (x^0)^T A y^1 \\ &= (m+n) - \Delta\gamma (x^0)^T A y^1. \end{aligned}$$

Thus,

$$\begin{aligned} \prod_{i=1}^{m} (x_i^0 s_i^1) \prod_{i=1}^{n} (z_i^0 y_i^1) &\le \left(\frac{(x^0)^T s^1 + (z^0)^T y^1}{m+n} \right)^{m+n} \\ &= \left(1 - \frac{\Delta\gamma (x^0)^T A y^1}{m+n} \right)^{m+n} \\ &\le \exp(-\Delta\gamma (x^0)^T A y^1). \end{aligned}$$

Using the proof of Theorem 6 and relation (15), we have

$$\begin{aligned} \Delta\gamma (x^0)^T A y^1 &= \Delta\gamma e^T X^0 A Y^1 e \\ &= \beta \min\{ \frac{1}{\|X^0 A y^0\|}, \frac{1}{\|Y^0 A^T x^0\|} \} e^T X^0 A Y^1 e \\ &\ge \beta \frac{e^T X^0 A Y^1 e}{e^T X^0 A Y^0 e} \\ &\ge \beta (1 - \frac{\delta + \sqrt{2}\beta}{1 - \delta - \sqrt{2}\beta}) \\ &= \beta \frac{1 - 2(\delta + \sqrt{2}\beta)}{1 - \delta - \sqrt{2}\beta}. \end{aligned}$$

Finally, we have

$$\begin{aligned}
\phi(s^1,y^1)-\phi(s^0,y^0) &= \sum_{i=1}^m \log\frac{s_i^1}{s_i^0}+\sum_{i=1}^n \log\frac{y_i^1}{y_i^0}\\
&= \sum_{i=1}^m \log(x_i^0 s_i^1)+\sum_{i=1}^n \log(z_i^0 y_i^1)-\sum_{i=1}^m \log(x_i^0 s_i^0)-\sum_{i=1}^n \log(z_i^0 y_i^0)\\
&\le -\beta\frac{1-2(\delta+\sqrt{2}\beta)}{1-\delta-\sqrt{2}\beta}-\sum_{i=1}^m \log(x_i^0 s_i^0)-\sum_{i=1}^n \log(z_i^0 y_i^0)\\
&\le -\beta\frac{1-2(\delta+\sqrt{2}\beta)}{1-\delta-\sqrt{2}\beta}+\frac{\delta^2}{2(1-\delta)} \quad \text{(from relations (12) and (13))}.
\end{aligned}$$

One can verify

$$\beta\frac{1-2(\delta+\sqrt{2}\beta)}{1-\delta-\sqrt{2}\beta}-\frac{\delta^2}{2(1-\delta)}=\frac{1}{8\sqrt{2}}-\frac{1}{264}>0.$$

This gives the desired result.

□

We now formally state the algorithm using an approximate center.

Algorithm 2.

Step 0. Let $\gamma^0=0$. Then generate a δ-approximate center y^0 of $\Gamma^0=\Gamma(\gamma^0)$. Set $k=0$.

Step 1. Let

$$\Delta\gamma^k=\beta\min\{\frac{1}{\|X^kAY^ke\|},\frac{1}{\|Y^kA^TX^ke\|}\}$$

for some constant $0<\beta<1$ and let $\gamma^{k+1}=\gamma^k+\Delta\gamma^k$. Let

$$\bar{s}=s^k-\Delta\gamma Ay^k \quad \text{and} \quad \bar{y}=y^k,$$

and

$$\bar{x}=x^k \quad \text{and} \quad \bar{z}=z^k+\Delta\gamma A^Tx^k.$$

Solve for $(\Delta s,\Delta y)$ and $(\Delta x,\Delta z)$:

$$\begin{aligned}
\bar{S}\Delta x+\bar{X}\Delta s &= e-\bar{X}\bar{s},\\
\bar{Y}\Delta z+\bar{Z}\Delta y &= e-\bar{Y}\bar{z},\\
(B-\gamma^{k+1}A)^T\Delta x+\Delta z &= 0,\\
\Delta s-(B-\gamma^{k+1}A)\Delta y &= 0.
\end{aligned}$$

Let

$$x^{k+1} = \bar{x} + \Delta x \quad \text{and} \quad z^{k+1} = \bar{z} + \Delta z,$$

and

$$s^{k+1} = \bar{s} + \Delta s \quad \text{and} \quad y^{k+1} = \bar{y} + \Delta y.$$

Step 2. If $\phi(s^{k+1}, y^{k+1}) > -O((m+n)(L + \min(m,n)t))$, then set $k := k+1$ and return to Step 1.

