Interior-Point Algorithms: 1994 Annual Progress Report on NSF Grant DDM-9207347

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1 Introduction

This document is the second year progress report on an optimization project funded by NSF Grant DDM-9207347. The goal of the project is to analyze and develop interior-point algorithms (e.g., [3][4][5][8] [9][10][11]) for solving large-scale linear, quadratic, and convex optimization problems that arise in production management and resource allocation. It is a continuation of my previous projects, and it is motivated by the demand to resolve several theoretical and practical issues concerning interior-point algorithms. In this project I do research to strengthen existing results and to explore new techniques for analyzing and improving interior-point algorithms. I also do research on implementation and application to solving problems in production scheduling and network planning. My research topics include the following.

1. Initialization and Termination: Develop efficient initialization and termination techniques to generate an initial point, and obtain an optimal basic solution or detect infeasibility.

2. Condition-Based Complexity: Study complexity issues for linear programming (LP) with data given by real numbers, analyze "condition" numbers that determine the degree of difficulty of LP problems, and "precondition" some problems and develop strongly polynomial interior-point algorithms.

3. Decomposition and Column Generation: Develop various decomposition and column generation schemes for solving large-scale or infinite-dimensional practical optimization problems.

4. Nonlinear Optimization: Extend LP techniques to solving nonlinear optimization problems arising from engineering design and control.

5. Implementation and Application: Implement the state of the art interior-point algorithm, and apply the algorithm to solving real-world problems.

Progresses have been made in each of these areas since my last annual report. The results are better than what I expected.

- Several new publications and working papers have been resulted from the project: seven of them in *Operations Research Letters*, *ACM Symposium on Theory of Computing*, *Operations Research*, *SIAM J. on Optimization*, *Mathematics of Operations Research*, or *Annals of Operations Research.* These papers are listed in Section 5, where [J*] for journal publication and [W*] for working paper.

- A computational optimization laboratory (COL) has been established in my department at Iowa. Currently we have one post-doctor associate, and one visiting Ph.D. student fully supported by the Danish government. Another visiting student, supported by the Brazil government, will join us soon.

- Two implementation codes have been developed and installed in several major universities. One solves linear programming and the other solves nonlinear optimization.

- The MCI Real-Time-Restoration project, where my students and I did joint research work with MCI, has been implemented into operation. MCI has reported a significant improvement to its service.

Below, I provide partial results and discuss ongoing research activities.

2 Project Progress

2.1 Progress in Initialization and Termination

First, in working paper [W7] Xu and I generalized the simplified homogeneous and self-dual LP algorithm (see publication [J4] and [14]). Again, the algorithm does not need to start from a big-M initial point. In particular, the searching direction is obtained by solving a Newton equation system without infeasibility residual terms on its right hand side. This will eliminate any accumulate numerical error during the iterative process.

Second, in exact arithmetic the simplex method applied to a particular linear programming problem instance either shows that it is infeasible, shows that its dual is infeasible, or generates optimal solutions to both problems. Interior-point methods do not provide such clear-cut information. In [W2] Todd and I provide general tools (extensions of the Farkas Lemma) for concluding that a problem or its dual is likely (in a certain well-defined sense) to be infeasible, and apply them to develop stopping rules for a generic infeasible-interior-point method and for the homogeneous self-dual algorithm for linear programming. These rules allow precise conclusions to be drawn about the linear programming problem and its dual: either near-optimal solutions are produced, or we obtain "certificates" that all optimal solutions, or all feasible solutions to the primal or dual, must have large norm. Our rules thus allow more definitive interpretation of the output of such an algorithm than previous termination criteria. We give bounds on the number of iterations required before these rules apply. Our tools may also be useful for other iterative methods for linear programming.

Third, in paper [W5] Andersen and I propose an approach to combine linear programming (LP) interior-point and simplex pivoting algorithms, which is related to the work of Bixby and Saltzman [1]. We present a strongly polynomial procedure in any iteration of an interior-point algorithm to produce a basis. We show that this basis will eventually become optimal after a finite number of interior-point iterations. Our combined method naturally handles the case where the interior-point algorithm does not provide an exact optimal or feasible solution. In fact, our method allows either cross over to the simplex algorithm or to execute the two algorithms alternately. Furthermore, if LP data are integers, the total number of pivoting steps to generate an optimal basis can be bounded above by $O(n\sqrt{n}L)$, where L is the data size and n is the number of variables in the standard primal form. Later, I'll report partial computational results for this combined approach.

