Interior-Point Algorithms: Final Project Report
NSF Grant DDM-8922636

Principal Investigator: Yinyu Ye

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1. PART I—Project ID Information

See NSF Form 98A.

2. PART II—Summary of Completed Project

This project is to develop and analyze recent interior-point algorithms, a breakthrough development in optimization, for solving large-scale linear programming and nonlinear optimization problems that arise in production management and resource allocation. These algorithms have the potential to be faster than the traditional methods currently in use and could solve problems never solved before. In this project, we have done research to strengthen some of the existing results and to explore new techniques and tools for analyzing and improving algorithm's efficiency. These include average complexity analysis, asymptotic convergence rate, decomposition and column generation, and nonlinear and global optimization. We have also successively resolved several remaining difficult issues concerning interior-point algorithms such as initialization, termination and detection of unboundedness, and applied our theoretical results to solving some industrial problems.

The project resulted in 15 publications in several major journals of operations research and management sciences. We have also applied our results to solving two industrial problems. The educational beneficiaries of the project are the students at the University of Iowa. Several Ph.D. students are partially supported by this grant and they are actively involved in the project. Some results are part of their Ph.D. theses and published in top journals.

In general, progress in the area of developing efficient algorithms for solving large-scale optimization problems will be of great importance in improving the efficiency of manufacturing systems, communication networks, aircraft routing, multi-commodity-flow operations, and resources planning. Strengthening research in this area will definitely contribute to the national interest in industrial competitiveness and scientific knowledge.

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3. PART III—Technical Information

3.1. Introduction

This document is the final report on an operations research project funded by NSF Grant DDM-8922636. The project principally considers the interior-point algorithms for large-scale linear programming (LP), quadratic programming (QP), linear complementarity problem (LCP), and nonlinear programming (NP) problems arisen in production management and manufacturing. The discoveries and advances resulting from the project include the following.

1. Algorithm Initialization: we developed a homogeneous and self-dual LP algorithm that completely avoids any use of the Big M technique. The algorithm also detects unboundedness and infeasibility, which issue has not been addressed in satisfaction by the current interior-point algorithm implementation.
2. Algorithm Termination: we developed efficient termination techniques to "purify" an approximate optimal solution to an (exact) optimal solution, and improved the complexity result for a class of optimization problems.
3. Average Complexity: we analyzed the complexity of interior-point algorithms for solving random problems, and probabilisticly explained the excellent practical behavior of interior-point algorithms.
4. Asymptotic Convergence: we developed and analyzed superlinearly convergent interior-point algorithms, and improved the algorithms' global and asymptotic complexity results.
5. Decomposition and Column Generation: we developed various decomposition and column generation schemes for solving large-scale combinatorial, nonlinear and stochastic programming problems using interior-point algorithms.
6. Nonlinear and Global Optimization: we developed extensions of interior-point algorithms for nonlinear optimization, and established first polynomial complexity results for several important nonconvex decision problems.

We briefly summarize several results from this project. In what follows, the references [•] are listed in Section 6.

3.2. Algorithm Initialization

Recently, myself, Todd and Mizuno [W1] developed an $O(\sqrt{n}L)$-iteration homogeneous and self-dual LP algorithm. Our algorithm possesses the following features: 1) It solves the linear programming problem without any regularity assumption concerning the existence of optimal, feasible, or interior feasible solutions; 2) It can start at any positive primal-dual pair, feasible or infeasible, near the central ray of the positive orthant (cone), and it does not use any big M penalty parameter or lower bound; 3) Each iteration solves a system of linear equations whose dimension is almost the same as that solved in the standard (primal-dual) interior-point algorithms; 4) If the LP problem has a solution, the algorithm generates a sequence that approaches feasibility and optimality simultaneously, or if the problem is infeasible or unbounded, the algorithm will correctly detect infeasibility for at least one of the primal and dual problems.

This development resolved several key remaining issues concerning interior-point algorithms for LP. We are implementing this algorithm and developing a computer code.

3.3. Algorithm Termination

Unlike the simplex method for LP which terminates in *finite* time, interior-point algorithms are continuous optimization algorithms that generate a solution sequence converging to the optimal solution set in *infinite* time. Ever since the rebirth of interior-point algorithms, efforts have been made in finding efficient algorithm-termination techniques.

