On Homotopy-Smoothing Methods for Variational Inequalities

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Abstract

A variational inequality problem with a mapping $g : \Re^n \to \Re^n$ and lower and upper bounds on variables can be reformulated as a system of nonsmooth equations $F(x) = 0$ in $\Re^n$. Recently, several homotopy methods, such as interior-point and smoothing methods, have been employed to solve the problem. All of these methods use parametric functions and construct perturbed equations to approximate the problem. The solution to the perturbed system constitutes a smooth trajectory leading to the solution of the original variational inequality problem. The methods generate iterates to follow the trajectory. Among these methods Chen-Mangasarian and Gabriel-Moré proposed a class of smooth functions to approximate F. In this paper, we study several properties of the trajectory defined by solutions of these smooth systems. We propose a homotopy-smoothing method for solving the variational inequality problem, and show that the method converges globally and superlinearly under mild conditions. Furthermore, if the involved function g is an affine function, the method finds a solution of the problem in finite steps. Preliminary numerical results indicate that the method is promising.

Key words. smoothing approximation, variational inequalities, P_0 function, finite convergence, homotopy.
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1 Introduction

We consider the following variational inequality problem (VIP). Given a continuously differentiable mapping $g : \Re^n \to \Re^n$, $l \in \{\Re \cup -\infty\}^n$, and $u \in \{\Re \cup \infty\}^n$, where $l < u$. Find $x \in [l, u]$ such that

$$(y-x)^T g(x) \geq 0, \qquad \text{for any } y \in [l, u].$$

Such problem is also called a box-constrained or mixed complementarity problem in [5, 8, 9, 12, 14, 15, 16, 21]. We denote this problem by VI(l, u, g).

If $l_i = 0$ and $u_i = \infty$ for $i = 1, \dots, n$, VI(l, u, g) reduces to the nonlinear complementarity problem (NCP). During the past few decades, this problem has become a well-established and fruitful discipline within mathematical programming [14, 17, 22, 27]. In particular, several homotopy methods, such as interior-point (e.g.,[23]) and smoothing methods (e.g., [5]), have been employed to solve the NCP. All of these methods use parametric functions and construct perturbed equations to approximate the NCP. The solution to the perturbed system constitutes a smooth trajectory leading to the solution of the original NCP. These methods generate iterates to follow the trajectory.

In particular, the NCP can be formulated as a nonsmooth equations:

$$F(x) = x - \max(0, x - g(x)) = 0. \tag{1.1}$$

Chen and Mangasarian [5] introduced a class of parametric smooth approximation functions for the nonsmooth function F in (1.1). A family of homotopy continuation methods, including an interior-point path-following method, can be constructed to solve the NCP based on Chen-Mangasarian smooth approximation equations. Using the proving techniques developed in Kojima, Megiddo and Noma [22], Chen and Harker [4] established the existence, uniqueness and limiting properties of the trajectory defined by these approximation equations under certain conditions. Recently, Xu and Burke [33] gave a polynomial complexity bound for an interior-point smoothing method for the monotone linear complementarity problem (LCP) based on the Chen-Harker-Kanzow smoothing function [3, 20] and Tseng's convergence analysis for a class of infeasible interior point methods [32]. More recently, Burke-Xu [2] and Hotta-Yoshise [19] studied the global convergence of non-interior-point algorithms using the Chen-Harker-Kanzow smoothing function.

A number of important problems in engineering and economics have both lower and/or upper bounds on the variables, rather than just nonnegative constraints as in the NCP. The VI(l, u, g) model includes more general problems and attracts researchers' attention [5, 8, 9, 12, 14, 15, 16, 21]. One can add slacks to transform the problem into a NCP. However, in doing so the problem dimension will be multiplied and some undesired computational difficulties may occur.

Many researchers try to develop direct homotopy-smoothing methods for solving VI(l, u, g). This is possible since VI(l, u, g) itself is equivalent to the nonsmooth equations [10]

$$F(x) := x - \Pi_{[l,u]}(x - g(x)) = 0, \tag{1.2}$$

where $\Pi_{[l,u]}$ is the projection operator onto $[l, u]$ under the Euclidean norm. The special structure of box constraints $[l, u]$ provides a convenient way to study VI(l, u, g) with each component. Precisely, (1.2) can be written as

$$F(x) = x - \text{mid}(l, u, x - g(x)) = 0, \tag{1.3}$$

where mid$(\cdot)$ denotes the componentwise median operator. Note that for the NCP case (1.3) reduces to (1.1).

Gabriel and Moré [16] defined a parametric function $f : \Re^n \times R_{++} \to \Re^n$ to directly approximate function F in (1.3), where $f(x, \varepsilon)$ is continuously differentiable with respect to variable vector x and $\|F(x) - f(x, \varepsilon)\| \leq \mu\varepsilon$. Here μ is a positive constant. In this paper, we study several properties of the trajectory defined by solutions of the equation $f(x, \varepsilon) = r$, where r is a given n-dimensional vector.

We will reformulate VI(l, u, g) as a generalized complementarity problem (GCP) in $\Re^n$:

$$q(x) \geq 0, \quad p(x) \geq 0, \quad p(x)^T q(x) = 0,$$

where the set $\{x : \ p(x) \geq 0, \ q(x) \geq 0\}$ is usually called the feasible set of the GCP. We will show that for any $\varepsilon > 0$, if x is a solution of $f(x, \varepsilon) = 0$ then x is approximately feasible for the GCP — $q(x) \geq 0$ and $p(x) \geq -O(\varepsilon)e$, and for each i, $|p_i(x)q_i(x)| \leq O(\varepsilon^2)$ (which is stronger than the current result that $|p_i(x)q_i(x)|$ is bounded). Furthermore, if the VIP or GCP is merely an NCP, then any solution of $f(x, \varepsilon) = 0$ is an exactly feasible point of the NCP, i.e., $x \geq 0$ and $g(x) \geq 0$, and for each i, $x_i g_i(x) \leq O(\varepsilon^2)$.

Under the assumptions that g is a P_0 function and the level sets $\{x : \|F(x)\| \leq \Gamma\}$ with $\Gamma > 0$ are bounded, we will investigate the trajectory formed by solutions of $f(x, \varepsilon) = r$ as ε and r vary. Our trajectory analysis is related to the one established in Chen and Harker [4]. However, there are several fundamental distinctions. For example, the Gabriel-Moré approximation function is not convex. Consequently, Assumptions A1, A2, A4 in [4], which play an important role in their proof, do not hold for the Gabriel-Moré approximation function.

Using the Gabriel-Moré approximation function, we will also modify a homotopy or smoothing Newton method for solving VI(l, u, g), which was originally proposed in [8]. The Chen-Qi-Sun method solves a system of linear equations at each step and generates iterates converging to a solution of (1.3) globally and superlinearly when g is a P-uniform function. Global convergence of the method only requires that g be a P_0 function and the level sets $\{x : \|F(x)\| \leq \Gamma\}$ with $\Gamma > 0$ be bounded. We will show that the monotone complementarity problem with a feasible interior point satisfies these conditions on certain level sets.

The Chen-Qi-Sun method is not finitely convergent when the VIP is a linear variational inequality problem (LVI). Motivated by the finite convergence result of other algorithms [13, 31, 36] and our trajectory analysis, we will add a Newton step in the Chen-Qi-Sun method [8]. We call the new method a hybrid Newton-smoothing method. The new method converges globally, and superlinearly under weaker conditions. Furthermore, if the involved function g is an affine function, the method finds a solution of the LVI in finite steps.

We have tested the hybrid Newton-smoothing method with three approximate functions on five types of samples which contain 24 LVI problems. Our preliminary numerical results show that the new method is promising.

Some words about our notation are in order. Let $N = \{i \ : \ i = 1, 2, \ldots, n\}$. The $n \times n$ identity matrix is denoted by I. The i-th row of an $n \times n$ matrix M is denoted by M_i. We use $\|\cdot\|$ to denote the Euclidean norm. Let e denote the vector of ones, i.e.

$$e = (1, 1, \ldots, 1)^T.$$

We denote

$$\Re_+ = \{\varepsilon \ : \ \varepsilon \in \Re, \varepsilon \geq 0\} \quad \text{and} \quad \Re_{++} = \{\varepsilon \ : \ \varepsilon \in \Re, \varepsilon > 0\}.$$

2 Smoothing approximations

Let $\rho : \Re \to \Re_+$ be a density function with a bounded absolute mean, that is

$$\kappa := \int_{-\infty}^{\infty} |s|\rho(s)ds < \infty. \tag{2.1}$$

We also let ρ satisfy

$$\zeta^2 := \max\left\{\sup_{t\geq 0} \int_t^{\infty} ts\rho(s)ds, \quad \sup_{t\leq 0} \int_{-\infty}^{t} ts\rho(s)ds\right\} < \infty. \tag{2.2}$$

It is easy to verify that

$$\zeta^2 \leq \int_{-\infty}^{\infty} s^2\rho(s)ds.$$

The three approximate functions used in our computational experiment meet these two conditions. See section 4. Note that if the second moment of ρ is bounded, then ρ satisfies (2.1) and (2.2). The condition (2.2) is new. We introduce it to establish a strong complementarity error bound in Theorem 2.1 below.

The class of Gabriel-Moré approximation functions to function F in (1.3) is defined by

$$f_i(x, \varepsilon) = x_i - \int_{-\infty}^{\infty} \text{mid}(l_i, u_i, x_i - g_i(x) - \varepsilon s)\rho(s)ds, \quad i = 1, \ldots n. \tag{2.3}$$

By using (2.3), a smooth approximation function can be generated by an appropriate density function. The following proposition is known.