2.2 Progress in Condition-Based Complexity

Consider a linear programming (LP) problem in the standard form:

$$\begin{array}{lll}(LP) & \text{minimize} & c^T x \\ & \text{subject to} & Ax = b, \ x \geq 0,\end{array}$$

where $c \in R^n$, $A \in R^{m\times n}$ and $b \in R^m$ are given, $x \in R^n$, and ^{T} denotes transpose.

Based on our previous work [J1][J2][J3], Vavasis and I [W1] recently developed a **primadual** "layered-step" interior point (LIP) algorithm for linear programming. This algorithm follows the central path, either with short steps or with a new type of step called a "layered least squares" (LLS) step. The algorithm returns the exact global minimum after a finite number of steps—in particular, after $O(n^{1.5}|B||N|\chi(A))$ iterations, where $\chi(A)$ is a function of the coefficient matrix, and sets B and N represent the strictly complementary partition of the LP solutions; thereby $|B| + |N| = n$. Note that this bound is independent of b and c.

The LLS steps can be thought of as accelerating a path-following interior point method whenever near-degeneracies occur. One consequence of the new method is a new characterization of the central path: we show that it composed of at most $|B| * |N|$ alternating straight and curved segments. This is the first such result for LP with data given by real numbers. If the LIP algorithm is applied to integer data, we get as another corollary a new proof of a well-known theorem by Tardos that linear programming can be solved in strongly polynomial time provided that A contains small-integer entries.

Todd and I also proved a complexity lower bound for interior-point LP algorithms [W8]. We have shown that there is an LP problem that almost all polynomial interior-point algorithms need $\Omega(n^{1/3})$ iterations to solve it.

2.3 Progress in Column Generation

Methods, which allow for column generation—the addition of columns or constraints to the system only when needed—permit a great deal of flexibility for solving optimization problems in which the number of columns is either very large or not explicitly known. Thus, we need to develop interior-point algorithms that allow decomposition or column generation and are implementable with practical efficiency. This will advance the practical solvability for a large class of optimization problems, such as semi-infinite programming, convex nonlinear programming, and combinatorial optimization.

Several approaches for column generation or decomposition were proposed and developed. One method, the analytic center cutting plane method was developed and analyzed by several people, including myself ([2][12][13]), to solve general convex problems defined by a separation oracle. The oracle is called at the analytic center of a polytope, which contains a solution set and is given by the intersection of the linear inequalities previously generated from the oracle. If the center is not in the solution set, separating hyperplanes will be placed through the center to shrink the containing polytope. Its complexity or convergence result has been established in solving general convex problems by Goffin, Luo and me [J5], where we proved that the method is a pseudo-polynomial algorithm for solving general convex optimization problems.

However, the complexity bound holds only when one cutting plane is placed in each step. Thus, the question remains open when multiple cuts are added. Moreover, adding multiple cuts actually is a key to practical effectiveness and it presents theoretical difficulties in analyzing other cutting plane methods. In paper [W3], I overcome this difficulty and show that the analytic center cutting plane algorithm, with multiple cuts added in each step, still is a fully polynomial approximation algorithm.

2.4 Progress in Nonlinear Optimization

First, I have extended the LP homogeneous and self-dual algorithm to the monotone linear complementarity problem which is a quadratic optimization problem [W4]. The problem was an interesting open question. Thus, we can solve QP or detect infeasibility using interior-point algorithm without the big-M technique. More recently, Andersen and I have extended the algorithm to solving the monotone nonlinear complementarity problem, which includes general convex optimization problems.

Second, I have developed an algorithm to solving the von Neumann economic growth problem, which is a quasi-convex multi-fractional optimization problem [J7]. In this paper, I established the first polynomial algorithm for solving this problem.

Third, my colleagues Kortanek, Xu and I developed some theoretical and computational results for solving the posynomial geometric programming (GP) problem using an infeasible primal-dual interior-point algorithm. We applied the Newton method for a perturbed KKT system arising from the dual GP problem and develop effective techniques in choosing iterate direction and step length. Our computational results indicate that the algorithm, coupled with these new techniques, leads to an efficient and stable implementation for solving GP programs. Competitive numerical results are presented for 40 or so of the most challenging posynomial geometric programming problems in the field, where for a significant number of these we see similar numerical performance to linear programming ([6][7]).

2.5 Progress in Implementation

Here, I present our computational results of the approach that combines interior-point and simplex pivoting algorithms [W5]. The whole source code is written in ANSI C and is compiled on a HP 715/75 workstation using the HP C compiler and +O3 optimization option. All computational results are reported on the same workstation. All timing results are obtained with the C procedure "clock" which measures the time in number of cpu sec.