We have recently developed a new termination technique based on identifying the optimal face and projecting an interior point onto the face [J5], based on an early result of Güler and myself [J7]. In particular, 1) the termination criterion is guaranteed to work in polynomial time, 2) the projection procedure (solving one least-squares problem) is strongly polynomial and can be efficiently performed on *parallel* computers, 3) the termination identifies the optimal face which is useful in sensitivity analysis, and 4) it works very well in practice.

Mehrotra and I [J4] have given computational results to study the practical performance of the termination technique. A FORTRAN 77 implementation of an interior-point algorithm and **Netlib** test problems are used in the experiment. For 74 out of the 86 tested **Netlib** problems an interior point on the optimal face was found on the first attempt. More than two attempts are only needed for five problems. The following are the computational results for five problems, which are typical for all **Netlib** problems.

					Optimal Value	
Name	Rows	Cols	Nonzeros	BR	MINOS (on vax)	CPLEX
Scrs8	491	1169	4029		9.0429998619e+2	9.0429695380e+2
Scsd1	78	760	3148		8.6666666743e+0	8.6666666743e+1
Scsd6	148	1350	5666		5.0500000078e+1	5.0500000078e+1
Scsd8	398	2750	11334		9.0499999993e+2	9.0499999993e+2
Sctap1	301	480	2052		1.4122500000e+3	1.4122500000e+3

Table 1: NETLIB Problem Data Summary

				Objective Value	
Name	itr. start	itr. stop	number of tries	Primal	Dual
Scrs8	21	21	1	9.0429695380079E+2	9.0429695380079E+2
Scsd1	8	8	1	8.6666666743334E+1	8.6666666743334E+1
Scsd6	10	12	3	5.0500000077144E+1	5.0500000077144E+1
Scsd8	9	9	1	9.0499999992546E+2	9.0499999992546E+2
Sctap1	15	15	1	1.4122500000000E+3	1.4122500000000E+3

Table 2: Computational Results for the Face Finding

Name	$\frac{\lvert b^T y^* - c^T x^*\rvert}{1+\lvert b^T y^*\rvert}$	$\frac{\lVert b-Ax^*\rVert}{1+\lVert b\rVert}$	$\frac{\lVert c-A^T y^* - s^*\rVert}{1+\lVert c\rVert}$	$\max_i\{x_i^*\}$	$\min_i\{x_i^{*+}\}$	$\max_i\{s_i^*\}$	$\min_i\{s_i^{*+}\}$
Scrs8	5.0E-16	2.0E-15	8.6E-17	7.909E+2	7.772E-4	4.551E+3	6.028E-3
Scsd1	5.5E-16	1.0E-16	6.9E-17	5.270E-1	4.730E-3	8.000E+0	6.860E-3
Scsd6	0.	1.6E-16	7.9E-17	1.356E+0	2.074E-2	8.000E+0	8.113E-10
Scsd8	5.0E-16	6.5E-16	3.7E-16	2.250E+1	2.503E-2	7.304E+0	5.444E-2
Sctap1	6.4E-16	4.2E-16	4.1E-16	8.000E+1	5.216E-3	2.925E+2	2.225E-2

Table 3: Properties of the Interior Point Obtained on the Optimal Face

Table 1 describes the five problems. The optimal objective values using MINOS 5.3 and CPLEX are given. Table 2 gives the computational results of our termination procedure. The second column of this table gives the iteration number at which we start try to identify the optimal face for the first time. The third column gives the iteration at which the optimal face is identified and an optimal pair is found. Column 4 gives the number of projections. Columns 5 and 6 give primal and dual objective values recorded at the point generated on the optimal face.

Table 3 provides additional information on the quality of primal and dual optimal solutions generated by our procedure. The second column of this table gives the relative error in the objective value at optimality. Columns 3 and 4 give the relative error of primal feasibility and dual feasibility, respectively. Columns 5 to 8 give information on the range of positive primal and dual slacks for the generated optimal solutions.

Two papers in this topic are accepted for publication. Several practitioners tested the technique and began to use it in their software development. This work also lead to several results on the probabilistic analysis of the LP interior-point algorithms.

3.4. Average Complexity

Probabilistic or average complexity analysis of algorithm's performance is a major research topic in computer science and operations research. It has been a "hot" topic for the simplex method in the past decade. However, it is relatively new for interior-point algorithms. Some non-rigorous analyses were discussed, which are based on a probabilistic assumption that holds only in one iteration of the interior-point algorithms [J10][J16].