Proposition 2.1 *[6, 8, 16]. The function $f : \Re^n \times \Re_{++} \to \Re^n$ defined by (2.3) satisfies the following four conditions.*

(i) For any $(x, \varepsilon) \in \Re^n \times \Re_{++}$,

$$|F_i(x) - f_i(x, \varepsilon)| \leq \kappa\varepsilon, \quad i = 1, 2, \ldots, n. \tag{2.4}$$

(ii) f *is continuously differentiable with respect to the first variable, and for any* $(x, \varepsilon) \in \Re^n \times \Re_{++}$

$$f'_{\mathbf{1}}(x, \varepsilon) = I - D(x)(I - g'(x)), \tag{2.5}$$

where $D(x)$ *is a diagonal matrix whose diagonal elements are*

$$D_{ii}(x) = \int_{(x-g(x)-u)_i/\varepsilon}^{(x-g(x)-l)_i/\varepsilon} \rho(s)ds, \quad i = 1, 2, \ldots, n.$$

(iii) For any $x \in \Re^n$

$$\lim_{\varepsilon \downarrow 0} f'_{\mathbf{1}}(x, \varepsilon) = f^0(x), \tag{2.6}$$

where for $i \in N$,

$$f_i^0(x) = \begin{cases} g'_i(x) & \text{if } x_i - g_i(x) \in (l_i, u_i), \\ I_i & \text{if } x_i - g_i(x)_i \notin [l_i, u_i], \\ I_i - \left(\int_{-\infty}^0 \rho(s)ds\right)(I_i - g'_i(x)) & \text{if } x_i - g_i(x) = l_i, \\ I_i - \left(\int_0^\infty \rho(s)ds\right)(I_i - g'_i(x)) & \text{if } x_i - g_i(x) = u_i. \end{cases}$$

(iv) For any $x \in \Re^n$, f *satisfies the* directional derivative consistence property *at* x, *i.e.*

$$\lim_{h \to 0} \frac{F(x+h) - F(x) - f^0(x+h)h}{\|h\|} = 0. \tag{2.7}$$

We now develop the following proposition which will be used in our analysis.

Proposition 2.2 *For a given* $\varepsilon \in \Re_{++}$ *and a pair* $a, b \in \Re \cup \{-\infty, \infty\}$ *such that* $a < b$, *let* $\rho(s) = \rho(-s)$ *and*

$$h(t) = \int_{-\infty}^{\infty} \mathrm{mid}(a, b, t - \varepsilon s)\rho(s)ds.$$

Then $h(t)$ *is an increasing function,*

$$h'(t) \leq 1, \tag{2.8}$$

and

$$h(t) \geq \mathrm{mid}(a, b, t) \quad \Leftrightarrow \quad t \leq \frac{a+b}{2}. \tag{2.9}$$

Furthermore, if $supp\{\rho\} = \Re$, *then*

$$h'(t) < 1 \tag{2.10}$$

and

$$h(t) > \mathrm{mid}(a, b, t) \quad \Leftrightarrow \quad t < \frac{a+b}{2}. \tag{2.11}$$

Proof: The result on $h(t)$ and $h'(t)$ was proved in Lemma 2.3 of Gabriel and Moré [16]. We focus on proving the rest of the proposition.

Using $\rho(s) = \rho(-s)$, we have

$$\int_{-\infty}^{\infty} s\rho(s)ds = 0,$$

which implies

$$\int_{-\infty}^{\infty} (t - \varepsilon s)\rho(s)ds = t.$$

By the definition of the median operator, we obtain

$$\begin{aligned} h(t) &= \int_{-\infty}^{(t-b)/\varepsilon} b\rho(s)ds + \int_{(t-a)/\varepsilon}^{\infty} a\rho(s)ds + \int_{(t-b)/\varepsilon}^{(t-a)/\varepsilon} (t - \varepsilon s)\rho(s)ds \\ &= \int_{-\infty}^{(t-b)/\varepsilon} b\rho(s)ds + \int_{(t-a)/\varepsilon}^{\infty} a\rho(s)ds + \int_{-\infty}^{\infty} (t - \varepsilon s)\rho(s)ds \\ &\quad - \int_{-\infty}^{(t-b)/\varepsilon} (t - \varepsilon s)\rho(s)ds - \int_{(t-a)/\varepsilon}^{\infty} (t - \varepsilon s)\rho(s)ds \\ &= \int_{-\infty}^{(t-b)/\varepsilon} (b - t + \varepsilon s)\rho(s)ds + \int_{(t-a)/\varepsilon}^{\infty} (a - t + \varepsilon s)\rho(s)ds + t. \end{aligned} \tag{2.12}$$

Suppose that $t \le (a + b)/2$. Then there are only two cases: $\mathrm{mid}(a, b, t) = t$ or $\mathrm{mid}(a, b, t) = a$.

If $\mathrm{mid}(a, b, t) = t$, then from (2.12), $b - t \ge t - a \ge 0$ and $\rho(s) = \rho(-s)$, we have

$$\begin{aligned} &h(t) - \mathrm{mid}(a, b, t) \\ &= \int_{-\infty}^{(t-b)/\varepsilon} (a + b - 2t)\rho(s)ds + \int_{(t-a)/\varepsilon}^{(b-t)/\varepsilon} (a - t + \varepsilon s)\rho(s)ds \ge 0. \end{aligned} \tag{2.13}$$

If $\mathrm{mid}(a, b, t) = a$, then from (2.12) and $0 \ge t - a \ge t - b$, we have

$$\begin{aligned} &h(t) - \mathrm{mid}(a, b, t) \\ &= \int_{-\infty}^{(t-b)/\varepsilon} (b - a + \varepsilon s)\rho(s)ds + \int_{(t-b)/\varepsilon}^{\infty} (t - a)\rho(s)ds + \int_{(t-a)/\varepsilon}^{\infty} (a - t + \varepsilon s)\rho(s)ds \\ &= \int_{-\infty}^{(t-b)/\varepsilon} (b - a)\rho(s)ds + \int_{(t-b)/\varepsilon}^{(t-a)/\varepsilon} (t - a - \varepsilon s)\rho(s)ds \ge 0. \end{aligned} \tag{2.14}$$

Hence, we obtain the $\Leftarrow$ part of (2.9) from (2.13) and (2.14).

Assume that $t \ge (a + b)/2$. Then following the same argument above, we can show that $h(t) \le \mathrm{mid}(a, b, t)$. This implies the $\Rightarrow$ part of (2.9). So the proof of (2.9) is completed.

Notice that $\mathrm{supp}\{\rho\} = \Re$ implies that for any α, β satisfying $\alpha < \beta$,

$$\int_{\beta}^{\alpha} \rho(s)ds > 0.$$

Using this fact and following the above argument, we can prove (2.11).

∎

To study the path defined by the smooth equations $f(x,\varepsilon) = 0$ for $\varepsilon > 0$, we introduce a generalized complementarity problem. Let $z \in \Re^n$ and Σ be a diagonal matrix whose diagonal elements are

$$\Sigma_{ii}(z) = \begin{cases} 1, & z_i \geq 0 \\ -1, & z_i < 0, \quad i = 1, 2, \ldots, n. \end{cases}$$

It is easy to verify that a vector $x \in \Re^n$ solves (1.3) if and only if x solves the following complementarity problem

$$q(x) := \min(x - l, u - x) \geq 0 \tag{2.15}$$

$$p(x) := \Sigma(\frac{l+u}{2} - x)g(x) \geq 0 \tag{2.16}$$

$$q(x)^T p(x) = 0. \tag{2.17}$$

In the complementarity formulation (2.15)-(2.17), for $i \in N$ we regard $l_i \cdot 0 = 0$ and $u_i \cdot 0 = 0$ even l_i or u_i is infinity.

Based on the complementarity problem (2.15)-(2.17), we define a restricted feasible set of the variational inequality problem VI(l, u, g) by

$$X_0 = \{x \ : \ q(x) \geq 0, \ p(x) \geq 0\}.$$

$q(x) \geq 0$ implies that $x \in [l, u]$ which is desired, but $p(x) \geq 0$ implies that $g(x) \geq 0$ if $x \leq \frac{l+u}{2}$ and $g(x) \leq 0$ if $x > \frac{l+u}{2}$, which is strictly contained in the feasible set of the original VIP:

$$\{x \ : \ q(x) \geq 0, \ g_i(x) \geq 0 \text{ if } x_i = l_i; \ g_i(x) \leq 0 \text{ if } x_i = u_i, \ i = 1, \ldots, n\}.$$

Thus, we define another set X associated with X_0 as

$$X = \{x \ : \ q(x) \geq 0, \ \Sigma(\frac{l+u}{2} + g(x) - x)g(x) \geq 0\},$$

and its interior by

$$\text{int}X = \{x \ : \ q(x) > 0, \ \Sigma(\frac{l+u}{2} + g(x) - x)g(x) > 0\}.$$

Obviously, we have

$$X_0 \subseteq X,$$

and both X_0 and X contain the solution set of the VIP if it exists. Furthermore, if $l_i = 0$ and $u_i = \infty$, for each $i \in N$, then VI(l, u, g) reduces to the NCP and

$$X_0 = X = \{x \ : \ x \geq 0, \ g(x) \geq 0\},$$

which is known as the feasible set for the NCP.

The following theorem shows that the homotopy equation $f(x, \varepsilon) = 0$ possesses several desired features and is closely related to the interior-point homotopy equation for the GCP (2.15)–(2.17).