The NETLIB problems are relative small and may not really represent todays large-scale LP problems. Therefore, we evaluate the cross-over procedure for the so-called Kennington

Name	ROWS	COLS
CRE-A	3517	4067
CRE-B	9649	72447
CRE-C	3069	3678
CRE-D	8927	69980
KEN-07	2427	3602
KEN-11	14695	21349
KEN-13	28633	42659
OSA-07	1119	23949
OSA-14	2338	52460
OSA-30	4351	100024
PDS-02	2954	7535
PDS-06	9882	28655
PDS-10	16559	48763
Sum	108120	479168

Table 1: The Kennington problems.

Name	RGAP	ITE	PBI	DBI	TOPT	TCRO	RATIO
CRE-A	9.7E-11	30	4.9E-14	5.8E-11	19.83	1.98	0.10
CRE-B	4.4E-15	45	1.2E-13	5.8E-11	289.36	19.07	0.07
CRE-C	4.9E-09	29	2.3E-13	8.7E-11	16.89	1.56	0.09
CRE-D	6.0E-11	45	1.1E-13	2.9E-11	245.94	13.13	0.05
KEN-07	9.9E-11	16	0.0E+00	7.3E-12	5.48	0.36	0.07
KEN-11	3.9E-12	21	0.0E+00	4.5E-12	65.31	6.04	0.09
KEN-13	1.5E-11	25	0.0E+00	3.8E-12	200.17	29.96	0.15
OSA-07	3.7E-09	31	0.0E+00	6.8E-13	53.72	0.74	0.01
OSA-14	6.1E-10	18	0.0E+00	1.6E-12	78.34	2.19	0.03
OSA-30	3.2E-11	26	5.7E-14	2.0E-12	210.23	10.38	0.05
PDS-02	1.8E-11	27	0.0E+00	0.0E+00	21.52	1.76	0.08
PDS-06	2.0E-14	38	0.0E+00	0.0E+00	455.78	28.83	0.06
PDS-10	2.5E-13	44	0.0E+00	0.0E+00	2564.14	84.37	0.03
Sum		395			4226.71	200.37	0.05

Table 2: Termination results for the Kennington problems.

problems. In Table 1, we have shown some problem related statistics. The columns ROWS and COLS show the numbers of rows and columns of the constraint matrix in each of the 13 problems.

In Table 2 we have shown the computational results for solving the Kennington problems. The column RGAP shows the relative duality gap of the interior-point solution and ITE shows the number of interior-point iterations, when the cross-over starts. PBI and DBI are the maximal primal and dual bound infeasibilities, respectively. The columns TOPT and TCRO show the total time spent in optimizing and in doing cross-over, respectively. TCRO is included in TOPT. Finally, RATIO shows the ratio TCRO/TOPT.

We see that in all the cases we obtain an optimal basic solution to the original problem with very low PBI and DBI. Usually PBI and DBI are below 10^{-10}, which is a very good accuracy. Often the feasibility measures are much better. It can be seen that all the problems are solved in total 4227 cpu. sec. and out of these are 200 spent in the cross-over procedure. In other words, on average we obtain the optimal basic solution in less than 5% of the interior-point time.

2.6 Progress in Application

My students and I have been supported by an MCI grant on research for a Real-Time Restoration project. This project is to improve the restoration process for the MCI network during emergence situation. We have applied mathematical modeling and optimization techniques to reroute communication traffic as quickly and efficiently as possible. The project has been implemented and put into operation in 1993. The result? According to *MCI Network National*, February, 1994, "What used to take just more than an hour to reroute traffic during a catastrophic transmission outage, now only takes minutes with the implementation of Real-Time Restoration."

My colleagues and I [J6] have also applied an optimization algorithm to solving a popular production management problem: structured uncapacitated facility location problem. We developed a strongly polynomial algorithm with the currently best complexity bound.

3 Education and Human Resources

Some beneficiaries, educationally, of the project are the Ph.D. students at the University of Iowa. We have supported several Ph.D. students at UI from this NSF grant. Some results mentioned above are part of their Ph.D. theses.