Quite recently, we developed a rigorous analysis of interior-point algorithms for some randomly generated LP problems [J1]. We have shown that, with high probability (probability converges to 1 as the number of variables tends to ∞) the number of iterations required to solve a LP problem is bounded above by $O(\sqrt{n}\log(n))$, where n is the number of variables in the standard LP form. These random LP problems include Borgwardt's and Todd's models with standard Gauss distributions. This is the first rigorous probabilistic complexity result for interior-point LP algorithms.

The *Mathematics of OR* Associate Editor's report stated, "This is a very good paper, giving a new and interesting method to solve a large class of LP problems in strongly polynomial time ... This bound is sharper than all known bounds (as far as known to the experts I have consulted)." This work also motivated several other probabilistic complexity results of my colleagues and myself.

3.5. Asymptotic Convergence

The asymptotic convergence order is a widely used criterion judging the efficiency of an algorithm. Recently there have been some developments on the asymptotic superlinear convergence of interior-point algorithms for LP. Among them is quadratic convergence for general LP which assumes neither the convergence of the iteration sequence nor nondegeneracy. It was first established by myself, Güler, Tapia and Zhang [J6] (and independently by Mehrotra). The result is based on a predictor-corrector algorithm of Mizuno, Todd and myself [J10]. Each iteration of the algorithm needs to solve two systems of linear equations or two least squares problems—one in the predictor step and one in the corrector step. If one counts each iteration as solving one system of linear equations, as is usually done in the analysis of interior-point algorithms, the (average) order of convergence of the algorithm is only $\sqrt{2}$. More recently, I developed an $O(\sqrt{n}L)$ algorithm whose order of convergence exactly equals 2 [C1]. This result for LP has also been extended to solving a more complex problem, the linear complementary problem (LCP), by myself and Anstreicher [J2]. In the *Mathematical Programming* Associate Editor's report on our result, it was written, "Both referees are extremely positive about this paper, in content and exposition. One referee's letter states 'I give it my highest recommendation'." Our result was also one of the most recommended papers at the 1992 SIAM Optimization Conference at Chicago according to the leading experts in this field.

3.6. Decomposition and Column Generation

While various interior point algorithms have been developed for LP, they all share a common drawback: every column (constraint) is active in computation. In contrast, the (revised) simplex and ellipsoid methods do not require every column to be active. These methods allow for column generation— the addition of columns to the system only when needed. This technique permits a great deal of flexibility for solving optimization problems in which the number of columns is either very large or not explicitly known. We need to develop interior-point algorithms that allow decomposition or column generation and are implementable with practical efficiency. This will advance the practical solvability for a large class of optimization problems, such as semi-infinite programming, convex nonlinear programming, and combinatorial optimization.

We have developed and analyzed a potential reduction and column generation algorithm [J15]. When the number of columns is finite, we proved that it is a polynomial-time algorithm. Quite recently, we finally proved that the algorithm will terminate in finite time even when the number of columns is infinite [W5]. As a result, the algorithm can be used for solving general convex optimization problems. Several similar algorithms and implementations have been developed to solve a variety of nonlinear, combinatorial and stochastic optimization problems, and they performed quite well.

A decomposition approach was also proposed by Dantzig and myself [W4] which has the same convergence properties as the well known dual affine scaling algorithm but may require less computational effort. A variant [W4] for linear programming with column generation has been developed and implemented by John Kaliski, one of our Ph.D. students, on an IBM 3090-200S machine. Table 4 illustrates some computational results for solving the LP relaxation of some large-scale airline crew scheduling problems. In these problems $m = 837$ and n ranges from .5 to 12.7 million in the LP standard form. The second column is the number of interior-point algorithm iterations. The third column is the maximum number of columns being active in each iteration. The fourth column is the CPU time in seconds for solving the problem, including the use of the simplex method to find an optimal basic solution.

n	Number of Iterations	Number of Active Columns	Total CPU Time
.5 mil	30	5595	188
1 mil	30	4995	274
2 mil	29	4995	352
4 mil	28	4995	512
8 mil	28	4995	759
12.7 mil	30	4995	1159

Table 4: Solving LP Problems with Decomposition and Column Generation

When the interior-point algorithm stops, the objective value at the interior solution is usually within 10^{-4} relative accuracy of the optimal value. Then, the simplex method is used to purify the interior solution to an optimal basic solution. This hybrid procedure was first proposed by Bixby et al. Note that the number of interior-point iterations in our decomposition method is much lower than that of Bixby et al. (about 120 iterations). Also, the number of active columns in our approach is much smaller than that of Bixby et al. (approximately 40,000 columns). Of course, we have to do more pricing work, i.e., pricing out inactive columns in each iteration. But here vectorization and parallelization can effectively be used to reduce the impact of this extra pricing.