Theorem 2.1 *Suppose that $\rho(s)=\rho(-s)$. Let x be a solution of*

$$f(x,\varepsilon)=0.$$

Then

(i) For any $\varepsilon\in\Re_{++}$,

$$x\in X.$$

Furthermore, if supp$(\rho)=\{s:\rho(s)>0\}=\Re$, then

$$x\in \text{int}X\cup\{x\ :\ x_i=\frac{l_i+u_i}{2},\ \exists\ i\in N\}.$$

(ii) For $\varepsilon\in\Re_{++}$ satisfying $\varepsilon<\frac{1}{2\kappa}\min_{1\le i\le n}(u_i-l_i)$,

$$x\in X_\varepsilon=\{x\ :\ q(x)\ge 0,\quad p(x)\ge -\kappa\varepsilon e\}.$$

(iii) For any $\varepsilon\in\Re_{++}$,

$$|q_i(x)p_i(x)|\le\varepsilon^2(\zeta^2+\kappa^2/2),\qquad i=1,2,\ldots,n.$$

Proof: To simplify the proof, we denote $y=g(x)$.

(i) Since $l_i\le$mid$(l_i,u_i,x_i-y_i-\varepsilon s)\le u_i$ and $\rho(s)\ge 0$, we have

$$l_i\le x_i-f_i(x,\varepsilon)=x_i\le u_i,\quad i=1,2,\ldots,n.$$

Now we show

$$\Sigma_{ii}(\frac{l+u}{2}+y-x)y_i\ge 0,\quad i=1,2,\ldots,n. \tag{2.18}$$

By Proposition 2.2, $f(x,\varepsilon)=0$ implies that for any $i\in N$

$$x_i\ge \text{mid}(l_i,u_i,x_i-y_i)\quad\Leftrightarrow\quad x_i-y_i\le\frac{l_i+u_i}{2}. \tag{2.19}$$

Suppose that $x_i-y_i\le\frac{l_i+u_i}{2}$. Then from (2.19), we have either

$$x_i\ge x_i-y_i\ge l_i\quad\text{or}\quad x_i\ge l_i\ge x_i-y_i.$$

In either case, $y_i\ge 0$.

Suppose that $x_i-y_i\ge\frac{l_i+u_i}{2}$. Then from (2.19), we have either

$$x_i\le x_i-y_i\le u_i\quad\text{or}\quad x_i\le u_i\le x_i-y_i.$$

In either case, $y_i\le 0$.

Therefore (2.18) holds and hence $x\in X$.

The assumption supp$(\rho) = \Re$ implies that ρ vanishes in any nontrivial interval. In such case, from

$$x_i = x_i - f_i(x,\varepsilon)$$
$$= \int_{-\infty}^{\frac{x_i-y_i-u_i}{\varepsilon}} u_i\rho(s)ds + \int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} l_i\rho(s)ds + \int_{\frac{x_i-y_i-u_i}{\varepsilon}}^{\frac{x_i-y_i-l_i}{\varepsilon}} (x_i - y_i - \varepsilon s)\rho(s)ds,$$

we have $l_i < x_i < u_i$. Furthermore, using (2.11) and following the same argument above, we can show

$$\Sigma_{ii}(\frac{l+u}{2} + y - x)y_i > 0, \quad \text{for} \quad x_i - y_i \neq \frac{l_i + u_i}{2} \quad i = 1, 2, \ldots, n.$$

If $x_i - y_i = (l_i + u_i)/2$, then by (2.11) $y_i = 0$. Hence, we obtain $x \in$int $X \cup \{x \ : \ x_i = (l_i + u_i)/2, \exists\, i \in N\}$.

(ii) By (2.4) and $f(x,\varepsilon) = 0$, we have

$$|F_i(x)| = |x_i - \text{mid}(l_i, u_i, x_i - y_i)| \le \kappa\varepsilon.$$

This implies

$$x_i - \kappa\varepsilon \le \text{mid}(l_i, u_i, x_i - y_i) \le x_i + \kappa\varepsilon. \tag{2.20}$$

If $x_i \le (l_i + u_i)/2$, then from the second inequality in (2.20) and $\varepsilon < \frac{u_i - l_i}{2\kappa}$, we have either

$$\text{mid}(l_i, u_i, x_i - y_i) = l_i \tag{2.21}$$

or

$$\text{mid}(l_i, u_i, x_i - y_i) = x_i - y_i. \tag{2.22}$$

Moreover, (2.21) implies that $y_i \ge x_i - l_i \ge 0$, and (2.22) with (2.20) implies $y_i \ge -\kappa\varepsilon$.

If $x_i \ge (l_i + u_i)/2$, then from the first inequality in (2.20) and $\varepsilon < \frac{u_i - l_i}{2\kappa}$, we have either

$$\text{mid}(l_i, u_i, x_i - y_i) = u_i \tag{2.23}$$

or

$$\text{mid}(l_i, u_i, x_i - y_i) = x_i - y_i. \tag{2.24}$$

Moreover, (2.23) implies that $y_i \le x_i - u_i \le 0$, and (2.24) with (2.20) implies $y_i \le \kappa\varepsilon$.

Hence $p_i(x) = \Sigma_{ii}(\frac{l+u}{2} - x)y_i \ge -\kappa\varepsilon$.

(iii) For a fixed i, we consider three cases.

Case 1: $\text{mid}(l_i, u_i, x_i - y_i) = l_i$. We have

$$x_i - y_i \le l_i < u_i, \quad x_i - l_i \ge 0, \quad y_i \ge 0,$$

and

$$x_i - l_i \le \kappa\varepsilon.$$

Moreover,

$$
\begin{aligned}
& x_i - l_i \\
= & \int_{-\infty}^{\frac{x_i-y_i-u_i}{\varepsilon}} (u_i - x_i + y_i + \varepsilon s)\rho(s)ds - \int_{-\infty}^{\frac{x_i-y_i-l_i}{\varepsilon}} (l_i - x_i + y_i + \varepsilon s)\rho(s)ds \\
= & \varepsilon \left(\int_{-\infty}^{\frac{x_i-y_i-u_i}{\varepsilon}} (\frac{u_i - x_i + y_i}{\varepsilon} + s)\rho(s)ds - \int_{-\infty}^{\frac{x_i-y_i-l_i}{\varepsilon}} (\frac{l_i - x_i + y_i}{\varepsilon} + s)\rho(s)ds \right) \\
\leq & -\varepsilon \int_{-\infty}^{\frac{x_i-y_i-l_i}{\varepsilon}} (\frac{l_i - x_i + y_i}{\varepsilon} + s)\rho(s)ds \\
\leq & -\varepsilon \int_{-\infty}^{\frac{x_i-y_i-l_i}{\varepsilon}} s\rho(s)ds,
\end{aligned}
$$

since the first integral in the parenthesis has a negative value and $l_i - x_i + y_i \geq 0$. Thus,

$$
\begin{aligned}
y_i(x_i - l_i) & \leq (l_i - x_i + y_i + x_i - l_i)(-\varepsilon \int_{-\infty}^{\frac{x_i-y_i-l_i}{\varepsilon}} s\rho(s)ds) \\
& \leq (l_i - x_i + y_i + \kappa\varepsilon)(-\varepsilon \int_{-\infty}^{\frac{x_i-y_i-l_i}{\varepsilon}} s\rho(s)ds) \\
& = \varepsilon^2 \frac{x_i - y_i - l_i}{\varepsilon} \int_{-\infty}^{\frac{x_i-y_i-l_i}{\varepsilon}} s\rho(s)ds - \varepsilon^2\kappa \int_{-\infty}^{\frac{x_i-y_i-l_i}{\varepsilon}} s\rho(s)ds \\
& \leq \varepsilon^2(\zeta^2 + \kappa^2/2).
\end{aligned}
$$

Case 2: $\mathrm{mid}(l_i, u_i, x_i - y_i) = u_i$. The proof of Case 2 is similar to Case 1, where we have

$$l_i < u_i \leq x_i - y_i, \quad u_i - x_i \geq 0, \quad y_i \leq 0,$$

and

$$u_i - x_i \leq \kappa\varepsilon.$$

Case 3: $\mathrm{mid}(l_i, u_i, x_i - y_i) = x_i - y_i$ and $l_i < x_i - y_i < u_i$. We have

$$|y_i| \leq \kappa\varepsilon.$$

Moreover,

$$y_i = \int_{-\infty}^{\frac{x_i-y_i-u_i}{\varepsilon}} (u_i - x_i + y_i + \varepsilon s)\rho(s)ds + \int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} (l_i - x_i + y_i + \varepsilon s)\rho(s)ds.$$

Since the first integral has negative value and the second has positive value, we have

$$\int_{-\infty}^{\frac{x_i-y_i-u_i}{\varepsilon}} (u_i - x_i + y_i + \varepsilon s)\rho(s)ds \leq y_i \leq \int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} (l_i - x_i + y_i + \varepsilon s)\rho(s)ds.$$

Furthermore, since $u_i - x_i + y_i \geq 0$ and $l_i - x_i + y_i \leq 0$, we have

$$\int_{-\infty}^{\frac{x_i-y_i-u_i}{\varepsilon}} \varepsilon s\rho(s)ds \leq y_i \leq \int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} \varepsilon s\rho(s)ds.$$

Thus,

$$\begin{aligned}
\min(x_i - l_i, u_i - x_i)y_i &\leq (x_i - l_i)\int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} \varepsilon s\rho(s)ds \\
&= (x_i - l_i - y_i + y_i)\int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} \varepsilon s\rho(s)ds \\
&\leq (x_i - l_i - y_i + \kappa\varepsilon)\int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} \varepsilon s\rho(s)ds \\
&= \varepsilon^2 \frac{x_i - l_i - y_i}{\varepsilon}\int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} s\rho(s)ds + \varepsilon^2\kappa\int_{\frac{x_i-y_i-l_i}{\varepsilon}}^{\infty} s\rho(s)ds \\
&\leq \varepsilon^2(\zeta^2 + \kappa^2/2).
\end{aligned}$$

Similarly, we can prove

$$\min(x_i - l_i, u_i - x_i)y_i \geq -\varepsilon^2(\zeta^2 + \kappa^2/2).$$

Therefore,

$$|\min(x_i - l_i, u_i - x_i)y_i| \leq \varepsilon^2(\zeta^2 + \kappa^2/2).$$

∎

Statement (i) of the above theorem implies that $l \leq x \leq u$ ($l < x < u$ if $\mathrm{supp}(\rho) = \{s : \rho(s) > 0\} = \Re$); (ii) together with (i) implies that x might be an exterior point but is approximately feasible; (iii) implies the complementarity gap goes to zero quadratically in ε. Thus, the solution of $f(x, \varepsilon) = 0$, if it exists, converges to the solution set of the VIP as $\varepsilon \to 0$.