$\square$

It is well known that an initial δ-approximate center pair (z^0, y^0) and (x^0, z^0) can be generated in no more than $O((m+n)L)$ interior-point- algorithm iterations (Vaidya [18]). Thus, we conclude the following.

Theorem 11 . *Algorithm 2, with a suitable choice of δ and β, terminates in $k = O((m+n)(L+\min(m,n)t))$ iterations and each iteration solves a system of $(m+n)$ linear equations. The resulting γ^k satisfies*

$$\gamma^* - 2^{-t} \le \gamma^k < \gamma^*.$$

6 Further Remarks

Our algorithm also generates the optimal dual vector. More precisely, we prove the following result.

Proposition 12 . *Any limit point of*

$$\lim_{k\to\infty} \frac{x^k}{e^T x^k},$$

where x^k is generated in Algorithm 1 or 2, is a solution for the dual NEG problem with $\eta = \eta^*$ under the assumptions (A1), (A2) and (A3).

Proof. For simplicity, we assume that (x^k, y^k, s^k, z^k) is exactly centered, i.e., it satisfies relations (7), (8) and (9). Since at least one component of s^k converges to zero, $e^T x^k = e^T (S^k)^{-1} e$ tends to $+\infty$. Moreover, from (9) we have

$$(B - \gamma^k A)^T \frac{x^k}{e^T x^k} + \frac{z^k}{e^T x^k} = (m+n) \frac{e}{e^T x^k}.$$

Thus, the right-hand vector of the above equation converges to zero. Since $z^k / e^T x^k > 0$ for all k,

$$\lim_{k \to \infty} (B - \gamma^k A)^T \frac{x^k}{e^T x^k} = - \lim_{k \to \infty} \frac{z^k}{e^T x^k} \leq 0.$$

Furthermore, under the assumptions $(A1)$, $(A2)$ and $(A3)$ we have $\gamma^k \to \gamma^* = \eta^*$. Therefore, any limit point of the sequence of positive $x^k / e^T x^k$ is a solution for the dual.

□

Now we turn our attention to the question raised in the remark of Section 2, that is, what happens if $\Gamma(\gamma)$ has an empty (relative) interior for some $\gamma < \gamma^*$. A referee has pointed out to the author a nice duality theorem for the NEG problem, that is, under the assumptions (A1) and (A2), $\Gamma(\gamma)$ has an nonempty (relative) interior for all $\gamma < \eta^* \leq \gamma^*$ (see Kemeny et al. [5] and Gale [2]). Thus, the algorithm discussed in this paper will precisely generate η^* under only (A1) and (A2). Similarly, the η-level set of the dual has an nonempty (relative) interior for all $\eta > \gamma^* \geq \eta^*$. Thus, one can apply the algorithm to solving the dual to generate γ^* in the same manner. Thus, we can solve the NEG problem under only (A1) and (A2), which are basic assumptions for a meaningful economic growth model.

Of course, we can also identify the irrelevant good as we suggested earlier. One can show that if for some good i, $(b_i - \gamma a_i)y = 0$ for all $y \in \Gamma(\gamma)$ with $\gamma < \gamma^*$, then it will stay at zero when γ increases. In particular, both $b_i y^*$ and $a_i y^*$ stay at zero for all the optimal intensity vector y^*. Thus, we can run our algorithm and stop at a γ^1 and a temporary intensity vector y^1 such that the (relative) interior of $\Gamma(\gamma^1)$ becomes empty. We now identify those irrelevant goods such that $(B_1 - \gamma^1 A_1) y^1 = 0$, where B_1 and A_1 are submatrices (with full column) of B and A, respectively. Then, $(B_1 - \gamma^1 A_1) y = 0$ for all $y \in \Gamma(\gamma^1)$, due to the property of interior-point algorithms (Güler and Ye [4]). Then, using the same algorithm developed in the present paper we continue to solve a reduced NEG problem (both row and column dimensions may be reduced)

$$\max\{\gamma \mid \exists\, y \neq 0: \quad y \geq 0, \quad (B_2 - \gamma A_2)y \geq 0, \quad (B_1 + A_1)y = 0\}$$

where B_2 and A_2 are the remaining matrices of B and A after deleting B_1 and A_1, respectively. In at most m steps (each step reduce at least one row), we shall solve the general NEG problem.