The resulting paper [J3] has been accepted for publication in *Management Sciences*.

3.7. Nonlinear and Global Optimization

A number of my papers are concerned with nonconvex optimization, including nonconvex QP and Linear Complementarity Problem (LCP), most of which are NP-hard problems. My paper [J17] studied an interior point algorithm for nonconvex QP and proved its convergence. One of my papers [J20] is written with Kojima of Japan and Megiddo of IBM. The paper produced a complexity result for P-matrix LCP that has wide applications in engineering and economic equilibrium computation.

Later, I enhanced this result in another paper [J11] and developed a general class of LCP solvable in polynomial time in a paper [J19] with Pardalos. More recently, I proved the first fully polynomial approximation algorithm for computing a stationary point of the general LCP [J8]. Some computational experiences were given in [J13].

I first proved that the ball constrained quadratic optimization problem can be solvable in polynomial time [J17]. I further improved its computational complexity significantly. This work has been well received in the global optimization field [J14].

Kortanek, Potra and I have extended interior-point LP algorithms to solving convex nonlinear optimization problem [J9][J18]. These problems, including entropy problems, have wide applications in mechanical engineering, image construction, material engineering etc, [J12].

Recently, I also analyzed the first polynomial algorithm for the von Neumann Economic Growth problem [W2]. This work has been extended for optimization problems that arise in engineering design and control theory.

4. Industrial Applications

We have done research on two applications:

1) The airline crew scheduling problem. We will continue our research on the LP relaxation of the airline crew scheduling problems. The finite termination, parallel computation, and decomposition techniques are particularly useful in improving the method's performance for solving these problems.

2) The large-scale network flow problem. The large-scale network flow problem, including the multi-commodity flow problem, can be formulated as an LP problem with a large number of columns. We plan to investigate a decomposition variant of interior-point algorithms for solving this problem. We are currently working under a contract from MCI Telecommunication to do research on a real application of this problem. This application has significant importance to the reliability and performance of communication networks.

5. Education and Human Resources

As mentioned earlier, several Ph.D. students at UI are partially supported by the grant and they are actively involved in the project. Some results mentioned above are part of their Ph.D. theses.

John Kaliski is graduating this year. His thesis deals with decomposition and column generation techniques. He and others have written several technical reports, four of which have been accepted for publication in *Mathematical Programming*, *Operations Research Letters*, and *SIAM Journal on Matrix Analysis and Applications*. He is now working at IBM.

Paul Gray was a Master student in the Department of Mathematics of UI. He joined the first phase of this project and did many computer program codings. He is now a Ph.D student in the Applied Math Program of Michigan State University.

Siming Huang, another Ph.D. student, is interested in the probabilistic analysis. He has written two papers on this issue. Similarly, several new Ph.D. students, Goran Lesaja and Pi-Fang Hung, in the Applied Math Program at UI are continuing to do research in this project.

Another outgrowth of our research has been a new seminar, meeting once a week, among graduate students and faculty members from several departments. We have also brought some of the new research topics and results to the optimization and mathematical programming courses at the graduate level.

6. Publications and Technical Reports

6.1. Journal Papers

[J1] "Toward probabilistic analysis of interior-point algorithms for linear programming," Working Paper No. 91-13, College of Business Administration, The University of Iowa (1991), accepted in *Mathematics of Operations Research*

[J2] "On quadratic and $O(\sqrt{n}L)$ convergence of a predictor- corrector algorithm for LCP," (Y. Ye and K. Anstreicher) Working Paper No. 91-20, College of Business Administration, The University of Iowa (1991), accepted in *Mathematical Programming A*.

[J3] "A decomposition variant of the potential reduction algorithm for linear programming," (J. Kaliski and Y. Ye) Working Paper No. 91-11, College of Business Administration, The University of Iowa (1991), to appear in *Management Science*.