The following two assumptions will be used to establish a sufficient condition for the existence of a solution to $f(x, \varepsilon) = r$, where r is a given n-dimensional version.

A1. The level sets

$$D(\Gamma) = \{x \in \Re^n \ : \ \|F(x)\| \leq \Gamma\}$$

are bounded for all positive numbers Γ.

A2. For any $\varepsilon > 0$ and $x \in \Re^n$, $f_1'(x, \varepsilon)$ is nonsingular.

Assumptions related to A1 and A2 have been used in several papers on smoothing methods. We state some here.

Proposition 2.3 *One of the following conditions implies Assumption A1:*
(i) g is a P uniform function [8];
(ii) $-\infty < l_i < u_i < \infty$, for $i = 1, 2, \ldots, n$ [8];
(iii) $l_i = 0$, for $i = 1, 2, \ldots, n$, and g is a R_0 function [4].

One of the following conditions implies Assumption A2:
(i) g is a P uniform function [16];
(ii) g is a P_0 function and supp$\{\rho\} = \Re$ [16].

Many interior-point algorithms (e.g.[22, 34]) for solving the NCP use the following assumption

AIP. g is monotone and the set of all the strictly positive feasible solutions

$$S_{++}(g) = \{x \in \Re^n : g(x) > 0,\ x > 0\}$$

is nonempty.

The existence of a feasible interior point is the standard assumption for any interior point algorithm, and the monotonicity of g is necessary for LCP to have a convex objective [34]. It is interesting to see the relation between interior point methods and smoothing methods from the difference of their assumptions.

Since the monotonicity is stronger than the P_0 property, the assumption AIP implies A2 for smoothing functions with supp$\{\rho\} = \Re$. However, neither of the two assumptions A1 and AIP implies the other. A1 does not imply AIP, because the P uniform property satisfies A1 but fails the monotonicity. The following example shows that AIP does not imply A1. Let $g(x) \equiv q > 0$, where q is a given n-dimensional vector. Then g is monotone and $S_{++}(g)$ is nonempty. However, for $\Gamma \geq \min_{i\in N} q_i := q_{i_0}$,

$$\{x^k \ :\ x^k_{i_0} \to \infty, x^k_i = 0, i \neq i_0\} \subset D(\Gamma) = \{x \ :\ \|\min(x, g(x))\| \leq \Gamma\}.$$

Nevertheless, the following proposition shows that AIP implies the boundness of certain level sets.

Proposition 2.4 *Under the assumption AIP, the level set*

$$D(\Gamma) = \{x \in \Re^n \ :\ \|F(x)\| \leq \Gamma\}$$

are bounded for all positive numbers $\Gamma < \Gamma_0$, where

$$F(x) := \min(x, g(x))$$

and

$$\Gamma_0 := \sup\{\min_{i\in N} F_i(x),\ x \in S_{++}(g)\}.$$

Proof: Let $\Gamma < \Gamma_0$ and $\hat{x} \in S_{++}(g)$ satisfy

$$\Gamma < \min_{i\in N} F_i(\hat{x}).$$

This means that for each $i \in N$, $g_i(\hat{x}) > \Gamma$ and $\hat{x}_i > \Gamma$.

Suppose on the contrary that we have an unbounded sequence $\{x^k\} \subset D(\Gamma)$. Since $x^k \in D(\Gamma)$ implies that $|\min(x^k_i, g_i(x^k))| \leq \Gamma$ for each $i \in N$, there is no index

j such that $x_j^k \to -\infty$ or $g_j(x^k) \to -\infty$. Define the index sets $J = \{i \ : \ x_i^k \to \infty, \ i \in N\}$ and $L = \{\ell \ : \ g_\ell(x^k) \to \infty, \ \ell \in N\}$. By the unbounded assumption on $\{x^k\}$, the set J is nonempty, while the set L can be either empty or nonempty.

Then there is a $k' \geq 0$ such that for all $k \geq k'$,

$$|\min(x_i^k, g_i(x^k))| = |g_i(x^k)| \leq \Gamma, \quad \text{for} \quad i \in J$$

and

$$|\min(x_\ell^k, g_\ell(x^k))| = |x_\ell^k| \leq \Gamma, \quad \text{for} \quad \ell \in L \text{ (if } L \neq \emptyset).$$

Hence we have

$$x_i^k - \hat{x}_i \to \infty, \quad |g_i(x^k)| \leq \Gamma < g_i(\hat{x}) \quad \text{for} \quad k \geq k', \ i \in J$$

and

$$g_\ell(x^k) - g_\ell(\hat{x}) \to \infty, \quad |x_\ell^k| \leq \Gamma < \hat{x}_\ell \quad \text{for} \quad k \geq k', \ \ell \in L \text{ (if } L \neq \emptyset).$$

Note this results in a contradiction with the monotonicity of g, i.e,

$$(g(x^k) - g(\hat{x}))^T (x^k - \hat{x}) \geq 0.$$

Thus $D(\Gamma)$ must be bounded. ∎

Theorem 2.2 *Suppose that assumptions A1 and A2 hold, then for any $\varepsilon > 0$ and $r \in \Re^n$, there is a solution $x(\varepsilon, r)$ of $f(x, \varepsilon) = r$. Furthermore, if g is a P_0 function and supp$\{\rho\} = \Re$. Then, $x(\varepsilon, r)$ is unique.*

Proof: Let $\mu = \kappa\sqrt{n}$. We choose $x^0 \in \Re^n$ and a positive number Γ satisfying $\Gamma > \mu\varepsilon + \|r\|$ and $\Gamma - \mu\varepsilon - \|r\| > \|F(x^0)\|$. We define

$$C_{\varepsilon,r}(\Gamma) = \{x \ : \ \|f(x, \varepsilon) - r\| \leq \Gamma\}.$$

By (2.4), we have

$$\|f(x, \varepsilon) - r\| \leq \|F(x)\| + \mu\varepsilon + \|r\| \tag{2.25}$$

and

$$\|F(x)\| \leq \|f(x, \varepsilon) - r\| + \mu\varepsilon + \|r\|. \tag{2.26}$$

Since $x^0 \in D(\Gamma - \mu\varepsilon - \|r\|)$, (2.25) implies that $x^0 \in C_{\varepsilon,r}(\Gamma)$ and hence $C_{\varepsilon,r}(\Gamma)$ is nonempty. By (2.26), we have

$$C_{\varepsilon,r}(\Gamma) \subseteq D(\Gamma + \mu\varepsilon + \|r\|).$$

Hence, from Assumption A1, the level set $C_{\varepsilon,r}(\Gamma)$ is bounded.

Let $\theta(x) := \frac{1}{2}\|f(x, \varepsilon) - r\|^2$. The boundedness of $C_{\varepsilon,r}(\Gamma)$ and the continuous differentiability of $f(\cdot, \varepsilon)$ imply that θ has a global minimum point x^* in $C_{\varepsilon,r}(\Gamma)$

[26, 4.2.2]. Since $\theta(x^*) \le \theta(x^0)$ and $\|f(x^0,\varepsilon) - r\| < \Gamma$, $x^* \in \text{int } C_{\varepsilon,r}(\Gamma)$. Moreover, by Assumption A2, $f_{\mathbf{1}}'(x^*,\varepsilon)$ is nonsingular. Therefore, by [26, 4.1.3], $\theta'(x^*) = f_{\mathbf{1}}'(x^*,\varepsilon)^T(f(x^*,\varepsilon) - r) = 0$, and so $f(x^*,\varepsilon) = r$.

Now we prove the uniqueness of $x(\varepsilon, r)$.

Let $y = g(x)$, $t = x - y$, and

$$h_i(t) = \int_{-\infty}^{\infty} \text{mid}(l_i, u_i, t_i - \varepsilon s)\rho(s)ds, \quad i = 1, \dots n.$$

By Proposition 2.2, $h_i(t)$ is an increasing function; therefore, an increasing function in x_i and a decreasing function in y_i. Furthermore, by Lemma 2.3 in [16], we can claim $h_i'(t) \in (0,1)$.