Finally, we would like to emphasize that most complexity results for interior-point algorithms are based on the reduction of the objective value, the primal-dual gap, or an algebraic potential function. Comparing to those analyses, our proof technique in the present paper is more "geometric." If one formulates the NEG problem as a nonlinear optimization problem, then nonconvexity will occur and it will make those analyses difficult. Therefore, the analysis in the present paper will be useful for solving a class of geometric or equilibrium problems related to linear inequalities. In fact, the author has recently learnt that, using this reasoning analysis, Nemirovskii obtained a very nice theoretical complexity bound for general, nonlinear, and fractional optimization problems [10].

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References

[1] J. V. Burke, A. A. Goldstein, P. Tseng, and Y. Ye, "Translation cuts for minimization," in: P. Pardalos eds., *Complexity in Numerical Optimization* (World Scientific, New Jersey, 1993) 57-73.

[2] D. Gale, *The Theory of Linear Economic Models* (McGraw-Hill, New York, 1960).

[3] C. C. Gonzaga, "An algorithm for solving linear programming programs in $O(n^3L)$ operations," in: N. Megiddo, ed., *Progress in Mathematical Programming, Interior Point and Related Methods* (Springer-Verlag, New York, 1988) 1-28.

[4] O. Güler and Y. Ye, "Convergence behavior of interior-point algorithms," *Mathematical Programming* **60** (1993) 215-228.

[5] J. Kemeny, O. Morgenstern, and G. Thompson, "A generalization of the von Neumann model of an expanding economy," *Econometrica* **24** (1956) 115-135.

[6] M. Kojima, S. Mizuno and A. Yoshise, "A Polynomial-Time Algorithm for a Class of Linear Complementarity Problems," *Mathematical Programming* **44** (1989) 1-26.

[7] N. Megiddo, "Pathways to the optimal set in linear programming," in: N. Megiddo, ed., *Progress in Mathematical Programming, Interior Point and Related Methods* (Springer-Verlag, New York, 1988) 131-158.

[8] S. Mizuno, M. J. Todd and Y. Ye, "On adaptive-step primal-dual interior-point algorithms for linear programming," *Mathematics of Operations Research* **19** (1994) 53-67.

[9] R. C. Monteiro and I. Adler, "Interior path following primal-dual algorithms. Part I: Linear programming," *Mathematical Programming* **44** (1989) 27-42.

[10] A. Nemirovskii, "On polynomiality of the method of analytic centers for fractional problems," manuscript, Central Economic & Mathematical Institute of Russian Academy of Sciences (Moscow, Russia, 1993).

[11] J. Renegar, "A polynomial-time algorithm based on Newton's method for linear programming," *Mathematical Programming* **40** (1988) 59-94.

[12] S. Robinson, "A linearization technique for solving the irreducible von Neumann economic model," in: Loś and Loś, eds., *Mathematical Models in Economics* (PWN-Polish Scientific Publishers, Warszawa, 1974) 139-150.

[13] C. Roos and J. -Ph. Vial, "A polynomial methods of approximate centers for linear programming," *Mathematical Programming* **54** (1992) 295-306.

[14] A. Schrijver, *Theory of Linear and Integer Programming* (John Wiley & Sons, New York, 1986).

[15] G. Sonnevend, "An analytical center for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming," in: *Lecture Notes in Control and Information Sciences* 84 (Springer, New York, 1985) 866-876.

[16] K. C. Tan and R. M. Freund, "Newton's method for the general parametric center problem with applications," manuscript, Sloan School of Management, Massachusetts Institute of Technology (Cambridge, MA, 1991).

[17] M. J. Todd, "Improved bounds and containing ellipsoids in Karmarkar's linear programming algorithm," *Mathematics of Operations Research* **13** (1988) 650-659.

[18] P. M. Vaidya, "A locally well-behaved potential function and a simple Newton-type method for finding the center of a polytope," in: N. Megiddo, ed., *Progress in Mathematical Programming, Interior Point and Related Methods* (Springer-Verlag, New York, 1988) 79-90.

[19] Y. Ye, "A potential reduction algorithm allowing column generation," *SIAM J. on Optimization* **2** (1992) 7-20.

[20] Y. Ye, "Karmarkar's algorithm and the ellipsoid method," *Operations Research Letters* **4** (1987) 177-182.