[J4] "On finding an interior point on the optimal face of linear programs," (S. Mehrotra and Y. Ye) Technical Report, Department of IE and MS, Northwestern University (1991), to appear in *Mathematical Programming A*.

[J5] "On the finite convergence of interior-point algorithms for linear programming," *Mathematical Programming* **57** (1992) 325-335.

[J6] "A quadratically convergent $O(\sqrt{n}L)$-iteration algorithm for linear programming," (Y. Ye, O. Güler, R. Tapia and Y. Zhang) Working Paper No. 91-14, College of Business Administration, The University of Iowa (1991), to appear in *Mathematical Programming A*.

[J7] "Convergence behavior of some interior-point algorithms," (O. Güler and Y. Ye), Working Paper No. 91-4, College of Business Administration, The University of Iowa (1991), to appear in *Mathematical Programming A*.

[J8] "A fully polynomial-time approximation algorithm for computing a stationary point of the general LCP," Working Paper No. 90-10, College of Business Administration, The University of Iowa (1990), to appear in *Mathematics of Operations Research*

[J9] "An interior-point algorithm for solving entropy optimization problems with globally linear and locally quadratic convergence," (F. Potra and Y. Ye) Working Paper No. 90-22, College of Business Administration, The University of Iowa (1990), to appear in *SIAM J. on Optimization*.

[J10] "On adaptive-step primal-dual interior-point algorithms for linear programming," (S. Mizuno, M. Todd and Y. Ye) Technical Report No. 944, School of ORIE, Cornell University (1990), to appear in *Mathematics of Operations Research*

[J11] "A further result on potential reduction algorithm for the P-matrix linear complementarity problem," in P. Pardalos eds., *Advances in Optimization and Parallel Computing* (North-Holland, NY, 1992).

[J12] "Implementation of interior-point algorithms for some entropy optimization problems," (C. Han, P. Pardalos and Y. Ye) *Optimization Methods and Software* **1** (1992) 71-80.

[J13] "Solutions of P_0-matrix linear complementarity problems," (P. Pardalos, Y. Ye, C. Han and J. Kaliski) manuscript (1990), to appear in *SIAM J. on Matrix Analysis and Applications*.

[J14] "A new complexity result on minimization of a quadratic function with a sphere constraint," in C. Floudas and P. Pardalos eds., *Recent Advances in Global Optimization* (Princeton University Press, NJ, 1992).

[J15] "A potential reduction algorithm allowing column generation," *SIAM J. on Optimization* **2** (1992) 7-20.

[J16] "Comparative analysis of affine scaling algorithms for linear programming," *Mathematical Programming* **52** (1992) 405-414.

[J17] "On affine scaling algorithms for nonconvex quadratic programming," manuscript (revised July 1989), to appear in *Mathematical Programming A*.

[J18] "On some efficient interior point methods for nonlinear convex programming," (K. Kortanek, F. Potra and Y. Ye) *Linear Algebra and its Applications* **152** (1991) 169-189.

[J19] "A class of LCPs solvable in polynomial time," (Y. Ye and P. Pardalos) *Linear Algebra and its Applications* **152** (1991) 3-17.

[J20] "An interior point potential reduction algorithm for the linear complementarity problem," (M. Kojima, N. Megiddo and Y. Ye) *Mathematical Programming* **54** (1992) 267-279.

6.2. Conference Papers

[C1] "On the Q-order of convergence of interior-point algorithms for linear programming," in Wu Fang, ed., *Proc. of the 1992 Symp. on Applied Mathematics* (Institute of Applied Mathematics, Chinese Academy of Sciences, 1992).

6.3. Working Papers

[W1] "An $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming algorithm," (Y. Ye, M. Todd and S. Mizuno) Working Paper No. 92-11, College of Business Administration, The University of Iowa (1992).

[W2] "On the von Neumann economic growth problem," Working Paper No. 92-9, College of Business Administration, The University of Iowa (1992).

[W3] "Combining binary search and Newton's method to compute real roots for a class of real functions," Working Paper No. 91-7, College of Business Administration, The University of Iowa (1991).

[W4] "A 'build-up' interior method for linear programming," (G. Dantzig and Y. Ye) SOL Report, Department of Operations Research, Stanford University (1990).

[W5] "Convergence of a potential-reduction and column- generation algorithm for convex feasibility problems," Working Paper No., College of Business Administration, The University of Iowa (1993).