Suppose on the contrary that we have x^1 and x^2 are both solutions. Let $y^1 = g(x^1)$ and $y^2 = g(x^2)$. Since $g(x)$ is a P_0 function, we have an index i such that

$$x_i^1 \ne x_i^2 \quad \text{and} \quad (y_i^1 - y_i^2)(x_i^1 - x_i^2) \ge 0.$$

Assume that, without loss of generality, $x_i^1 > x_i^2$. Then we have $y_i^1 \ge y_i^2$. Moreover,

$$\begin{aligned} & x_i^1 - x_i^2 \\ &= h_i(x_i^1 - y_i^1) - h_i(x_i^2 - y_i^2) \\ &\le h_i(x_i^1 - y_i^2) - h_i(x_i^2 - y_i^2) \\ &= h_i'(\theta)(x_i^1 - x_i^2) \\ &< x_i^1 - x_i^2, \end{aligned}$$

where θ is between $x_i^1 - y_i^2$ and $x_i^2 - y_i^2$ according to the Taylor theorem. Note this results in a contradiction. Thus, we must have $x^1 = x^2$.

∎

3 Algorithm and convergence

In this section, we propose a hybrid Newton-smoothing algorithm for solving VI(l,u,g). We show that the algorithm converges globally and superlinearly under assumptions A1 and A2. Furthermore, the method is finite convergence, if

$$g(x) = Mx + q,$$

where M is an $n \times n$ matrix and q is an n-dimensional vector. In this case, we denote the linear variational inequality by VI(l,u,M,q).

We denote

$$\Theta(x) = \frac{1}{2}\|F(x)\|^2$$

and

$$\theta_k(x) = \frac{1}{2}\|f(x,\varepsilon_k)\|^2.$$

Algorithm 3.1 *Given $\rho, \alpha, \eta \in (0,1)$, and a starting point $x^0 \in \Re^n$. Choose a scalar $\sigma \in (0, \frac{1}{2}(1-\alpha))$. Let $\nu = \frac{\alpha}{2\sqrt{n}\kappa}$. Let $\beta_0 = \|F(x^0)\|$ and $\varepsilon_0 = \nu\beta_0$.*
For $k \geq 0$:

1. *Find a solution $\hat{d}^k$ of the system of linear equations*

$$F(x^k) + f^0(x^k)d = 0. \tag{3.1}$$

If $\|F(x^k + \hat{d}^k)\| \leq \eta\beta_k$, let $x^{k+1} = x^k + \hat{d}^k$. Otherwise perform Step 2.

2. *Find a solution d^k of the system of linear equations*

$$F(x^k) + f_{\mathbf{1}}'(x^k, \varepsilon_k)d = 0. \tag{3.2}$$

Let m_k be the smallest nonnegative integer m such that

$$\theta_k(x^k + \rho^m d^k) - \theta_k(x^k) \leq -2\sigma\rho^m \Theta(x^k). \tag{3.3}$$

Set $t_k = \rho^{m_k}$ and $x^{k+1} = x^k + t_k d^k$.

3. *3.1 If $\|F(x^{k+1})\| = 0$, terminate.*

   3.2 If

$$0 < \|F(x^{k+1})\| \leq \max\{\eta\beta_k, \alpha^{-1}\|F(x^{k+1}) - f(x^{k+1}, \varepsilon_k)\|\}, \tag{3.4}$$

let

$$\beta_{k+1} = \|F(x^{k+1})\| \quad \text{and} \quad \varepsilon_{k+1} = \min\{\nu\beta_{k+1}, \frac{\varepsilon_k}{2}\}.$$

   3.3 Otherwise, let $\beta_{k+1} = \beta_k$ and $\varepsilon_{k+1} = \varepsilon_k$.

Theorem 3.1 *Suppose that assumptions A1 and A2 hold. Then for any starting point $x^0 \in \Re^n$, Algorithm 3.1 is well defined and the generated sequence $\{x^k\}$ remains in $D_0 := D((1+\alpha)\|F(x^0)\|)$ and satisfies*

$$\lim_{k\to\infty} \|F(x^k)\| = 0. \tag{3.5}$$

Proof: By Lemma 3.1 in [8], there exists a finite nonnegative integer m_k such that (3.3) holds. Hence Algorithm 3.1 is well defined.

Let

$$K = \{k \ : \ \|F(x^k + \hat{d}^k)\| \leq \eta\beta_k, \quad k \geq 0\}.$$

By the construction of Algorithm 3.1, for $k \in K$, $x^{k+1} = x^k + \hat{d}^k$ and $\beta_{k+1} = \|F(x^{k+1})\| \leq \eta\beta_k \leq (1+\alpha)\|F(x^0)\|$. Hence $x^{k+1} \in D_0$ for $k \in K$. Moreover, following the proof of Theorem 3.1 in [8], we can show $x^{k+1} \in D_0$ for $k \notin K$. Hence $\{x^k\}$ remains in D_0.

Now we prove (3.5).

If K is finite, then there exists $k_0 \geq 0$ such that Step 2 is performed for all $k \geq k_0$. By Theorem 3.1 in [8]

$$\lim_{\substack{k\to\infty \\ k\geq k_0}} \|F(x^k)\| = 0.$$

Hence the whole sequence $\{x^k\}$ satisfies (3.5).

If K is infinite, then by the construction of Algorithm 3.1,

$$\lim_{\substack{k\to\infty \\ k\in K}} \|F(x^k)\| = 0.$$

Let

$$\bar{K} = \{k \ : \ \|F(x^{k+1})\| \leq \max\{\eta\beta_k, \alpha^{-1}\|F(x^{k+1}) - f(x^{k+1}, \varepsilon_k)\|\}\}.$$

Then $K \subseteq \bar{K}$. Assume that $\bar{K}$ consists of $k_0 = 0 < k_1 < k_2 < \ldots$. Let k be an arbitrary nonnegative integer. Let k_j be the largest number in $\bar{K}$ such that $k_j \leq k$. Then $\varepsilon_k = \varepsilon_{k_j}$ and $\beta_k = \beta_{k_j}$. By the line search rule

$$\|f(x^k, \varepsilon_{k_j})\| \leq \|f(x^{k_j}, \varepsilon_{k_j})\|.$$

Hence by (2.4), for $j \geq 0$

$$\begin{aligned}
\|F(x^k)\| &\leq \|f(x^k, \varepsilon_k)\| + \|F(x^k) - f(x^k, \varepsilon_k)\| \\
&= \|f(x^k, \varepsilon_{k_j})\| + \|F(x^k) - f(x^k, \varepsilon_{k_j})\| \\
&\leq \|f(x^{k_j}, \varepsilon_{k_j})\| + \mu\varepsilon_{k_j} \\
&\leq \|F(x^{k_j})\| + 2\mu\varepsilon_{k_j} \\
&= \beta_{k_j} + 2\mu\varepsilon_{k_j},
\end{aligned}$$

where $\mu = \kappa\sqrt{n}$.

Since K is infinite, $\bar{K}$ is infinite and hence $\beta_{k_j} \to 0$ and $\varepsilon_{k_j} \to 0$, as $j \to \infty$. Therefore, the whole sequence $\{x^k\}$ satisfies (3.5). ∎

Corollary 3.1 *Suppose that assumption AIP holds and $l_i = 0, u_i = \infty$ for each $i \in N$. Then for any smoothing function f with supp$\{\rho\} = \Re$ and any staring point $x^0 \in \Re^n$, Algorithm 3.1 is well defined and the generated sequence $\{x^k\}$ remains in $D_0 := D((1+\alpha)\|F(x^0)\|)$. Furthermore, if x^0 satisfies*

$$(1+\alpha)\|F(x^0)\| < \sup\{\min_{i\in N} F_i(x),\, x \in S_{++}(g)\}, \tag{3.6}$$

then $\lim_{k\to\infty} \|F(x^k)\| = 0$.

Proof: By Theorem 4.2 in [16], the monotonicity of g implies the assumption A2. By Lemma 3.1 in [8], there is a finite nonnegative integer m_k such that (3.3) holds. Hence Algorithm 3.1 is well defined. Following the proof of Theorem 3.1 in [8] and Theorem 3.1 here, we can show $\{x^k\} \subset D_0$ for any starting point $x^0 \in \Re^n$. By

Proposition 2.4, if x^0 satisfies (3.6), then D_0 is bounded. Therefore, we can show that $\lim_{k\to\infty} \|F(x^k)\| = 0$ by following the proof of Theorem 3.1. ∎

The following lemma shows that if an iterate is sufficiently close to a solution of VI(l, u, M, q), Algorithm 3.1 finds the solution in one step. We definite

$$\gamma(x) = \min_{1\le i,j\le n}\{|x - Mx - q - l|_i, |x - Mx - q - u|_j \ : \ (x - Mx - q - l)_i \neq 0, (x - Mx - q - u)_j \neq 0\}.$$

Since $l_i < u_i$ for all $i \in N$, $\gamma(x) > 0$ for any $x \in \Re^n$.

Lemma 3.1 *Let $x^* \in \Re^n$ be a solution of VI(l, u, M, q). Let*

$$S := \begin{cases} \{x \in \Re^n \ : \ \|x - x^*\|_\infty \le \gamma(x^*)/\|I - M\|_\infty\} & \textit{if } I \neq M \\ \Re^n & \textit{otherwise.} \end{cases} \tag{3.7}$$

Then for any $x \in S$

$$F(x) + f^0(x)(x^* - x) = 0. \tag{3.8}$$

Proof: If $M = I$, then $f^0(x) = I$ for any $x \in \Re^n$. Moreover, $x^* =$mid$(l, u, -q)$ is a solution of $F(x) = 0$. Hence, in this case for any $x \in \Re^n$,

$$F(x) + f^0(x)(x^* - x) = x - \text{mid}(l, u, -q) + (x^* - x) = 0.$$

Suppose that $I \neq M$ and $x \in S$. Then

$$\begin{aligned} |x - Mx - q - x^* + Mx^* + q|_i &\le \|I - M\|_\infty \|x - x^*\|_\infty \\ &\le \gamma(x^*). \end{aligned} \tag{3.9}$$

For a fixed i, we consider two cases.