In particular, my student Hung dealt with path-following algorithms, which generate a sequence of points within certain neighborhoods of a central-path. They prevent iterates from prematurely getting too close to the boundary of the feasible region. Depending on their norm used, these neighborhoods have different sizes. A paradox is that among all existing (infeasible or feasible) path-following algorithms, the theoretical iteration complexity, $O(\sqrt{n}L)$, of small-neighborhood algorithms is significantly better than the complexity, $O(nL)$, of wide-neighborhood algorithms, while in practice wide-neighborhood algorithms outperform small-neighborhood ones by a big margin. Here again, n is the number of LP variables and L is the LP data length. In paper [W6], we present an $O(n^{\frac{n+1}{2n}}L)$-iteration (infeasible) primal-dual high-order algorithm that uses wide neighborhoods. Note that this iteration bound is asymptotical $O(\sqrt{n}L)$, i.e., the best bound for small-neighborhood algorithms, as n increases.

Another outgrowth of our research has been a new seminar, meeting once a week, among graduate students and faculty members from several departments.

I have also brought some of the new research topics and results to the optimization and mathematical programming course. The following is a sample of my syllabus:

TOPICS: 6K287 emphasizes recent progresses in linear and nonlinear optimization. In addition to classical topics:

- Kuhn-Tucker optimality conditions,
- Farkas' lemma and duality theorems,
- convergence efficiency and complexity,
- Newton's method, and several other popular computational methods,

we will cover and analyze more recent developments:

- analytic center, central path, potential theory for convex cones and convex sets,
- condition-based complexity and accelerated path-following algorithms for linear programming,

- homogeneous and "infeasible" interior-point algorithms for convex programming and monotone complementarity problem,
- positive-definite programming: optimization in positive semi-definite matrix space,
- cutting-plane and column-generation methods for semi-infinite and fractional programming, and
- their applications in engineering, management and combinatorial optimization.

The course will also discuss future research directions in each of these areas.

4 Other Research Support

I took a sabbatical leave from the University of Iowa during the 1993-1994 academic year. I visited Cornell University in Fall 1993, supported by the Theory Center of Cornell University for $19,000. In Spring 1994, I visited Delft University, Holland, where I was supported by a fellowship $12,000 from Dutch Organization for Scientific Research (NWO).

5 Related Publication and Working Papers

5.1 New Publications

[J1] S. Vavasis and Y. Ye, "An accelerated interior-point method whose running time depends only on A," *Proc. of the Twenty-Sixth ACM Symposium on Theory of Computing* (1994) 512-521.
[J2] S. Vavasis and Y. Ye, "Condition numbers for polyhedra with real number data," to appear in *Operations Research Letters.*
[J3] S. Vavasis and Y. Ye, "Identifying an optimal basis in linear programming," to appear in *Annals of Operations Research.*
[J4] X. Xu, P. Hung, and Y. Ye, "A simplified homogeneous and self-dual linear programming algorithm and its implementation," to appear in *Annals of Operations Research.*
[J5] J. Goffin, Z. Luo and Y. Ye, "Complexity analysis of a potential reduction and column generation algorithm for convex programming," to appear in *SIAM J. Optimization.*
[J6] P. Jones, T. Lowe, G. Muller, N. Xu, Y. Ye, and J. Zydiak, "Specially structured uncapacitated facility location problems," to appear in *Operations Research.*
[J7] Y. Ye, "On the von Neumann economic growth problem," to appear in *Mathematics of Operations Research.*

5.2 New Working Papers

[W1] "An primal-dual accelerated interior-point method whose running time depends only on A," (S. Vavasis and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (September, 1994).
[W2] "Approximate Farkas lemmas and stopping rules for iterative infeasible-point algorithms for linear programming," (M. J. Todd and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (September, 1994).

[W3] "Complexity analysis of the analytic center cutting plane method that uses multiple cuts," Working Paper, Department of Management Sciences, The University of Iowa (September, 1994).
[W4] "On homogeneous and self-dual algorithm for LCP," Working Paper, Department of Management Sciences, The University of Iowa (July, 1994).
[W5] "Combining interior-point and pivoting algorithms for linear programming," (E. D. Andersen and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (May, 1994).
[W6] "A asymptotical $O(\sqrt{n}L)$-iteration path-following linear programming algorithm that uses long steps," (P. Hung and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (March, 1994).
[W7] "A generalized homogeneous and self-dual linear programming algorithm," (X. Xu and Y. Ye), Working Paper, Department of Management Sciences, The University of Iowa (1994).
[W8] "A lower bound on the number of iterations of long-step and polynomial interior-point linear programming algorithms," (M. J. Todd and Y. Ye), Technical Report, School of ORIE, Cornell University (1994).

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