Case 1. $(x^* - Mx^* - q)_i \neq l_i$ and $(x^* - Mx^* - q)_i \neq u_i$.

By the definition $\gamma(x)$ and (3.9), we have

$$(x^* - Mx^* - q)_i < l_i \quad \Rightarrow (x - Mx - q)_i < l_i \tag{3.10}$$

$$l_i < (x^* - Mx^* - q)_i < u_i \quad \Rightarrow l_i < (x - Mx - q)_i < u_i \tag{3.11}$$

and

$$(x^* - Mx^* - q)_i > u_i \quad \Rightarrow (x - Mx - q)_i > u_i. \tag{3.12}$$

If $(x^* - Mx^* - q)_i \in (l_i, u_i)$, then $(Mx^* + q)_i = 0$. By (3.11) and (2.6), $f_i^0(x) = M_i$ and

$$F_i(x) + f_i^0(x)(x^* - x) = x_i - (x - Mx - q)_i + M_i(x^* - x) = (Mx^* + q)_i = 0. \tag{3.13}$$

If $(x^* - Mx^* - q)_i < l_i$, then $x_i^* = l_i$. By (3.10) and (2.6), $f_i^0(x) = I_i$ and

$$F_i(x) + f_i^0(x)(x^* - x) = x_i - l_i + I_i(x^* - x) = x_i^* - l_i = 0. \tag{3.14}$$

Similarly, we can show

$$F_i(x) + f_i^0(x)(x^* - x) = x_i^* - u_i = 0,$$

for $(x^* - Mx^* - q)_i > u_i$.

Case 2. $(x^* - Mx^* - q)_i = l_i$ or $(x^* - Mx^* - q)_i = u_i$.

If $(x^* - Mx^* - q)_i = l_i$, then $x_i^* = l_i$ and $(Mx^* + q)_i = 0$.

If $(x - Mx - q)_i > l_i$ or $(x - Mx - q)_i < l_i$, we can find $F_i(x) + f_i^0(x)(x^* - x) = 0$ by following the same argument in (3.13) and (3.14). If $(x - Mx - q)_i = l_i$, then $f_i^0(x) = I_i - \lambda(I_i - M_i)$, where $\lambda \in (0, 1)$. Furthermore, we have

$$\begin{aligned}
&F_i(x) + f_i^0(x)(x^* - x) \\
&= x_i - l_i + (I_i - \lambda(I_i - M_i))(x^* - x) \\
&= \lambda(M_i - I_i)(x^* - x) \\
&= \lambda(x_i - l_i + (Mx^* + q - Mx - q)_i) \\
&= \lambda((x - l - Mx - q)_i + (Mx^* + q)_i) = 0.
\end{aligned}$$

The proof of the case $(x^* - Mx^* - q)_i = u_i$ is similar.

∎

Theorem 3.2 *Suppose that A1 and A2 hold. Assume that for any accumulation point $\bar{x}$ of $\{x^k\}$, there are an open ball $S := S(\bar{x}, \bar{r}) = \{x : \|x - x^*\| < \bar{r}\}$ and a positive number Υ such that for any $x \in S$, $f^0(x)$ is nonsingular and $\|f^0(x)^{-1}\| \le \Upsilon$. Then $\{x^k\}$ converges to a solution x^* of (1.3) superlinearly. Moreover, if g has a locally Lipschitz continuous derivative around x^*, the convergence rate is quadratic. In addition, if g is an affine function, the convergence is finite.*

Proof: Let $\bar{x}$ be an accumulation point of $\{x^k\}$. By Theorem 3.1, $\bar{x}$ is a solution of (1.3). By Lemma 2.4 in [6], $\bar{x}$ is the unique solution of (1.3) in S. Hence $\{x^k\}$ only has a finite number of accumulation points. Let x^* be an accumulation point and K_* be a subsequence of $\{0, 1, \ldots, \}$ such that

$$\lim_{\substack{k \to \infty \\ k \in K_*}} x^k = x^*.$$

By the directional derivative consistence property of f at x^*, for $k \in K_*$,

$$\begin{aligned}
&\|x^k + \hat{d}^k - x^*\| \\
&= \|x^k - f^0(x^k)^{-1} F(x^k) - x^*\| \\
&= \|f^0(x^k)^{-1}(f^0(x^k)(x^k - x^*) - F(x^k) + F(x^*))\| \\
&\le \Upsilon \|F(x^k) - F(x^*) - f^0(x^k)(x^k - x^*)\| \\
&= o(\|x^k - x^*\|).
\end{aligned} \tag{3.15}$$

Following the proof of Theorem 3.1 in [28],

$$\|F(x^k + \hat{d}^k)\| = o(\|F(x^k)\|).$$

This implies that there is $k_* \in K_*$ such that for all $k \geq k_*$

$$\|F(x^k + \hat{d}^k)\| \leq \eta \|F(x^k)\|$$

and $x^{k+1} = x^k + \hat{d}^k$. Hence $\{x^k\}$ converges to x^* superlinearly.

Moreover, if g has a locally Lipschitz continuous derivative, then F is strongly semi-smooth at x^* [28, 8]. Since f has the Jacobian consistence property [8] and $f^0(x) \in \partial_C F(x)$ for any $x \in \Re^n$ [6],

$$\|F(x^k) - F(x^*) - f^0(x^k)(x^k - x^*)\| = O(\|x^k - x^*\|^2).$$

Following the argument in (3.15), we obtain the quadratic convergence rate.

The finite convergence follows from Lemma 3.1 and that $k \in K_*$ for all $k \geq k_*$. ∎

Remark 3.1 If g is a P-uniform function, by Theorem 4.3 in [16], for any $x \in \Re^n$, $f^0(x)$ is nonsingular. Hence, by Proposition 2.3, Algorithm 3.1 converges globally and superlinearly for VI(l, u, g), and converges globally and finitely for VI(l, u, M, q), assuming only that g is a P-uniform function. It is notable that global convergence of Algorithm 3.1 only requires that g be a P_0 function and the level sets $\{x \ : \ \|F(x)\| \leq \Gamma\}$ with $\Gamma > 0$ be bounded. Furthermore, Algorithm 3.1 converges to the solution set of the NCP, if g is monotone, $S_{++}(g)$ is nonempty and the starting point is in a certain level set.

4 Numerical experiments

We numerically tested Algorithm 3.1 with five examples containing 24 linear box constrained variational inequality problems. We use three smooth approximation functions which are generated by (2.3) and density functions ρ satisfying (2.1) and (2.2). (cf. [4, 5, 6, 7, 8, 12, 16, 29]).

Let

$$z = x - Mx - q.$$

Neural networks smooth approximation function (S1)

Let the density function be

$$\rho(x) = \frac{e^{-s}}{(1 + e^{-s})^2}.$$

The smooth approximation function is

$$f_i(x, \varepsilon) = x_i - u_i - \varepsilon \log(1 + e^{(l_i - z_i)/\varepsilon}) + \varepsilon \log(1 + e^{(u_i - z_i)/\varepsilon})$$

and $f_1'(x, \varepsilon) = I - D(x)(I - M)$ where

$$d_{ii}(x) = e^{(l_i - z_i)/\varepsilon}/(1 + e^{(l_i - z_i)/\varepsilon}) - e^{(u_i - z_i)/\varepsilon}/(1 + e^{(u_i - z_i)/\varepsilon}).$$

Chen-Harker-Kanzow-Smale smooth approximation function (S2)

Let the density function be

$$\rho(s) = \frac{2}{(s^2+4)^{\frac{3}{2}}}.$$

The smooth approximation function is

$$f_i(x,\varepsilon) = x_i - \frac{1}{2}(\sqrt{(l_i-z_i)^2+4\varepsilon^2} - \sqrt{(u_i-z_i)_i^2+4\varepsilon^2} + u_i + l_i)$$

and $f'_{\mathbf{1}}(x,\varepsilon) = I - D(x)(I-M)$ where

$$d_{ii}(x) = \frac{1}{2}(\frac{z_i-l_i}{\sqrt{(z_i-l_i)^2+4\varepsilon^2}} - \frac{z_i-u_i}{\sqrt{(z_i-u_i)^2+4\varepsilon^2}}).$$

Uniform smooth approximation function (S3)

Let the density function be

$$\rho(s) = \begin{cases} 1, & |s| \le 0.5 \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < \varepsilon \le \min_{1\le i\le n}\{u_i - l_i\}$,

$$f_i(x,\varepsilon) = \begin{cases} \frac{1}{2}(Mx+q+x)_i + \frac{1}{2\varepsilon}(u_i-z_i)^2 + \frac{\varepsilon}{8} - \frac{u_i}{2} & \text{if } |u_i-z_i| \le \frac{\varepsilon}{2}, \\ \frac{1}{2}(Mx+q+x)_i - \frac{1}{2\varepsilon}(l_i-z_i)^2 - \frac{\varepsilon}{8} - \frac{l_i}{2} & \text{if } |l_i-z_i| \le \frac{\varepsilon}{2}, \\ F_i(x) & \text{otherwise,} \end{cases}$$

and

$$f'_{\mathbf{1}}(x,\varepsilon) = \begin{cases} \frac{1}{2}(I_i+M_i) + \frac{1}{\varepsilon}(u_i-z_i)(M_i-I_i) & \text{if } |u_i-z_i| \le \frac{\varepsilon}{2}, \\ \frac{1}{2}(I_i+M_i) - \frac{1}{\varepsilon}(l_i-z_i)(M_i-I_i) & \text{if } |l_i-z_i| \le \frac{\varepsilon}{2}, \\ F'_i(x) & \text{otherwise.} \end{cases}$$

The three density functions have a common property: $\rho(-s) = \rho(s)$. By (2.6), this implies that for any $x \in \Re^n$, the derivatives $f'_{\mathbf{1}}(x,\varepsilon)$ of the three smooth approximation functions have the same limit $f^0(x)$ as $\varepsilon \downarrow 0$, where

$$f_i^0(x) = \begin{cases} M_i & \text{if } z_i \in (l_i,u_i), \\ I_i & \text{if } z_i \notin [l_i,u_i], \\ \frac{1}{2}(I_i+M_i) & \text{otherwise.} \end{cases} \tag{4.16}$$

Furthermore, the three density functions satisfy (2.1) and (2.2).

We chose $x^0 = e$, $\rho = 0.75$, $\alpha = 0.56$, $\eta = 0.87$ and $\sigma = 0.2$ in Algorithm 3.1. The stopping criterion was $\|F(x^k)\| \le 10^{-8}$. Numerical results were obtained using Matlab 4.2c on a Sun 2000 workstation. These testing problems are constructed from some testing problems for LCP. We added different lower and upper bounds on the variables. We report the iterations k, the total numbers of line search steps $\sum_{i=1}^k m_i$ and CUP time in Table 1 – Table 5 for solving these problems.

Table 1: Example 4.1, iterations, total number of line search steps, CPU(sec.)

$l = 0$, $u = 10^7 e$				
n	100	200	400	500
S1	2, 0, 0.3	2, 0, 2.1	2, 0, 20.5	2, 0, 45.0
S2	2, 0, 0.2	2, 0, 1.6	2, 0, 18.3	2, 0, 45.5
S3	2, 0, 0.2	2, 0, 1.5	2, 0, 10.0	2, 0, 30.9
$l = 0.5e$, $u = e$				
n	50	100	200	400
S1	2, 0, 0.1	2, 0, 0.3	2, 0, 1.8	2, 0, 12.5
S2	2, 0, 0.0	2, 0, 0.2	2, 0, 2.0	4, 0, 24.3
S3	2, 0, 0.0	2, 0, 0.2	2, 0, 1.9	2, 0, 7.3
$l = -e$, $u = e$				
n	50	100	200	300
S1	12, 32, 0.8	16, 57, 3.4	22, 103, 25.3	28, 143, 114.1
S2	12, 32, 0.3	18, 57, 2.2	22, 97, 20.8	28, 142, 106.0
S3	13, 35, 0.5	16, 58, 2.4	24, 107, 21.9	27, 141, 66.9
$l = -e$, $u = 0$				
n	50	100	200	400
S1	3, 0, 0.1	3, 0, 0.4	3, 0, 3.0	3, 0, 29.6
S2	3, 0, 0.1	3, 0, 0.3	3, 0, 2.9	3, 0, 29.6
S3	3, 0, 0.1	3, 0, 0.3	3, 0, 2.6	3, 0, 18.9
$l = 0$, $u_i = 1$ for i is even, $u_i = 10^7$ for i is odd.				
n	50	100	150	180
S1	2, 0, 0.1	2, 0, 0.2	2, 0, 0.8	2, 0, 1.5
S2	2, 0, 0.0	2, 0, 0.2	2, 0, 0.7	2, 0, 1.4
S3	2, 0, 0.0	2, 0, 0.2	2, 0, 0.8	2, 0, 1.0
$l_i = -1.0$, $u_i = 0.5$ for i is even, $l_i = -0.5$, $u_i = 1.0$ for i is odd.				
n	50	100	150	180
S1	14, 32, 0.7	18, 58, 3.0	25, 108, 26.4	26, 104, 17
S2	14, 30, 0.3	17, 53, 1.7	24, 101, 21.4	7, 10, 3.4
S3	14, 32, 0.5	18, 58, 2.2	25, 108, 19.2	45, 307, 24.4

Example 4.1 *(Murty[25])*

$$M = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad q = -e.$$

Example 4.2 *(Fathi[11])*

$$M = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 2 & 5 & 6 & \dots & 6 \\ 2 & 6 & 9 & \dots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \dots & 4(n-1)+1 \end{pmatrix}, \qquad q = -e.$$

Table 2: Example 4.2, iterations, total number of line search steps, CPU(sec.)

$l=0$, $u=10^7e$				
n	100	200	300	400
S1	8, 21, 1.1	9, 34, 7.7	8, 19, 24.5	9, 22, 81.0
S2	8, 21, 0.9	9, 34, 7.8	8, 19, 26.0	9, 22, 89.7
S3	8, 21, 0.8	9, 34, 5.3	8, 19, 12.9	9, 22, 40.9
$l=0.5e$, $u=e$				
n	100	200	300	400
S1	2, 0, 0.2	2, 0, 1.4	2, 0, 5.6	2, 0, 16.2
S2	2, 0, 0.2	2, 0, 1.3	2, 0, 5.4	2, 0, 16.6
S3	2, 0, 0.2	2, 0, 1.0	2, 0, 3.0	2, 0, 9.2
$l=-e$, $u=0$				
n	100	200	300	400
S1	3, 0, 0.3	3, 0, 2.2	3, 0, 8.4	3, 0, 24.2
S2	3, 0, 0.2	3, 0, 2.0	3, 0, 8.1	3, 0, 25.5
S3	3, 0, 0.2	3, 0, 1.4	3, 0, 4.5	3, 0, 13.4
$l=-10e$, $u=5$				
n	64	128	256	320
S1	17, 69, 1.4	26, 147, 10.3	15, 207, 43.1	26, 187, 140.1
S2	12, 57, 0.5	11, 108, 3.2	25, 164, 66.3	20, 153, 100.4
S3	17, 71, 1.0	10, 101, 2.8	25, 180, 41.5	16, 236, 43.5
$l=0$, $u_i=1$ for i is even, $u_i=10^7$ for i is odd.				
n	32	64	128	256
S1	5, 3, 0.1	7, 13, 0.4	6, 4, 2.0	8, 14, 19.1
S2	5, 3, 0.1	7, 13, 0.3	6, 4, 1.9	8, 14, 20.6
S3	5, 3, 0.1	7, 13, 0.3	6, 4, 1.3	8, 14, 10.1
$u=0$, $l_i=-1.0$ for i is even, $l_i=-10^7$ for i is odd.				
n	32	64	128	256
S1	4, 0, 0.1	4, 0, 0.2	6, 0, 2.1	4, 0, 10.8
S2	4, 0, 0.0	4, 0, 0.2	6, 0, 1.9	4, 0, 10.4
S3	4, 0, 0.1	4, 0, 0.2	6, 0, 1.6	4, 0, 6.7

Example 4.3 *(Ahn[1])*

$$M = \begin{pmatrix} 4 & -2 & 0 & 0 & \dots & 0 \\ 1 & 4 & -2 & 0 & \dots & 0 \\ 0 & 1 & 4 & -2 & 0\dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 & 4 \end{pmatrix}, \qquad q = -4e.$$

Example 4.4 *We randomly generated 100% dense $A \in \Re^{n\times n}$ and $q \in \Re^n$ whose elements distributed in $(-5,5)$. We used the QR decomposition of A to get a upper triangular matrix N. Then we replaced the diagonal elements of N by their absolute values and obtained a triangular matrix M with positive diagonal elements. The matrix M is P matrix.*

Example 4.5 *(Harker and Pang [18]) We randomly generated 100% dense $A \in \Re^{n\times n}$, $B \in \Re^{\frac{n}{2}\times\frac{n}{2}}$, $d \in \Re^n$ and $q \in \Re^n$, where $a_{ij} \in (-5,5)$, $b_{i,j} \in (-5,5)$, $d_i \in (0,0.3)$*

Table 3: Example 4.3, iterations, total number of line search steps, CPU(sec.)

$l = 0,\ u = 10^7 e$				
n	100	200	300	400
S1	3, 0, 0.3	3, 0, 2.2	3, 0, 8.8	3, 0, 24.7
S2	3, 0, 0.3	3, 0, 2.1	3, 0, 8.5	3, 0, 25.3
S3	3, 0, 0.2	3, 0, 1.4	3, 0, 4.6	3, 0, 14.8
$l = 0,\ u = e$				
n	100	200	300	400
S1	4, 2, 0.5	4, 2, 3.1	4, 2, 11.7	4, 2, 33.2
S2	4, 2, 0.3	4, 2, 2.8	4, 2, 11.3	4, 2, 33.0
S3	4, 2, 0.3	4, 2, 2.0	4, 2, 6.2	4, 2, 20.9
$l = -e,\ u = e$				
n	100	200	300	400
S1	2, 0, 0.2	2, 0, 1.3	2, 0, 5.7	2, 0, 16.2
S2	2, 0, 0.2	2, 0, 1.3	2, 0, 5.6	2, 0, 16.3
S3	2, 0, 0.1	2, 0, 0.9	2, 0, 3.0	2, 0, 8.6
$l = -e,\ u = 0$				
n	100	200	300	400
S1	3, 0, 0.3	3, 0, 2.2	3, 0, 8.6	3, 0, 24.0
S2	3, 0, 0.3	3, 0, 2.1	3, 0, 8.4	3, 0, 24.5
S3	3, 0, 0.2	3, 0, 1.5	3, 0, 4.6	3, 0, 13.8
$l_i = -1,\ u_i = 1$ for i is even, $l_i = 0,\ u_i = 10^7$ for i is odd.				
n	100	200	300	400
S1	2, 0, 0.2	2, 0, 1.6	2, 0, 6.0	2, 0, 22.3
S2	2, 0, 0.2	2, 0, 1.4	2, 0, 5.8	2, 0, 21.5
S3	2, 0, 0.2	2, 0, 1.0	2, 0, 3.1	2, 0, 12.9
$l_i = -1.0,\ u_i = 0.5$ for i is even, $l_i = -0.5,\ u_i = 1.0$ for i is odd.				
n	100	200	300	400
S1	3, 0, 0.3	3, 0, 2.3	3, 0, 8.6	3, 0, 30.6
S2	3, 0, 0.3	3, 0, 2.1	3, 0, 10.6	3, 0, 33.3
S3	3, 0, 0.2	3, 0, 1.4	3, 0, 4.6	3, 0, 18.6

and $q_i \in (-500, 500)$. *We define a P-matrix as*

$$M = A^T A + \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix} + \mathrm{diag}(d).$$

Remark 4.1. Both Lemke's complementarity pivot algorithm and the Cottle and Danzig's principal pivoting method are known to run in exponential time for Example 4.1 and Example 4.2 with $l_i = 0, u_i = \infty$ (i.e. the LCP). Table 1 and Table 2 show that the hybrid Newton-smoothing method takes only a few iterations for the LCP. It is notable that the LCP is relatively easier than the problem having different lower and upper bounds. To see the reason, we consider Example 4.1. The solution for the LCP is

$$x^* = (0, 0, \dots, 0, 1)^T,$$

and

$$Mx^* + q = (1, 1, \dots, 1, 0)^T.$$

In this case $q_i(x) + p_i(x) > 0$, $i = 1, 2, \dots, n$, i.e., x^* is a strictly complementary solution. The function F is differentiable at x^*.

Table 4: Example 4.4, iterations, total number of line search steps, CPU(sec.)

$l = 0$, $u = 10^7 e$					
n		100	200	300	400
	Max	8, 12, 1.1	8, 3, 8.3	7, 0, 28.1	7, 0, 69.1
S1	Avg	7, 2, 0.8	7, 0, 5.2	7, 0, 27.7	7, 0, 67.1
	Min	5, 1, 0.6	6, 2, 6.4	6, 0, 24.8	6, 0, 62.5
	Max	9, 5, 0.8	9, 7, 8.9	8, 4, 31.4	7, 0, 67.6
S2	Avg	7, 5, 0.7	8, 6 6.6	7, 0, 28.6	7, 0, 67.5
	Min	6, 1, 0.6	7, 0, 4.9	6, 0, 23.9	6, 0, 60.0
	Max	8, 12, 0.8	12, 17, 10.4	7, 1, 21.1	7, 0, 39.1
S3	Avg	7, 4, 0.7	7, 2, 6.7	7, 0, 19.1	7, 0, 38.9
	Min	5, 1, 0.5	7, 0, 3.6	6, 0, 18.4	6, 0, 38.7
$l_i = -1, u_i = 1.0$ for i is even, $l_i = 0, u_i = 10^7$ for i is odd					
n		100	200	300	400
	Max	24, 78, 4.0	33, 180, 33.7	30, 133, 115.7	42, 202, 421.3
S1	Avg	22, 87, 3.9	32, 126, 31.4	28, 119, 109.2	41, 200, 342.1
	Min	22, 53, 3.4	28, 113, 26.8	27, 119, 103.5	41, 196, 337.1
	Max	20, 67, 1.8	35, 160, 26.4	42, 188, 157.4	41, 196, 398.7
S2	Avg	19, 58, 1.8	32, 135, 24.2	34, 146, 126.1	38, 193, 300.0
	Min	15, 30, 1.3	26, 101, 20.6	32, 145, 117.3	31, 147, 243.8
	Max	24, 90, 2.8	32, 141, 21.2	33, 149, 85.4	43, 203, 338.8
S3	Avg	19, 75, 2.4	27, 100, 16.8	32, 138, 82.4	40, 183, 206.1
	Min	14, 37, 1.6	25, 99, 15.9	27, 113, 70.2	37, 174, 190.1
$l_i = -100, u_i = 0.0$ for i is even, $l_i = 0, u_i = 100$ for i is odd					
n		100	200	300	400
	Max	9, 7, 1.2	10, 2, 7.3	13, 16, 40.4	16, 32, 132.2
S1	Avg	7, 0, 0.8	8, 0, 5.7	12, 14, 35.4	12, 18, 100.0
	Min	6, 0, 0.7	7, 1, 5.1	10, 14, 30.3	10, 4, 82.4
	Max	8, 5, 0.7	9, 4, 6.0	11, 8, 31.4	12, 18, 95.3
S2	Avg	7, 0, 0.6	8, 0, 5.4	10, 7, 29.4	12, 7, 93.5
	Min	6, 0, 0.5	7, 0, 4.8	10, 3, 27.9	11, 2, 88.2
	Max	7, 2, 0.8	10, 6, 5.5	15, 27, 27.1	16, 33, 69.6
S3	Avg	7, 1, 0.7	8, 1, 4.4	12, 18, 21.4	15, 31, 67.6
	Min	6, 0, 0.5	7, 1, 3.9	12, 16, 21.4	13, 14, 62.0

For $0 < l < u = 1$, the solution for $\mathrm{VI}(l, u, M, q)$ is

$$x^* = (l_1, \ldots, l_{n-1}, u),$$

and

$$(Mx^* + q)_i > 0, \quad i = 1, \ldots, n-1, \quad (Mx^* + q)_n = 0.$$

In this case, $q_i(x) + p_i(x) > 0$, $i = 1, 2, \ldots, n-1$ and $q_n(x) + p_n(x) = 0$.

This problem is relatively easy to solve, if $l \geq 0$ or $u \leq 0$. However, this problem becomes hard if $l < 0$ but $u > 0$. For example, if $l_i = -1$ and $u_i = 1$ for each $i \in N$, the solution is

$$x^* = (1, -1, 1, -1, \ldots, 1), \quad \text{if } n \text{ is odd}$$

or

$$x^* = (-1, 1, -1, 1, \ldots, 1), \quad \text{if } n \text{ is even},$$

and

$$Mx^* + q = 0.$$

Table 5: Example 4.5. iterations,total number of line search steps,CPU(sec.)

$l = 0$, $u = 10^7 e$					
n		50	100	200	300
S1	Max	15, 114, 1.3	19, 132, 4.4	7, 0, 6.8	7, 0, 30.7
	Avg	9, 33, 0.6	9, 25, 2.1	6, 0, 6.7	7, 0, 29.3
	Min	5, 1, 0.2	7, 0, 1.2	6, 0, 5.9	6, 0, 24.8
S2	Max	31, 271, 1.0	48, 534, 7.1	7, 0, 6.4	7, 0, 30.2
	Avg	22, 210, 1.0	36, 295, 6.1	6, 0, 6.0	7, 0, 29.0
	Min	19, 144, 0.6	7, 0, 1.1	6, 0, 5.7	6, 0, 25.5
S3	Max	14, 66, 0.9	15, 67, 2.8	7, 0, 5.4	7, 0, 22.2
	Avg	6, 1, 0.2	10, 7, 2.1	6, 0, 4.9	7, 0, 20.2
	Min	5, 0, 0.2	7, 0, 1.1	6, 0, 4.7	6, 0, 16.4
$l = -10^7$, $u = 0$					
n		100	200	300	400
S1	Max	23, 171, 6.0	8, 0, 10.7	10, 19, 36.0	10, 21, 101.3
	Avg	14, 60, 3.0	8, 0, 10.0	7, 0, 24.4	8, 0, 77.3
	Min	7, 0, 0.9	7, 0, 9.4	7, 0, 23.7	7, 0, 23.7
S2	Max	18, 57, 24.5	8, 0, 9.9	10, 10, 34.5	10, 3, 99.3
	Avg	12, 26, 1.3	8, 0, 9.5	7, 0, 23.6	8, 0, 72.9
	Min	7, 0, 0.7	7, 0, 8.4	7, 0, 23.6	8, 0, 72.7
S3	Max	23, 145, 4.5	8, 0, 10.1	10, 19, 22.5	10, 21, 58.8
	Avg	14, 55, 2.2	8, 0, 9.8	7, 0, 15.0	8, 0, 45.5
	Min	7, 0, 0.8	7, 0, 9.2	7, 0, 14.6	8, 0, 42.3
$l_i = 0$, $u_i = 10^7$ for i is even, $l_i = -10^7$, $u_i = 0$ for i is odd.					
n		100	200	300	400
S1	Max	7, 0, 0.9	7, 0, 8.9	7, 0, 24.2	8, 0, 78.4
	Avg	6, 0, 0.8	7, 0, 8.8	7, 0, 23.9	8, 0, 77.2
	Min	6, 0, 0.7	6, 0, 7.5	6, 0, 20.4	7, 0, 68.2
S2	Max	7, 0, 0.8	7, 0, 8.3	7, 0, 23.5	8, 0, 73.7
	Avg	6, 0, 0.7	7, 0, 8.2	7, 0, 22.9	8, 0, 73.1
	Min	6, 0, 0.6	6, 0, 7.5	6, 0, 20.1	7, 0, 61.4
S3	Max	7, 0, 0.8	7, 0, 8.9	7, 0, 15.7	8, 0, 46.8
	Avg	6, 0, 0.7	7, 0, 8.7	7, 0, 14.4	8, 0, 45.4
	Min	6, 0, 0.6	6, 0, 7.1	6, 0, 12.7	7, 0, 40.4

This means $p_i(x) + q_i(x) = 0$ for $i = 1, \dots, n$.

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